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Approximation solvability of two nonlinear optimization problems involving monotone operators

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Abstract

Fixed points of strict pseudocontractions and zero points of two monotone operators are investigated based on a viscosity iterative method. A strong convergence theorem of common solutions is established in the framework of Hilbert spaces. The results obtained in this paper improve and extend many corresponding results announced recently.

Keywords: Iterative process, quasi-variational inclusion, nonexpansive mapping, fixed point.
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1. Introduction and Preliminaries

Quasi-variational inclusion problem was recently extensively investigated by many authors. The problem has emerged as an interesting branch of applied mathematics with a wide range of applications in industry, finance, economics, optimization, and medicine; see [1, 2, 4, 8, 9, 17, 18] and the references therein. The ideas and techniques for solving quasi-variational inclusion problem are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. Fixed point methods are efficient and powerful to solving the inclusion problem. In this paper, we use a viscosity fixed point method, which first was introduced by Moudafi [13], to study a quasi-variational inclusion problem. Strong convergence theorems are established without any compact assumptions imposed on the framework of the spaces and the operators.

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. From now on, we use $\rightarrow$ and $\rightharpoonup$

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to denote the strong convergence and weak convergence, respectively. Recall that a space is said to satisfy Opial's condition [14] if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$. Indeed, the above inequality is equivalent to the following

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Let S be a mapping. We use $F(S)$ to stand for the fixed point set of S; that is, $F(S) := \{x \in C : x = Sx\}$. Recall that S is said to be α-contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is known that the fixed point set of the mapping is not empty if the subset C is bounded in the framework of Hilbert spaces.

S is said to be κ-strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\forall x, y \in C, \qquad \|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(x - Sx) - (y - Sy)\|^2.$$

The class of κ-strictly pseudocontractive mappings was introduced by Browder and Petryshyn [5]. Note that the class of κ-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings. That is, S is nonexpansive iff $\kappa = 0$. The class of κ-strict pseudocontractions has been extensively investigated based on viscosity iterative methods since it has a close relation with monotone operators; see [5] and the references therein.

A multivalued operator $B : H \to 2^H$ with the domain $Dom(B) = \{x \in H : Bx \neq \emptyset\}$ and the range $Ran(B) = \{Bx : x \in Dom(B)\}$ is said to be monotone if for $x_1 \in Dom(B)$, $x_2 \in Dom(B)$, $y_1 \in Bx_1$ and $y_2 \in Bx_2$, we have $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. A monotone operator B is said to be maximal if its graph $Graph(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator. Let I denote the identity operator on H and $B : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $\lambda > 0$, a nonexpansive single valued mapping $(I + \lambda B)^{-1}$. It is called the resolvent of B. We know that $B^{-1}0 = F((I + \lambda B)^{-1})$ for all $\lambda > 0$. We also know that $(I + \lambda B)^{-1}$ is firmly nonexpansive; see [7, 10, 15] and the references therein.

Let $A : C \to H$ be a single-valued mapping. Recall that A is said to be monotone iff

$$\forall x, y \in C, \qquad \langle Ax - Ay, x - y\rangle \geq 0.$$

A is said to be ξ-strongly monotone iff there exists a constant $\xi > 0$ such that

$$\forall x, y \in C, \qquad \langle Ax - Ay, x - y\rangle \geq \xi\|x - y\|^2.$$

A is said to be ξ-inverse-strongly monotone iff there exists a constant $\xi > 0$ such that

$$\forall x, y \in C, \qquad \langle Ax - Ay, x - y\rangle \geq \xi\|Ax - Ay\|^2.$$

It is not hard to see that ξ-inverse-strongly monotone mappings are Lipschitz continuous. It is also obvious that every operator is ξ-inverse-strongly monotone iff its inverse is ξ-strongly monotone.

Recall that the classical variational inequality, denoted by $VI(C, A)$, is to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$, $\forall v \in C$. It is known that the variational inequality is equivalent to a fixed point problem. This equivalence plays an important role in the studies of the variational inequalities and related optimization problems.

In this paper, we are concerned with the problem of finding a common element in the intersection: $F(S)\cap(A+B)^{-1}(0)$, where $F(S)$ denotes the fixed point set of κ-strict pseudocontraction S and $(A+B)^{-1}(0)$ denotes the zero point set of the sum of the operator A and the operator B. The results obtain in this paper mainly improve the corresponding results in [3, 6, 7, 11, 16, 19],[21]-[26].

