



Splitting methods for a convex feasibility problem in Hilbert spaces

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Abstract

In this paper, we investigate a convex feasibility problem based on a splitting method. Strong convergence theorems are established without the aid of metric projections in the framework of real Hilbert spaces. ©2016 All rights reserved.

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1. Introduction

A very interesting problem in diverse areas of physics and mathematics consists of trying to find a special solution. This problem is referred to as the convex feasibility problem. It can be described as follows: D_1, D_2, \dots, D_N , where N denotes some positive integer, are finitely many closed convex nonempty subsets of a Hilbert space with $D := \bigcap_{i=1}^N D_i \neq \emptyset$. Convex feasibility problem is to find a solution in D . Closely related subjects of the problem are variational inequality problems, zero point problems, fixed point problems and equilibrium problem; see [1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 23, 24, 26, 27, 28, 29, 30] and the references therein.

Many problems of convex programming can be reduced to that of finding a zero point x of a maximal monotone operator B on a Hilbert space H ; see [21, 22, 26]. A fundamental technique for solving a zero point equation involving a monotone operator is the proximal point algorithm. The resolvent $J_s = (I +$

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$sB)^{-1}$, where I stands for the identity mapping and s is some positive real number has many important properties that make it a central tool in monotone operator theory and its applications. Especially, it is firmly nonexpansive. In the context of monotone operator theory, what is known as the Douglas-Rachford algorithm is a splitting scheme initially proposed in [12] for finding a zero of the sum of two monotone operators. Splitting algorithms for problems involving the sum of two monotone operators give some applications to the obstacle problems and minimization problems. In this paper, we investigate a convex feasibility problem based on a splitting method. Strong convergence theorems are established without the aid of metric projections in the framework of real Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H .

Let $S : C \rightarrow C$ be a mapping. $F(S)$ stands for the fixed point set of S . Recall that S is said to be contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be firmly nonexpansive iff $\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y \rangle$; S is said to be strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was introduced by Browder and Petryshyn [6]. It is clear that nonexpansive mappings are strictly pseudocontractive mappings with $\kappa = 0$.

Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction, where \mathbb{R} denotes the set of real numbers. Consider the following equilibrium problem in the terminology of Blum and Oettli [5]

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (2.1)$$

In this paper, the solution set of problem (2.1) is denoted by $EP(F)$.

To study equilibrium problem (2.1), we may assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Let $A : C \rightarrow H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that A is inverse-strongly monotone if and only if the inverse of A is strongly monotone.

A set-valued mapping $B : H \rightarrow 2^H$ is said to be monotone if for all $x, y \in H, f \in Bx$ and $g \in By$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $B : H \rightarrow 2^H$ is maximal if the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(B)$ implies $f \in Bx$. Next, we use $D(B)$ to denote the domain of B . If B is maximal monotone, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \rightarrow H$, where r is some positive constant. The single-valued operator is called the resolvent of B for the constant r .

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 ([2]). *Let C be a closed convex subset of a real Hilbert space H and let A be a maximal monotone operator on H . For any $\lambda > 0$ and $\mu > 0$, we have $J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x \right)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 2.2 ([6]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \rightarrow C$ be a strictly pseudocontractive mapping. Then $I - S$ is demiclosed at zero.*

Lemma 2.3 ([5]). *Let C be a closed convex subset of a real Hilbert space H and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$rF(z, y) + \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : rF(z, y) + \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

- (a) T_r is single-valued;
- (b) $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;
- (c) $F(T_r) = EP(F)$;
- (d) $EP(F)$ is closed and convex.

Lemma 2.4 ([6]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \rightarrow H$ be a strictly pseudocontractive mapping with the constant κ . Define a mapping T by $T = \delta I + (1 - \delta)S$, where δ is a constant in $[0, 1]$. If $\delta \in [\kappa, 1)$ then T is nonexpansive with $F(T) = F(S)$.*

Lemma 2.5 ([25]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and let $\{\beta_n\}$ be a sequence in $(0, 1)$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.6 ([16]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n + e_n$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$, $\{e_n\}$ and $\{\delta_n\}$ are sequences such that*

- (i) $\sum_{n=1}^\infty \gamma_n = \infty, \sum_{n=1}^\infty e_n < \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^\infty |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Lemma 2.7 ([3]). *Let C be a closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be a mapping and let B be a maximal monotone operator on H . Then $F(J_s(I - sA)) = (A + B)^{-1}(0)$, where $J_s = (I + sB)^{-1}$.*

