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Some common coupled fixed point results in two S-metric spaces and applications to integral equations

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Abstract

The purpose of this paper is to prove some new coupled common fixed point theorems for mappings defined on a set equipped with two S-metrics. We also provide illustrative examples in support of our new results. Meantime, we give an existence and uniqueness theorem of solution for a class of nonlinear integral equations by using the obtained result.

Keywords: S-metric space, contractive mappings, coupled coincidence point, coupled common fixed point, mixed g-monotone property.
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1. Introduction and Preliminaries

In 2006, Mustafa and Sims [14] introduced a new concept of metric spaces, which is a generalization of metric spaces; briefly G-metric space. In 2007, Sedghi, Rao and Shobe [19] investigated the concept of D-metric space proposed by Dhage[6], and introduced the concept of D^*-metric spaces, pointed out the basic properties of D^*-metric space. Very recently, Sedghi, Shobe and Aliouche [20] extended the notions of G-metric spaces and D^*-metric spaces, proposed the concept of S-metric spaces as follow:

Definition 1.1 ([20])**.** Let X be a nonempty set and $S : X \times X \times X \to \mathbb{R}^+$ be a function, such that for all $x, y, z, a \in X$, we have the following:

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(S1) $S(x, y, z) = 0 \Leftrightarrow x = y = z$;
(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
Then the function S is called a S-metric in X, the pair (X, S) is called a S-metric spaces.

Remark 1.2. For each S-metric, it definitely is a G-metric, and each G-metric definitely is a D^*-metric, vice untrue, the counter-example can be found in [12].

In [20], the author introduced some basic properties in S-metric spaces, and showed a new common fixed point theorem for contractive mapping in such spaces. Since then, many mathematicians such as Sedghi and Dung[18], Afra [2, 3] and Hieu, Ly and Dung [11] proposed several fixed point theorems under different contractive conditions in S-metric spaces, which is a generalization of the results in [20]. In 2013, Chouhan and Malviya [5] studied the expansive mappings in S-metric space, proved some new fixed point theorems. Recently, Kim, Sedghi and Shobkolaei [12] introduced the concepts of weak commutativity and R-weak commutativity mappings in S-metric spaces, and proved some new common fixed point theorems. Rahman, Sarwar and Rahman [15] established the common fixed point theorem of Altman integral contractive type mappings by using the notion of φ-weak commutativity mappings.

Many scholars such as Dung [7], Raj and Hooda [16], Dung, Hieua and Radojević [8], Raj and Hooda [17] and Afra [1] as well as Gupta and Deep [10] discussed the problems for common coupled coincidence point and coupled common fixed point in S-metric spaces, obtained some new coupled common fixed points theorems.

In 2013, Gu [9] discussed some coupled common fixed point problems in two G-metric spaces, and prove some new coupled common fixed point theorems. Inspired by the above corresponding results, In this paper, we study coupled common fixed point problems in two S-metric spaces and establish some new coupled common fixed point theorems. Furthermore, we also provide illustrative examples in support of our new results. As an application of our main result, we also prove the existence and uniqueness theorem of solution for a class of nonlinear integral equations in S-metric spaces.

In this section, we first introduce some basic notions and known results.

Lemma 1.3 ([14]). *Let (X, S) be a S-metric space, then we have $S(x, x, y) = S(y, y, x)$ $\forall x, y, z \in X$.*

Lemma 1.4 ([2, 16]). *Let (X, S) be a S-metric space, then $\forall x, y, z \in X$, we have*

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z),$$
$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y).$$

Lemma 1.5. *Let (X, S) be a S-metric space. Then, for all $x, y, z \in X$ it follows that:*
(1) $S(x, y, y) \leq S(x, x, y)$;
(2) $S(x, y, x) \leq S(x, x, y)$;
(3) $S(x, y, z) \leq S(x, x, z) + S(y, y, z)$;
(4) $S(x, y, z) \leq S(x, x, y) + S(z, z, y)$;
(5) $S(x, y, z) \leq S(y, y, x) + S(x, x, z)$;
(6) $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$;
(7) $S(x, y, z) \leq \frac{2}{3}[S(x, x, y) + S(y, y, z) + S(z, z, x)]$.

Proof. First, it follows from (S2) and Lemma 1.3, we can easily obtain (1)-(5). Now we prove (6) and (7) also hold.

By virtue of Lemma 1.3 and Lemma 1.4, we have

$$\begin{aligned} 2S(x, x, z) &= S(x, x, z) + S(z, z, x) \\ &\leq [2S(x, x, y) + S(y, y, z)] + [2S(z, z, y) + S(x, x, y)] \\ &= 3[S(y, y, z) + S(y, y, x)]. \end{aligned}$$

Consequently, $S(x, x, z) \leq \frac{3}{2}[S(y, y, z) + S(y, y, x)]$. Then we have (6).