Lemma 1.1 ([20])**.** *Suppose that H is a real Hilbert space and $0 < p \le t_n \le q < 1$ for all $n \ge 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.2 ([3])**.** *Let C be a nonempty, closed, and convex subset of H, $A : C \to H$ a mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $F((I+\lambda B)^{-1}(I-\lambda A)) = (A+B)^{-1}(0)$.*

Lemma 1.3 ([12])**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.4 ([5])**.** *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a strictly psedocontractive mapping. Then S is Lipschitz continuous and $I - S$ is demiclosed at zero.*

2. Main results

Theorem 2.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a ξ-inverse-strongly monotone mapping and let B be a maximal monotone operator on H such that $Dom(B) \subset C$. Let f be a fixed α-contractive mapping on C and let S be κ-quasi-strict pseudocontraction on C. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{\alpha_{n,1}\}$, $\{\alpha_{n,2}\}$, $\{\alpha_{n,3}\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_{n,1} z_n + \alpha_{n,2} f(x_n) + \alpha_{n,3} x_n, \\ x_{n+1} = (1-\beta_n)((1-\gamma_n)Sy_n + \gamma_n y_n) + \beta_n x_n, \quad \forall n \ge 1, \end{cases}$$

where $z_n \approx (I+\lambda_n B)^{-1}(x_n - \lambda_n A x_n)$, the criterion for the approximate computation is $\|z_n - (I+\lambda_n B)^{-1}(x_n - \lambda_n A x_n)\| \le e_n$. Assume that the sequences $\{\alpha_{n,1}\}$, $\{\alpha_{n,2}\}$, $\{\alpha_{n,3}\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy the following restrictions: $\alpha_{n,1} + \alpha_{n,2} + \alpha_{n,3} = 1$, $0 < a \le \beta_n \le b < 1$; $\kappa \le \gamma_n \le c < 1$, $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$; $\lim_{n\to\infty} \alpha_{n_2} = \lim_{n\to\infty} \alpha_{n_3} = 0$, $\sum_{n=1}^{\infty} \alpha_{n,2} = \infty$; $0 < d \le \lambda_n \le e < 2\xi$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\lim_{n\to\infty} \|e_n\| = 0$, where a, b, c, d and e are some real numbers. If $\mathcal{F} = Fix(S) \cap (A+B)^{-1}(0) \ne \emptyset$, then sequence $\{x_n\}$ converges strongly to $\bar{x}$, where $\bar{x}$ solves the following variational inequality $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \ge 0$, $\forall x \in \mathcal{F}$.

Proof. First, we show that $\{x_n\}$ is bounded. Since A is inverse-strongly monotone, we have

$$\begin{aligned}&\|(I-\lambda_n A)y-(I-\lambda_n A)x\|^2\\&={\lambda_n}^2\|Ax-Ay\|^2+\|x-y\|^2-2\lambda_n\langle x-y,Ax-Ay\rangle\\&\le\lambda_n(\lambda_n-2\xi)\|Ax-Ay\|^2+\|x-y\|^2.\end{aligned}$$

Using the restriction imposed on $\{\lambda_n\}$, one has $\|x-y\|\ge\|(I-\lambda_n A)x-(I-\lambda_n A)y\|$. That is, $I-\lambda_n A$ is nonexpansive. Fixing $p\in\mathcal{F}$, we find from Lemma 1.2 that

$$p=(I+\lambda_n B)^{-1}(p-\lambda_n Ap).$$

Since both $(I+\lambda_n B)^{-1}$ and $I-\lambda_n A$ are nonexpansive, we have $\|z_n-p\|\le\|e_n\|+\|x_n-p\|$. It follows that

$$\begin{aligned}\|y_n-p\|&\le\alpha_{n,1}\|z_n-p\|+\alpha_{n,2}\|f(x_n)-p\|+\alpha_{n,3}\|x_n-p\|\\&\le\alpha_{n,1}\|e_n\|+\alpha_{n,1}\|x_n-p\|+\alpha_{n,2}\alpha\|x_n-p\|+\alpha_{n,2}\|f(p)-p\|+\alpha_{n,3}\|x_n-p\|\\&\le(1-\alpha_{n,2}(1-\alpha))\|x_n-p\|+\alpha_{n,2}\|f(p)-p\|+\|e_n\|.\end{aligned}\tag{2.1}$$