3. Main results

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an inverse-strongly monotone mapping with the positive constant α and let B be a maximal monotone operator such that $D(B) \subset C$. Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let $S : C \rightarrow H$ be a strictly pseudocontractive mapping with the constant $\kappa \in [0, 1)$ and let f be a contractive mapping on H with the constant $\beta \in (0, 1)$. Assume that $\Omega = F(S) \cap (A + B)^{-1}(0) \cap EP(F)$ is nonempty. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in H$, $y_n = (I + s_n B)^{-1}(z_n - s_n A z_n + e_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \delta_n y_n + (1 - \delta_n) \gamma_n S y_n$, where $\{e_n\}$ is a sequence in H and $\{z_n\}$ is a sequence in C such that $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$. Assume that the control sequences satisfy the following conditions: $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$; $\sum_{n=1}^{\infty} \|e_n\| < \infty$, $\lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$ and $\kappa \leq \delta_n \leq \delta < 1$; $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ and $\liminf_{n \rightarrow \infty} r_n > 0$; $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$, and $0 < s \leq s_n \leq s' < 2\alpha$, where δ, s, s' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{\Omega} f(q)$.*

Proof. By using the conditions imposed on $\{s_n\}$, we find that

$$\|(I - s_n A)x - (I - s_n A)y\|^2 \leq \|x - y\|^2 - s_n(2\alpha - s_n)\|Ax - Ay\|^2.$$

This implies that $I - s_n A$ is nonexpansive. Put $J_{s_n} = (I + s_n B)^{-1}$ and $S_n = \delta_n I + (1 - \delta_n)S$. It follows from Lemma 2.4 that S_n is nonexpansive with $F(S_n) = F(S)$ and J_{s_n} is firmly nonexpansive. Letting $p \in \Omega$ be fixed arbitrarily, we have

$$\begin{aligned} \|y_n - p\| &\leq \|(z_n - s_n A z_n + e_n) - (p - s_n A p)\| \\ &\leq \|T_{r_n} x_n - p\| + \|e_n\| \\ &\leq \|x_n - p\| + \|e_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|S_n y_n - p\| \\ &\leq \alpha_n \beta \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \alpha_n \beta \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n (\|x_n - p\| + \|e_n\|) \\ &\leq (1 - \alpha_n(1 - \beta)) \|x_n - p\| + \alpha_n \|f(p) - p\| + \|e_n\| \\ &\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \beta}\} + \|e_n\|. \end{aligned}$$

It follows that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \beta}\} + \sum_{n=1}^{\infty} \|e_n\|.$$

This shows that $\{x_n\}$ is bounded. Since the mapping $P_{\Omega} f$ is contractive, there exists an unique fixed point. Next, we denote the unique fixed point by q . Now, we are in a position to show $\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle \leq 0$. To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \lim_{i \rightarrow \infty} \langle f(q) - q, x_{n_i} - q \rangle.$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly some point x . We may assume, without loss of generality, that $\{x_{n_{i_j}}\}$ converges weakly to x .

In view of $z_n = T_{r_n} x_n$, one has $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$ and $F(z_{n+1}, z) + \frac{1}{r_{n+1}} \langle z - z_{n+1}, z_{n+1} - x_{n+1} \rangle \geq 0, \forall z \in C$. Hence, one has $F(z_n, z_{n+1}) + \frac{1}{r_n} \langle z_{n+1} - z_n, z_n - x_n \rangle \geq 0$ and $F(z_{n+1}, z_n) +$

$\frac{1}{r_{n+1}} \langle z_n - z_{n+1}, z_{n+1} - x_{n+1} \rangle \geq 0, \forall z \in C$. From the monotonicity of F , one has $\langle z_{n+1} - z_n, \frac{z_n - x_n}{r_n} - \frac{z_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0$. It follows that

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_{n+1}\|.$$

Putting $\rho_n = z_n - s_nAz_n + e_n$, we find that

$$\begin{aligned} \|\rho_{n+1} - \rho_n\| &\leq \|z_{n+1} - z_n\| + \|Az_n\| |s_{n+1} - s_n| + \|e_{n+1} - e_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_{n+1}\| + \|Az_n\| |s_{n+1} - s_n| + \|e_{n+1} - e_n\|. \end{aligned}$$