By virtue of (3)-(5) and Lemma 1.3, then we have

$$\begin{aligned}3S(x,y,z) &\le [S(x,x,z)+S(y,y,z)]+[S(x,x,y)+S(z,z,y)]+[S(y,y,x)+S(z,z,x)]\\ &=2[S(x,x,y)+S(y,y,z)+S(z,z,x)].\end{aligned}$$

Which implies $S(x,y,z)\le \frac{2}{3}[S(x,x,y)+S(y,y,z)+S(z,z,x)]$. Thus (7) is obtained. □

Definition 1.6 ([2])**.** Suppose that $\{x_n\}$ is the sequence in S-metric space (X,S). $\{x_n\}$ is called to be a S-Cauchy sequence in X, if

$$\lim_{n,m\to\infty} S(x_n,x_n,x_m)=0 \quad \forall a\in X.$$

Definition 1.7 ([2])**.** A sequence $\{x_n\}$ is said to be S-convergent sequence in S-metric space (X,d), if $\exists x\in X$ satisfying the following condition

$$\lim_{n\to\infty} S(x_n,x_n,x)=0.$$

Then the sequence $\{x_n\}$ is said to be S-convergent to x, noting $\lim_{n\to\infty} x_n=x$, that is $x_n\to x(n\to\infty)$.

Lemma 1.8 ([2])**.** *Let (X,S) be a S-metric space and $\{x_n\}$ and $\{y_n\}$ be two convergent subsequence in X such that $\lim\limits_{n\to\infty} x_n=x$, $\lim\limits_{n\to\infty} y_n=y$, thus we have*

$$\lim_{n\to\infty} S(x_n,x_n,y_n)=S(x,x,y).$$

In particular, by taking $y_n\equiv y$, then we have

$$\lim_{n\to\infty} S(x_n,x_n,y)=S(x,x,y).$$

Definition 1.9 ([14])**.** The S-metric space (X,d) is called to be S-complete, if each S-Cauchy sequence in X is S-convergent to some point in X.

Definition 1.10 ([4])**.** An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F:X\times X\to X$, if $F(x,y)=x, F(y,x)=y$.

Definition 1.11 ([13])**.** An element $(x,y)\in X\times X$ is called a coupled coincidence point of mappings $F:X\times X\to X$ and $g:X\to X$, if $F(x,y)=gx, F(y,x)=gy$, and in this case, (gx,gy) is called a coupled point of coincidence.

Definition 1.12 ([13])**.** An element $(x,y)\in X\times X$ is called a common coupled fixed point of mappings $F:X\times X\to X$ and $g:X\to X$, if $F(x,y)=gx=x, F(y,x)=gy=y$.

Definition 1.13 ([13])**.** Let X is a nonempty set. A pair of mappings $F:X\times X\to X$ and $g:X\to X$ is said to be w-compatible, if $F(x,y)=gx$ and $F(y,x)=gy$, then we have $gF(x,y)=F(gx,gy)$.

2. Main Results

Theorem 2.1. *Let X be a nonempty set and S_1,S_2 are two S-metrics on X such that $S_2(x,y,z)\le S_1(x,y,z)$ $\forall x,y,z\in X$. Suppose that the mappings $F:X\times X\to X$ and $g:X\to X$ satisfy the following contractive condition*

$$\begin{aligned}&S_1(F(x,y),F(u,v),F(s,t))\\ &\quad\le k_1S_2(gx,gu,gs)+k_2S_2(gy,gv,gt)\\ &\qquad+k_3S_2(gx,gu,F(s,t))+k_4S_2(gy,gv,F(t,s))\\ &\qquad+k_5S_2(gx,F(u,v),gs)+k_6S_2(gy,F(v,u),gt)\\ &\qquad+k_7S_2(F(x,y),gu,gs)+k_8S_2(F(y,x),gv,gt)\\ &\qquad+k_9S_2(gx,F(u,v),F(s,t))+k_{10}S_2(gy,F(v,u),F(t,s))\\ &\qquad+k_{11}S_2(F(x,y),gu,F(s,t))+k_{12}S_2(F(y,x),gv,F(t,s))\\ &\qquad+k_{13}S_2(F(x,y),F(u,v),gs)+k_{14}S_2(F(y,x),F(v,u),gt)\end{aligned}\tag{2.1}$$

for all $(x,y),(u,v),(s,t) \in X \times X$*, where* $k_i \geq 0$ *for* $i = 1,2,\cdots,14$ *with*

$$0 \leq k_1 + k_2 + 3(k_3+k_4) + k_5 + k_6 + k_7 + k_8 + 2(k_9 + k_{10} + k_{11} + k_{12}) + k_{13} + k_{14} < 1. \tag{2.2}$$

If $F(X \times X) \subseteq gX$ *and* gX *is a* S_1*-complete subspace of* (X, S_1) *then* F *and* g *have a unique coupled point of coincidence* $(gx, gy) \in X \times X$*, which satisfy* $gx = F(x,y) = gy = F(y,x)$*.*