Since S is κ-quasi-strictly pseudocontractive on C, one finds from (2.1) that

$$\begin{aligned}&\|\gamma_n y_n+(1-\gamma_n)Sy_n-p\|^2\\&=(1-\gamma_n)\|Sy_n-Sp\|^2+\gamma_n\|y_n-p\|^2-\gamma_n(1-\gamma_n)\|(y_n-p)-(Sy_n-Sp)\|^2\\&\le\gamma_n\|y_n-p\|^2+(1-\gamma_n)(\|y_n-p\|^2+\kappa\|(y_n-p)-(Sy_n-Sp)\|^2)\\&\quad-\gamma_n(1-\gamma_n)\|(y_n-p)-(Sy_n-Sp)\|^2\\&=\|y_n-p\|^2-(1-\gamma_n)(\gamma_n-\kappa)\|(y_n-p)-(Sy_n-Sp)\|^2\\&\le\|y_n-p\|^2.\end{aligned}\tag{2.2}$$

Using (2.1) and (2.2), we find

$$\|\gamma_n y_n+(1-\gamma_n)Sy_n-p\|\le(1-\alpha_{n,2}(1-\alpha))\|x_n-p\|+\alpha_{n,2}\|f(p)-p\|+\|e_n\|.$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|&\le(1-\beta_n)\|\gamma_n y_n+(1-\gamma_n)Sy_n-p\|+\beta_n\|x_n-p\|\\&\le\alpha_{n,2}(1-\beta_n)\|f(p)-p\|+(1-\beta_n)\big(1-\alpha_{n,2}(1-\alpha)\big)\|x_n-p\|\\&\quad+(1-\beta_n)e_n+\beta_n\|x_n-p\|\\&\le\alpha_{n,2}(1-\beta_n)\|f(p)-p\|+\big(1-\alpha_{n,2}(1-\alpha)(1-\beta_n)\big)\|x_n-p\|+e_n\\&\le\max\{\frac{\|f(p)-p\|}{1-\alpha},\|x_n-p\|\}+e_n\\&\le\cdots\\&\le\max\{\frac{\|f(p)-p\|}{1-\alpha},\|x_1-p\|\}+\sum_{i=1}^{\infty}e_i<\infty.\end{aligned}$$

This proves that $\{x_n\}$ is bounded. Since f is an α-contractive, we find

$$\begin{aligned}\|y_{n+1}-y_n\|&\le\alpha_{n+1,1}\|z_{n+1}-z_n\|+\alpha_{n+1,2}\|f(x_{n+1})-f(x_n)\|+\alpha_{n+1,3}\|x_{n+1}-x_n\|\\&\quad+|\alpha_{n+1,1}-\alpha_{n,1}|\|z_n\|+|\alpha_{n+1,2}-\alpha_{n,2}|\|f(x_n)\|+|\alpha_{n+1,3}-\alpha_{n,3}|\|x_n\|\\&\le\alpha_{n+1,1}\|z_{n+1}-z_n\|+\alpha_{n+1,2}\alpha\|x_{n+1}-x_n\|+\alpha_{n+1,3}\|x_{n+1}-x_n\|\\&\quad+|\alpha_{n+1,1}-\alpha_{n,1}|\|z_n\|+|\alpha_{n+1,2}-\alpha_{n,2}|\|f(x_n)\|+|\alpha_{n+1,3}-\alpha_{n,3}|\|x_n\|.\end{aligned}\tag{2.3}$$

Putting $J_{\lambda_n} = (I + \lambda_n B)^{-1}$, one has

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \|e_{n+1}\| + \|J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1} - J_{\lambda_n}(x_n - \lambda_n Ax_n\| + \|e_n\| \\ &\leq \|(x_n - \lambda_n Ax_n) - (x_{n+1} - \lambda_{n+1}Ax_{n+1})\| \\ &\quad + \|J_{\lambda_n}(x_n - \lambda_n Ax_n) - J_{\lambda_{n+1}}(x_n - \lambda_n Ax_n)\| + \|e_{n+1}\| + \|e_n\| \\ &\leq |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \|x_{n+1} - x_n\| \\ &\quad + \|J_{\lambda_n}(x_n - \lambda_n Ax_n) - J_{\lambda_{n+1}}(x_n - \lambda_n Ax_n)\| + \|e_{n+1}\| + \|e_n\|.\end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.3), we find