On the other hand, we find from Lemma 2.1 that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \left\| J_{s_n} \left(\frac{s_n}{s_{n+1}} \rho_{n+1} + \left(1 - \frac{s_n}{s_{n+1}}\right) J_{s_{n+1}} \rho_{n+1} \right) - J_{s_n} \rho_n \right\| \\ &\leq \left\| \frac{s_n}{s_{n+1}} (\rho_{n+1} - \rho_n) + \left(1 - \frac{s_n}{s_{n+1}}\right) (J_{s_{n+1}} \rho_{n+1} - \rho_n) \right\| \\ &\leq \|\rho_{n+1} - \rho_n\| + \frac{|s_{n+1} - s_n|}{s} \|J_{s_{n+1}} \rho_{n+1} - \rho_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_{n+1}\| + \|Az_n\| |s_{n+1} - s_n| \\ &\quad + \|e_{n+1} - e_n\| + \frac{|s_{n+1} - s_n|}{s} \|J_{s_{n+1}} \rho_{n+1} - \rho_{n+1}\|. \end{aligned} \tag{3.1}$$

Putting $\lambda_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$, we have

$$\begin{aligned} \|\lambda_{n+1} - \lambda_n\| &\leq \frac{\alpha_{n+1} \|f(x_{n+1}) - S_{n+1}y_{n+1}\|}{1 - \beta_{n+1}} + \frac{\alpha_n \|f(x_n) - S_n y_n\|}{1 - \beta_n} \\ &\quad + \|S_{n+1}y_{n+1} - S_n y_n\| \\ &\leq \frac{\alpha_{n+1} \|f(x_{n+1}) - S_{n+1}y_{n+1}\|}{1 - \beta_{n+1}} + \frac{\alpha_n \|f(x_n) - S_n y_n\|}{1 - \beta_n} \\ &\quad + \|y_{n+1} - y_n\| + |\delta_{n+1} - \delta_n| \|y_n - S_n y_n\|. \end{aligned} \tag{3.2}$$

Combining (3.1) with (3.2) finds that

$$\begin{aligned} \|\lambda_{n+1} - \lambda_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1} \|f(x_{n+1}) - S_{n+1}y_{n+1}\|}{1 - \beta_{n+1}} + \frac{\alpha_n \|f(x_n) - S_n y_n\|}{1 - \beta_n} \\ &\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_{n+1}\| + \|Az_n\| |s_{n+1} - s_n| \\ &\quad + \|e_{n+1} - e_n\| + \frac{|s_{n+1} - s_n|}{s} \|J_{s_{n+1}} \rho_{n+1} - \rho_{n+1}\| \\ &\quad + |\delta_{n+1} - \delta_n| \|y_n - S_n y_n\|. \end{aligned}$$

By using conditions imposed on the control sequences, one has

$$\limsup_{n \rightarrow \infty} (\|\lambda_{n+1} - \lambda_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 2.5 that $\lim_{n \rightarrow \infty} \|\lambda_n - x_n\| = 0$, which in turn implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Since T_{r_n} is firmly nonexpansive, we find that

$$\begin{aligned} \|z_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2, \\ &\leq \frac{1}{2}(\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2). \end{aligned}$$

That is, $\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_n\|^2$. It follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n y_n - p\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|z_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|)) \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|x_n - z_n\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|), \end{aligned}$$

which implies that

$$\begin{aligned} \gamma_n\|x_n - z_n\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad + \|e_n\|(\|e_n\| + 2\|z_n - p\|). \end{aligned}$$

By using conditions imposed on the control sequences, we find from (3.3) that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{3.4}$$

Hence, $\{z_{n_i}\}$ converges weakly to $x \in C$.

Next, we show $x \in EP(F)$. Notice that

$$F(z_n, z) + \frac{1}{r_n}\langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C.$$

By using the monotonicity of F , we see that $\frac{1}{r_n}\langle z - z_n, z_n - x_n \rangle \geq F(z, z_n), \forall z \in C$. Replacing n by n_i , we arrive at $\langle z - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq F(z, z_{n_i}), \forall z \in C$. It follows from (3.4) that $0 \geq F(z, x)$. For each t with $0 < t \leq 1$, let $z_t = tz + (1 - t)x$, where $z \in C$. It follows that $z_t \in C$. Hence, we have hence $F(z_t, x) \leq 0$. It follows that

$$0 = F(z_t, z_t) \leq tF(z_t, z) + (1 - t)F(z_t, x) \leq tF(z_t, z),$$

which yields that $F(z_t, z) \geq 0, \forall z \in C$. Letting $t \downarrow 0$, we obtain that $F(x, z) \geq 0, \forall z \in C$. This implies that $x \in EP(F)$.