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \big(1 - \alpha_{n+1,2}(1-\alpha)\big)\|x_{n+1} - x_n\| \\ &\quad + \alpha_{n+1,1}\|J_{\lambda_n}(x_n - \lambda_n Ax_n) - J_{\lambda_{n+1}}(x_n - \lambda_n Ax_n)\| + \|e_{n+1}\| + \|e_n\| \\ &\quad + |\alpha_{n+1,1} - \alpha_{n,1}|\|z_n\| + |\alpha_{n+1,2} - \alpha_{n,2}|\|f(x_n)\| + |\alpha_{n+1,3} - \alpha_{n,3}|\|x_n\|.\end{aligned} \tag{2.5}$$

Putting $u_n = x_n - \lambda_n Ax_n$, we see that

$$0 \geq \langle J_{\lambda_{n+1}}u_n - J_{\lambda_n}u_n, \frac{u_n - J_{\lambda_n}u_n}{\lambda_n} - \frac{u_n - J_{\lambda_{n+1}}u_n}{\lambda_{n+1}}\rangle.$$

It follows that

$$\|J_{\lambda_n}u_n - J_{\lambda_{n+1}}u_n\|^2 \leq \langle (1 - \frac{\lambda_{n+1}}{\lambda_n})(J_{\lambda_n}u_n - u_n), J_{\lambda_n}u_n - J_{\lambda_{n+1}}u_n\rangle.$$

Hence, we have

$$\|J_{\lambda_n}u_n - J_{\lambda_{n+1}}u_n\| \leq \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_n}\|J_{\lambda_n}u_n - u_n\|. \tag{2.6}$$

From (2.5) and (2.6), one has

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \big(1 - \alpha_{n+1,2}(1-\alpha)\big)\|x_{n+1} - x_n\| \\ &\quad + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_n}\|J_{\lambda_n}u_n - u_n\| + \|e_{n+1}\| + \|e_n\| \\ &\quad + |\alpha_{n+1,1} - \alpha_{n,1}|\|z_n\| + |\alpha_{n+1,2} - \alpha_{n,2}|\|f(x_n)\| + |\alpha_{n+1,3} - \alpha_{n,3}|\|x_n\|.\end{aligned} \tag{2.7}$$

Putting $T_n = (1-\gamma_n)S + \gamma_n I$, one has

$$\begin{aligned}\|T_n x - T_n y\|^2 &\leq (1-\gamma_n)\|Sx - Sy\|^2 + \gamma_n\|x-y\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|(Sx - Sy) - (x-y)\|^2 \\ &\leq \gamma_n\|x-y\|^2 + (1-\gamma_n)(\|x-y\|^2 \\ &\quad + \kappa\|(x-y) - (Sx-Sy)\|^2) \\ &\quad - \gamma_n(1-\gamma_n)\|(x-y) - (Sx-Sy)\|^2 \\ &= \|x-y\|^2 - (1-\gamma_n)(\gamma_n - \kappa)\|(x-y) - (Sx-Sy)\|^2 \\ &\leq \|x-y\|^2, \quad \forall x, y \in C.\end{aligned}$$

It follows that

$$\begin{aligned}\|T_n y_n - T_{n+1}y_{n+1}\| &\leq \|T_{n+1}y_{n+1} - T_{n+1}y_n + T_{n+1}y_n - T_n y_n\| \\ &\leq \|y_{n+1} - y_n\| + \|(\gamma_{n+1}y_n + (1-\gamma_{n+1})Sy_n) \\ &\quad - (\gamma_n y_n + (1-\gamma_n)Sy_n)\| \\ &\leq \|y_{n+1} - y_n\| + \|\gamma_{n+1} - \gamma_n|(\|y_n\| + \|Sy_n\|).\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), one has

$$\begin{aligned}&\|T_ny_n - T_{n+1}y_{n+1}\| - \|x_{n+1} - x_n\|\\ \le\ & |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_n}\|J_{\lambda_n}u_n - u_n\| + \|e_{n+1}\| + \|e_n\|\\ &+ |\alpha_{n+1,1} - \alpha_{n,1}|\|z_n\| + |\alpha_{n+1,2} - \alpha_{n,2}|\|f(x_n)\| + |\alpha_{n+1,3} - \alpha_{n,3}|\|x_n\|\\ &+ \|\gamma_{n+1} - \gamma_n|(\|y_n\| + \|Sy_n\|).\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}(\|T_{n+1}y_{n+1} - T_ny_n\| - \|x_{n+1} - x_n\|) \le 0.$$

In view of Lemma 1.1, one has $\lim_{n\to\infty}\|x_n - T_ny_n\| = 0$. This finds

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.9}$$

Put $\mu_n = J_{\lambda_n}(x_n - \lambda_nAx_n)$. Since $\|\cdot\|^2$ is convex, we see

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1-\beta_n)\|T_ny_n - p\|^2 + \beta_n\|x_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + \alpha_{n,2}\|f(x_n) - p\|^2 + \alpha_{n,1}(1-\beta_n)\|z_n - p\|^2 + \alpha_{n,3}(1-\beta_n)\|x_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + \alpha_{n,2}\|f(x_n) - p\|^2 + \|e_n\|^2 + \alpha_{n,1}(1-\beta_n)\|\mu_n - p\|^2 + 2\|\mu_n - p\|\|e_n\|\\ &\quad + \alpha_{n,3}(1-\beta_n)\|x_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + \alpha_{n,2}\|f(x_n) - p\|^2 + \|e_n\|^2 + \alpha_{n,1}(1-\beta_n)\|x_n - p\|^2\\ &\quad - \alpha_{n,1}(1-\beta_n)\lambda_n(2\xi - \lambda_n)\|Ax_n - Ap\|^2 + 2\|\mu_n - p\|\|e_n\| + \alpha_{n,3}(1-\beta_n)\|x_n - p\|^2\\ &\le \big(1 - \alpha_{n,2}(1-\beta_n)\big)\|x_n - p\|^2 + \alpha_{n,2}\|f(x_n) - p\|^2 + \|e_n\|^2\\ &\quad - \alpha_{n,1}(1-\beta_n)\lambda_n(2\xi - \lambda_n)\|Ax_n - Ap\|^2 + 2\|\mu_n - p\|\|e_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\alpha_{n,1}(1-\beta_n)\lambda_n(2\xi - \lambda_n)\|Ax_n - Ap\|^2 \le\ & \alpha_{n,2}(1-\beta_n)\|x_n - p\|^2 + \alpha_{n,2}\|f(x_n) - p\|^2 + \|e_n\|^2\\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\|\mu_n - p\|\|e_n\|.\end{aligned}$$

Using the restrictions imposed on the control sequences, we have

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{2.10}$$

Since J_{λ_n} is firmly nonexpansive, we have

$$\begin{aligned}\|\mu_n - p\|^2 &= \|J_{\lambda_n}(p - \lambda_nAp) - J_{\lambda_n}(x_n - \lambda_nAx_n)\|^2\\ &\le \langle \mu_n - p, (x_n - \lambda_nAx_n) - (p - \lambda_nAp)\rangle\\ &= \frac{1}{2}\big(\|\mu_n - p\|^2 + \|(x_n - \lambda_nAx_n) - (p - \lambda_nAp)\|^2\\ &\quad - \|(x_n - \lambda_nAx_n) - (p - \lambda_nAp) - (\mu_n - p)\|^2\\ &\le \frac{1}{2}\big(\|x_n - p\|^2 + \|\mu_n - p\|^2 - \|x_n - \mu_n - \lambda_n(Ax_n - Ap)\|^2\big)\\ &\le \frac{1}{2}\big(\|x_n - p\|^2 + \|\mu_n - p\|^2 - \|x_n - \mu_n\|^2 + 2\lambda_n\|x_n - \mu_n\|\|Ax_n - Ap\|\big).\end{aligned}$$

It follows that

$$\|\mu_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - \mu_n\|^2 + 2\lambda_n\|x_n - \mu_n\|\|Ax_n - Ap\|.$$

Hence, we have

$$\begin{aligned}\|z_n-p\|^2 &\le \|e_n\|^2+\|\mu-p\|^2+2\|e_n\|\|\mu-p\|\\ &\le \|e_n\|^2+\|x_n-p\|^2-\|x_n-\mu_n\|^2+2\lambda_n\|x_n-\mu_n\|\|Ax_n-Ap\|+2\|e_n\|\|\mu_n-p\|.\end{aligned}$$