Since A is inverse-strongly monotone, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n y_n - p\|^2, \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|(z_n - s_n Az_n) - (p - s_n Ap)\|^2 \\ &\quad + \|e_n\|(\|e_n\| + 2\|z_n - p\|)) \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|z_n - p\|^2 - s_n(2\alpha - s_n)\|Az_n - Ap\|^2 \\ &\quad + \|e_n\|(\|e_n\| + 2\|z_n - p\|)) \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - s_n(2\alpha - s_n)\gamma_n\|Az_n - Ap\|^2 \\ &\quad + \|e_n\|(\|e_n\| + 2\|z_n - p\|). \end{aligned}$$

This yields that

$$\begin{aligned} s_n(2\alpha - s_n)\gamma_n\|Az_n - Ap\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\|^2 + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| \\ &\quad + \|e_n\|(\|e_n\| + 2\|z_n - p\|). \end{aligned}$$

In view of conditions imposed on the control sequences that

$$\lim_{n \rightarrow \infty} \|Az_n - Ap\| = 0. \tag{3.5}$$

Since J_{s_n} is firmly nonexpansive, one has

$$\begin{aligned} \|y_n - p\|^2 &\leq \langle (I - s_n A)z_n + e_n - (I - s_n A)p, y_n - p \rangle \\ &= \frac{1}{2} \{ \|(I - s_n A)z_n + e_n - (I - s_n A)p\|^2 + \|y_n - p\|^2 \\ &\quad - \|(I - s_n A)z_n + e_n - (I - s_n A)p - (y_n - p)\|^2 \} \\ &\leq \frac{1}{2} \{ \|z_n - p\|^2 + L_n + \|y_n - p\|^2 - \|z_n - y_n - (s_n(Az_n - Ap) - e_n)\|^2 \} \\ &\leq \frac{1}{2} \{ \|x_n - p\|^2 + L_n + \|y_n - p\|^2 - \|z_n - y_n\|^2 \\ &\quad + 2\|z_n - y_n\| \|s_n(Az_n - Ap) - e_n\| - \|s_n(Az_n - Ap) - e_n\|^2 \}, \end{aligned}$$

where $L_n = \|e_n\|(\|e_n\| + 2\|z_n - p\|)$. This yields that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 + L_n - \|z_n - y_n\|^2 + 2s_n\|z_n - y_n\| \|Az_n - Ap\| \\ &\quad + 2\|z_n - y_n\| \|e_n\|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|S_n y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 + L_n - \gamma_n \|z_n - y_n\|^2 \\ &\quad + 2s_n \gamma_n \|z_n - y_n\| \|Az_n - Ap\| + 2\|z_n - y_n\| \|e_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \gamma_n \|z_n - y_n\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + L_n \\ &\quad + 2s_n \gamma_n \|z_n - y_n\| \|Az_n - Ap\| + 2\|z_n - y_n\| \|e_n\|. \end{aligned}$$

By using (3.3) and (3.5), we find that

$$\lim_{n \rightarrow \infty} \|z_n - y_n\| = 0. \tag{3.6}$$

Hence, $\{y_{n_i}\}$ converges weakly to $x \in C$.

Now, we show that $x \in (A+B)^{-1}(0)$. In view of $y_n = J_{s_n}(z_n - s_n Az_n + e_n)$, one has $\frac{z_n - y_n + e_n}{s_n} - Az_n \in B y_n$. Since B is maximal monotone, we get, for any $(\mu, \nu) \in B$,

$$\langle y_n - \mu, \frac{z_n - y_n + e_n}{s_n} - Az_n - \nu \rangle \geq 0.$$

Replacing n by n_i and letting $i \rightarrow \infty$, we get from (3.6) that

$$\langle x - \mu, -Ax - \nu \rangle \geq 0.$$

This gives $-Ax \in Bx$, that is, $0 \in (A+B)(x)$. This show that $x \in (A+B)^{-1}(0)$.