Since $\|\cdot\|^2$ is convex, we see that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\beta_n)\|T_ny_n-p\|^2+\beta_n\|x_n-p\|^2\\ &\le (1-\beta_n)\|y_n-p\|^2+\beta_n\|x_n-p\|^2\\ &\le (1-\beta_n)\alpha_{n,1}\|z_n-p\|^2+(1-\beta_n)\alpha_{n,2}\|f(x_n)-p\|^2\\ &\quad+(1-\beta_n)\alpha_{n,3}\|x_n-p\|^2+\beta_n\|x_n-p\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|e_n\|^2+(1-\beta_n)\alpha_{n,1}\|x_n-p\|^2-(1-\beta_n)\alpha_{n,1}\|x_n-\mu_n\|^2\\ &\quad+(1-\beta_n)\alpha_{n,1}2\lambda_n\|x_n-\mu_n\|\|Ax_n-Ap\|+2\|e_n\|\|\mu_n-p\|\\ &\quad+(1-\beta_n)\alpha_{n,2}\|f(x_n)-p\|^2+(1-\beta_n)\alpha_{n,3}\|x_n-p\|^2+\beta_n\|x_n-p\|^2\\ &\le \|e_n\|^2-(1-\beta_n)\alpha_{n,1}\|x_n-\mu_n\|^2+2\lambda_n\|x_n-\mu_n\|\|Ax_n-Ap\|+2\|e_n\|\|\mu_n-p\|\\ &\quad+\alpha_{n,2}\|f(x_n)-p\|^2+\big(1-\alpha_{n,2}(1-\beta_n)\big)\|x_n-p\|^2,\end{aligned}$$

which further implies

$$\begin{aligned}(1-\beta_n)\alpha_{n,1}\|x_n-\mu_n\|^2 &\le \|e_n\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n\|x_n-\mu_n\|\|Ax_n-Ap\|\\ &\quad+2\|e_n\|\|\mu_n-p\|+\alpha_{n,2}\|f(x_n)-p\|^2.\end{aligned}$$

This gives from (2.9) and (2.10) that $\lim_{n\to\infty}\|x_n-\mu_n\|=0$, which in turn implies that $\lim_{n\to\infty}\|x_n-z_n\|=0$. Since $\alpha_n\to 0$, we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{2.11}$$

Since $Proj_{\mathcal{F}}f$ is α-contractive, we see that there exists a unique fixed point. Next, we use $\bar{x}$ to denote the unique fixed point. $\limsup_{n\to\infty}\langle f(\bar{x})-(\bar{x}), y_n-x\rangle\le 0$. To show it, we can choose a subsequence $\{y_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, y_n-\bar{x}\rangle=\lim_{i\to\infty}\langle f(\bar{x})-\bar{x}, y_{n_i}-\bar{x}\rangle.$$

Since y_{n_i} is bounded, we can choose a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly some point $\bar{y}$. We may assume, without loss of generality, that y_{n_i} converges weakly to $\bar{y}$, so is x_{n_i} . First, we show $\bar{x}\in F(S)$. Note that

$$\begin{aligned}\|x_n-\big(\gamma_nx_n+(1-\gamma_n)Sx_n\big)\| &\le \|\big(\gamma_nx_n+(1-\gamma_n)Sx_n\big)-\big(\gamma_ny_n+(1-\gamma_n)Sy_n\big)\|\\ &\quad+\|\big(\gamma_ny_n+(1-\gamma_n)Sy_n\big)-x_n\|\\ &\le \|x_n-y_n\|+\|\big(\gamma_ny_n+(1-\gamma_n)Sy_n\big)-x_n\|\\ &\le \|x_n-y_n\|+\|T_ny_n-x_n\|.\end{aligned}$$

This implies from (2.11) that $\lim_{n\to\infty}\|x_n-Sx_n\|=0$. Now, we are in a position to show $\bar{x}\in F(S)$. Assume that $\bar{x}\notin F(S)$. In view of Opial's condition, we find from Lemma 1.4 that

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i}-\bar{x}\| &< \liminf_{i\to\infty}\|x_{n_i}-S\bar{x}\|\\ &= \liminf_{i\to\infty}\|x_{n_i}-Sx_{n_i}+Sx_{n_i}-S\bar{x}\|\\ &\le \liminf_{i\to\infty}\|x_{n_i}-\bar{x}\|.\end{aligned}$$

This is a contradiction. That is, $\bar{x} = S\bar{x}$. This shows that $\bar{x} \in F(S)$.