Next, we show $x \in F(S)$. Notice that

$$\|x_n - S_n y_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - S_n y_n\| + \beta_n \|x_n - S_n y_n\|.$$

By using (3.3) one has

$$\lim_{n \rightarrow \infty} \|x_n - S_n y_n\| = 0. \tag{3.7}$$

It follows that

$$\|S_n x_n - x_n\| \leq \|x_n - z_n\| + \|z_n - y_n\| + \|S_n y_n - x_n\|.$$

By using (3.4), (3.6) and (3.7), one has

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{3.8}$$

Notice that

$$\begin{aligned} \|Sx_n - x_n\| &\leq \|Sx_n - (\delta_n x_n + (1 - \delta_n)Sx_n)\| + \|S_n x_n - x_n\| \\ &\leq \delta_n \|Sx_n - x_n\| + \|S_n x_n - x_n\|. \end{aligned}$$

This implies from (3.8), one has

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0.$$

By using Lemma 2.2, we find $x \in F(S)$. This completes the proof that $x \in \Omega$. It follows that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle \leq 0.$$

This implies from (3.3) that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_{n+1} - q \rangle \leq 0.$$

Notice that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle + \beta_n \|x_n - q\| \|x_{n+1} - q\| + \gamma_n \|S_n y_n - q\| \|x_{n+1} - q\| \\ &\leq \alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle + \beta_n \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + \gamma_n \|(z_n - s_n A z_n + e_n) - (p - s_n A p)\| \|x_{n+1} - q\| \\ &\leq \alpha_n \|f(x_n) - f(q)\| \|x_{n+1} - q\| + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle + \beta_n \|x_n - q\| \|x_{n+1} - q\| \\ &\quad + \gamma_n \|z_n - p\| \|x_{n+1} - q\| + \|e_n\| \|x_{n+1} - q\| \\ &\leq (1 - \alpha(1 - \beta)) \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle + \|e_n\| \|x_{n+1} - q\|. \end{aligned}$$

It follows that

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n(1 - \beta)) \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle + 2\|e_n\| \|x_{n+1} - q\|.$$

By using Lemma 2.6, we find that $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$. This completes the proof. □

Remark 3.2. Let C be a nonempty closed and convex subset of H and $A : C \rightarrow H$ be a mapping. Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{3.9}$$

Projection methods have been recently investigated for solving variational inequality (3.9). It is known that x is a solution to (3.9) iff x is a fixed point of the mapping $Proj_C(I - rA)$, where I denotes the identity on H . If A is strongly monotone and Lipschitz, then problem (3.9) has a unique solution. If A is inverse-strongly monotone, then $Proj_C(I - rA)$ is nonexpansive. Moreover, if C is bounded, closed and convex, then the existence of solutions of the variational inequality is guaranteed by the nonexpansivity of the mapping $Proj_C(I - rA)$. Let i_C be a function defined by $i_C(x) = 0, x \in C, i_C(x) = \infty, x \notin C$. It is easy to see that i_C is a proper lower and semicontinuous convex function on H , and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent $J_s := (I + s\partial i_C)^{-1}$ of the subdifferential operator ∂i_C . Letting $x = J_s y$, we find that

$$\begin{aligned} y \in x + s\partial i_C x &\iff y \in x + sN_C x \\ &\iff \langle y - x, v - x \rangle \leq 0, \forall v \in C \\ &\iff x = Proj_C y, \end{aligned}$$

where $N_C x := \{e \in H : \langle e, v - x \rangle, \forall v \in C\}$.

Putting $B = \partial_i C$ in Theorems 3.1, we find the following result.

Corollary 3.3. *Let C be a closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an inverse-strongly monotone mapping with the positive constant α and let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let $S : C \rightarrow H$ be a strictly pseudocontractive mapping with the constant $\kappa \in [0, 1)$ and let f be a contractive mapping on H with the constant $\beta \in (0, 1)$. Assume that $\Omega = F(S) \cap VI(C, A) \cap EP(F)$ is nonempty. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in H$, $y_n = P_C(z_n - s_n A z_n + e_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \delta_n y_n + (1 - \delta_n) \gamma_n S y_n$, where $\{e_n\}$ is a sequence in H and $\{z_n\}$ is a sequence in C such that $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$. Assume that the control sequences satisfy the following conditions: $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$; $\sum_{n=1}^{\infty} \|e_n\| < \infty$, $\lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$ and $\kappa \leq \delta_n \leq \delta < 1$; $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ and $\liminf_{n \rightarrow \infty} r_n > 0$; $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$, $0 < s \leq s_n \leq s' < 2\alpha$, where δ, s, s' are real constants. Then $\{x_n\}$ converges strongly to $q \in \Omega$, which is also a unique solution to the variational inequality $\langle f(x) - x, x - y \rangle \geq 0, \forall y \in C$.*

Putting $\delta_n = 0$ and $S = I$, we have the following results.

Corollary 3.4. *Let C be a closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an inverse-strongly monotone mapping with the positive constant α and let B be a maximal monotone operator on H . Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let f be a contractive mapping on H with the constant $\beta \in (0, 1)$. Assume that $\Omega = (A + B)^{-1}(0) \cap EP(F)$ is nonempty. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in H$, $y_n = (I + s_n B)^{-1}(z_n - s_n A z_n + e_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, where $\{e_n\}$ is a sequence in H and $\{z_n\}$ is a sequence in C such that $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$. Assume that the control sequences satisfy the following conditions: $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$; $\sum_{n=1}^{\infty} \|e_n\| < \infty$; $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ and $\liminf_{n \rightarrow \infty} r_n > 0$; $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$, $0 < s \leq s_n \leq s' < 2\alpha$, where δ, s, s' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{\Omega} f(q)$.*

Corollary 3.5. *Let C be a closed convex subset of a real Hilbert space H . Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let $S : C \rightarrow H$ be a strictly pseudocontractive mapping with the constant $\kappa \in [0, 1)$ and let f be a contractive mapping on H with the constant $\beta \in (0, 1)$. Assume that $\Omega = F(S) \cap EP(F)$ is nonempty. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in H$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \delta_n z_n + (1 - \delta_n) \gamma_n S z_n$, where $\{z_n\}$ is a sequence in C such that $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$. Assume that the control sequences satisfy the following conditions: $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$; $\lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$ and $\kappa \leq \delta_n \leq \delta < 1$; $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ and $\liminf_{n \rightarrow \infty} r_n > 0$; , where δ is a real constant. Then $\{x_n\}$ converges strongly to $q = P_{\Omega} f(q)$.*

Put $r_n = 1$ and $F(x, y) = 0$ for any $x, y \in C$. By taking the initial in C , we find the following result.

Corollary 3.6. *Let C be a closed convex subset of a real Hilbert space H . Let $A : C \rightarrow H$ be an inverse-strongly monotone mapping with the positive constant α and let B be a maximal monotone operator such that $D(B) \subset C$. Let S be a strictly pseudocontractive mapping on C with the constant $\kappa \in [0, 1)$ and let f be a contractive mapping on C with the constant $\beta \in (0, 1)$. Assume that $\Omega = F(S) \cap (A + B)^{-1}(0)$ is nonempty. Let $\{s_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the process: $x_1 \in C$, $y_n = (I + s_n B)^{-1}(x_n - s_n A x_n + e_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \delta_n y_n + (1 - \delta_n) \gamma_n S y_n$, where $\{e_n\}$ is a sequence in H . Assume that the control sequences satisfy the following conditions: $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$; $\sum_{n=1}^{\infty} \|e_n\| < \infty$, $\lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$ and $\kappa \leq \delta_n \leq \delta < 1$; $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$, $0 < s \leq s_n \leq s' < 2\alpha$, where δ, s, s' are real constants. Then $\{x_n\}$ converges strongly to $q = P_{\Omega} f(q)$.*

Remark 3.7. Minimize the following l_1 -least square problem: $\min_{x \in R^3} \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (1, 2, 3)x - 5$, where $x = (x_1, x_2, x_3)^T$. Let $f(x) = \frac{1}{2}\|x\|_2^2 + (1, 2, 3)x - 5$ and $Bx = \|x\|_1$. Then ∇f is 1-inverse-strongly monotone and $\nabla = (x_1 + 1, x_2 + 2, x_3 + 3)^T$. Suppose that $Q \in R^{64 \times 64}$ is a positive semidefinite matrix that the maximum eigenvalue κ_{\max} of Q is positive and that $b \in R^{64}$. Define a quadratic function $f : R^{64} \rightarrow R$ by $f(x) = \frac{1}{2}\langle x, Qx \rangle + \langle b, x \rangle, \forall x \in R^{64}$. Then $\nabla f(\cdot) = Q(\cdot) + b$ is κ_{\max} -Lipschitz continuous and $\frac{1}{\kappa_{\max}}$ -inverse-strongly monotone. For firmly nonexpansive mappings $(I + \frac{\kappa}{(n+1)^a}B)^{-1} : R^{64} \rightarrow R^{64}$, applying the algorithm, we find the following method: $x_1 \in R^{64}$ and $x_1 \in R^{64}$, and $y_n = (I + \frac{\kappa}{(n+1)^a}B)^{-1}(z_n - \frac{\kappa}{(n+1)^a}Az_n + \frac{\kappa}{(n+1)^a})$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \delta_n y_n + (1 - \delta_n)\gamma_n S y_n$, where $\kappa > 0$, $a \in [1, \infty)$ and $\{z_n\}$ is a sequence in C such that $F(z_n, z) + \frac{1}{r_n}\langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$.

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