Next, we show that $\bar{x} \in (A+B)^{-1}(0)$. Since $\mu_n = J_{\lambda_n}(x_n - \lambda_n A x_n)$, we find that

$$x_n - \lambda_n A x_n \in (I + \lambda_n B)\mu_n.$$

That is,

$$\frac{x_n - \mu_n}{\lambda_n} - Ax_n \in B\mu_n.$$

Since B is monotone, we get, for any $(\mu, \nu) \in B$, that

$$\langle \mu_n - \mu, \frac{x_n - \mu_n}{\lambda_n} - Ax_n - \nu \rangle \geq 0.$$

Replacing n by n_i and letting $i \to \infty$, we obtain that

$$\langle \bar{x} - \mu, -A\bar{x} - \nu \rangle \geq 0.$$

This means $-A\bar{x} \in B\bar{x}$, that is, $0 \in (A+B)(\bar{x})$. Hence we get $\bar{x} \in (A+B)^{-1}(0)$. This completes the proof that $\bar{x} \in \mathcal{F}$. It follows that

$$\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle \leq 0.$$

Notice that

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &= \alpha_{n,1}\langle z_n - \bar{x}, y_n - \bar{x}\rangle + \alpha_{n,2}\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \alpha_{n,3}\langle x_n - \bar{x}, y_n - \bar{x}\rangle \\
&\leq \alpha_{n,1}\|z_n - \bar{x}\|\|y_n - \bar{x}\| + \alpha_{n,2}\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \alpha_{n,3}\|x_n - \bar{x}\|\|y_n - \bar{x}\| \\
&\leq \frac{\alpha_{n,1}}{2}(\|z_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2) + \alpha_{n,2}\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \frac{\alpha_{n,3}}{2}(\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &\leq \alpha_{n,1}\|z_n - \bar{x}\|^2 + 2\alpha_{n,2}\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \alpha_{n,3}\|x_n - \bar{x}\|^2 \\
&\leq \alpha_{n,1}(\|e_n\| + \|x_n - \bar{x}\|)^2 + 2\alpha_{n,2}\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \alpha_{n,3}\|x_n - \bar{x}\|^2 \\
&\leq (1 - \alpha_{n,2})\|x_n - \bar{x}\|^2 + 2\alpha_{n,2}\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \|e_n\|^2 + 2\|x_n - \bar{x}\|\|e_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\leq (1 - \beta_n)\|T_n y_n - \bar{x}\|^2 + \beta_n\|x_n - \bar{x}\|^2 \\
&\leq (1 - \beta_n)\|y_n - \bar{x}\|^2 + \beta_n\|x_n - \bar{x}\|^2 \\
&\leq \big(1 - \alpha_{n,2}(1 - \beta_n)\big)\|x_n - \bar{x}\|^2 + 2\alpha_{n,2}(1 - \beta_n)\langle f(x_n) - \bar{x}, y_n - \bar{x}\rangle + \|e_n\|^2 + 2\|x_n - \bar{x}\|\|e_n\|.
\end{aligned}$$

Using Lemma 1.3, we have $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$. This completes the proof that $\{x_n\}$ converges strongly to $\bar{x}$. □

From Theorem 2.1, we have the following results immediately.

Corollary 2.2. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a ξ-inverse-strongly monotone mapping and let B be a maximal monotone operator on H such that $Dom(B) \subset C$. Let f be a fixed α-contractive mapping on C and let S be κ-quasi-strict pseudocontraction on C. Let $\{\lambda_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence in C generated in the following iterative process*

$$\begin{cases}
x_1 \in C, \\
y_n = (1 - \alpha_n)z_n + \alpha_n f(x_n), \\
x_{n+1} = (1 - \beta_n)((1 - \gamma_n)Sy_n + \gamma_n y_n) + \beta_n x_n, \quad \forall n \geq 1,
\end{cases}$$

where $z_n \approx (I+\lambda_n B)^{-1}(x_n-\lambda_n Ax_n)$, *the criterion for the approximate computation is* $\|z_n-(I+\lambda_n B)^{-1}(x_n-\lambda_n Ax_n)\| \leq e_n$. *Assume that the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *satisfy the following restrictions:* $0 < a \leq \beta_n \leq b < 1$; $\kappa \leq \gamma_n \leq c < 1$, $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$; $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < d \leq \lambda_n \leq e < 2\xi$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\lim_{n\to\infty} \|e_n\| = 0$, *where* a, b, c, d *and* e *are some real numbers. If* $\mathcal{F} = Fix(S) \cap (A + B)^{-1}(0) \neq \emptyset$, *then sequence* $\{x_n\}$ *converges strongly to* $\bar{x}$, *where* $\bar{x}$ *solves the following variational inequality* $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$, $\forall x \in \mathcal{F}$.

Corollary 2.3. *Let* H *be a real Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $A : C \to H$ *be a* ξ*-inverse-strongly monotone mapping and let* B *be a maximal monotone operator on* H *such that* $Dom(B) \subset C$. *Let* f *be a fixed* α*-contractive mapping on* C. *Let* $\{\lambda_n\}$ *be a positive real number sequence. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0, 1)$. *Let* $\{x_n\}$ *be a sequence in* C *generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = (1-\alpha_n)z_n + \alpha_n f(x_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, \quad \forall n \geq 1, \end{cases}$$

where $z_n \approx (I+\lambda_n B)^{-1}(x_n-\lambda_n Ax_n)$, *the criterion for the approximate computation is* $\|z_n-(I+\lambda_n B)^{-1}(x_n-\lambda_n Ax_n)\| \leq e_n$. *Assume that the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *satisfy the following restrictions:* $0 < a \leq \beta_n \leq b < 1$; $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < d \leq \lambda_n \leq e < 2\xi$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\lim_{n\to\infty} \|e_n\| = 0$, *where* a, b, d *and* e *are some real numbers. If* $\cap(A + B)^{-1}(0) \neq \emptyset$, *then sequence* $\{x_n\}$ *converges strongly to* $\bar{x}$, *where* $\bar{x}$ *solves the following variational inequality* $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$, $\forall x \in \cap(A + B)^{-1}(0)$.

Finally, we give a result on a variational inequality problem.

Let H be a Hilbert space and $f : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y\rangle, \quad z \in H\}, \quad \forall x \in H.$$

From Rockafellar [17], [18], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y\in H} f(y)$. Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{2.12}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Let C be a nonempty closed convex subset of a real Hilbert space H, $Proj_C$ the metric projection from H onto C, ∂I_C the subdifferential of I_C, where I_C is as defined in (2.12) and $J_\lambda = (I + \lambda\partial I_C)^{-1}$. From [23], we have

$$y = J_\lambda x \Longleftrightarrow y = Proj_C x, \quad x \in H, y \in C.$$

Theorem 2.4. *Let* H *be a real Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $A : C \to H$ *be a* ξ*-inverse-strongly monotone mapping. Let* f *be a fixed* α*-contractive mapping on* C *and let* S *be* κ*-quasi-strict pseudocontraction on* C. *Let* $\{\lambda_n\}$ *be a positive real number sequence. Let* $\{\alpha_{n,1}\}$, $\{\alpha_{n,2}\}$, $\{\alpha_{n,3}\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real number sequences in* $(0, 1)$. *Let* $\{x_n\}$ *be a sequence in* C *generated in the following iterative process*

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_{n,1} z_n + \alpha_{n,2} f(x_n) + \alpha_{n,3} x_n, \\ x_{n+1} = (1-\beta_n)((1-\gamma_n)Sy_n + \gamma_n y_n) + \beta_n x_n, \quad \forall n \geq 1, \end{cases}$$

where $z_n \approx P_C(x_n - \lambda_n Ax_n)$*, the criterion for the approximate computation is* $\|z_n - P_C(x_n - \lambda_n Ax_n)\| \leq e_n$*. Assume that the sequences* $\{\alpha_{n,1}\}$*,* $\{\alpha_{n,2}\}$*,* $\{\alpha_{n,3}\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *satisfy the following restrictions:* $\alpha_{n,1} + \alpha_{n,2} + \alpha_{n,3} = 1$*,* $0 < a \leq \beta_n \leq b < 1$*;* $\kappa \leq \gamma_n \leq c < 1$*,* $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$*;* $\lim_{n\to\infty} \alpha_{n_2} = \lim_{n\to\infty} \alpha_{n_3} = 0$*,* $\sum_{n=1}^{\infty} \alpha_{n,2} = \infty$*;* $0 < d \leq \lambda_n \leq e < 2\xi$*,* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$*,* $\lim_{n\to\infty} \|e_n\| = 0$*, where* a, b, c, d *and* e *are some real numbers. If* $\mathcal{F} = Fix(S) \cap VI(C, A) \neq \emptyset$*, then sequence* $\{x_n\}$ *converges strongly to* $\bar{x}$*, where* $\bar{x}$ *solves the following variational inequality* $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$*,* $\forall x \in \mathcal{F}$*.*

Proof. Put $Bx = \partial I_C$. Next, we show that $VI(C, A) = (A + \partial I_C)^{-1}(0)$. Notice that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}(0) &\iff 0 \in Ax + \partial I_C x \\ &\iff -Ax \in \partial I_C x \\ &\iff \langle Ax, y - x\rangle \geq 0 \\ &\iff x \in VI(C, A). \end{aligned}$$

Hence, we conclude the desired conclusion immediately. □

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