Moreover, if F *and* g *are* w*-compatible, then* F *and* g *have a unique common coupled fixed point of the form* $(u,u) \in X \times X$*, satisfying* $u = gu = F(u,u)$*.*

Proof. Suppose that $(x_0, y_0) \in X \times X$. Since $F(X \times X) \subseteq g(X)$, $\exists (x_1, y_1) \in X \times X$ such that $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$. Similarly, $\exists (x_2, y_2) \in X \times X$, such that $gx_2 = T(x_1, y_1)$, $gy_2 = T(y_1, x_1)$. Continuing this process, then we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X, defined by

$$gx_{n+1} = F(x_n, y_n), \quad gy_{n+1} = F(y_n, x_n) \ \ \forall n \geq 0. \tag{2.3}$$

In (2.1), by taking $(x,y) = (u,v) = (x_n, y_n)$ and $(s,t) = (x_{n+1}, y_{n+1})$, also by (2.3), we have

$$\begin{aligned} S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) =& S_1(F(x_n,y_n), F(x_n,y_n), F(x_{n+1},y_{n+1})) \\ \leq& k_1 S_2(gx_n, gx_n, gx_{n+1}) + k_2 S_2(gy_n, gy_n, gy_{n+1}) \\ &+ k_3 S_2(gx_n, gx_n, F(x_{n+1}, y_{n+1})) + k_4 S_2(gy_n, gy_n, F(y_{n+1}, x_{n+1})) \\ &+ k_5 S_2(gx_n, F(x_n,y_n), gx_{n+1}) + k_6 S_2(gy_n, F(y_n,x_n), gy_{n+1}) \\ &+ k_7 S_2(F(x_n,y_n), gx_n, gx_{n+1}) + k_8 S_2(F(y_n,x_n), gy_n, gy_{n+1}) \\ &+ k_9 S_2(gx_n, F(x_n,y_n), F(x_{n+1},y_{n+1})) \\ &+ k_{10} S_2(gy_n, F(y_n,x_n), F(y_{n+1},x_{n+1})) \\ &+ k_{11} S_2(F(x_n,y_n), gx_n, F(x_{n+1},y_{n+1})) \\ &+ k_{12} S_2(F(y_n,x_n), gy_n, F(y_{n+1},x_{n+1})) \\ \leq& k_1 S_2(gx_n, gx_n, gx_{n+1}) + k_2 S_2(gy_n, gy_n, gy_{n+1}) \\ &+ k_3 S_2(gx_n, gx_n, gx_{n+2}) + k_4 S_2(gy_n, gy_n, gy_{n+2}) \\ &+ k_5 S_2(gx_n, gx_{n+1}, gx_{n+1}) + k_6 S_2(gy_n, gy_{n+1}, gy_{n+1}) \\ &+ k_7 S_2(gy_{n+1}, gx_n, gx_{n+1}) + k_8 S_2(gy_{n+1}, gy_n, gy_{n+1}) \\ &+ k_9 S_2(gx_n, gx_{n+1}, gx_{n+2}) + k_{10} S_2(gy_n, gy_{n+1}, gy_{n+2}) \\ &+ k_{11} S_2(gx_{n+1}, gx_n, gx_{n+2}) + k_{12} S_2(gy_{n+1}, gy_n, gy_{n+2}) \\ &+ k_{13} S_2(gx_{n+1}, gx_{n+1}, gx_{n+1}) + k_{14} S_2(gy_{n+1}, gy_{n+1}, gy_{n+1}). \end{aligned} \tag{2.4}$$

By using Lemma 1.3, Lemma 1.5 (1), (2), (4), (5) and (6), it follows from (2.4) that

$$\begin{aligned} S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) \leq& k_1 S_2(gx_n, gx_n, gx_{n+1}) + k_2 S_2(gy_n, gy_n, gy_{n+1}) \\ &+ k_3 \cdot \frac{3}{2}[S_2(gx_n, gx_n, gx_{n+1}) + S_2(gx_{n+1}, gx_{n+1}, gx_{n+2})] \\ &+ k_4 \cdot \frac{3}{2}[S_2(gy_n, gy_n, gy_{n+1}) + S_2(gy_{n+1}, gy_{n+1}, gy_{n+2})] \\ &+ k_5 S_2(gx_n, gx_n, gx_{n+1}) + k_6 S_2(gy_n, gy_n, gy_{n+1}) \\ &+ k_7 S_2(gy_n, gx_n, gx_{n+1}) + k_8 S_2(gy_n, gy_n, gy_{n+1}) \\ &+ k_9 [S_2(gx_n, gx_n, gx_{n+1}) + S_2(gx_{n+1}, gx_{n+1}, gx_{n+2})] \\ &+ k_{10} [S_2(gy_n, gy_n, gy_{n+1}) + S_2(gy_{n+1}, gy_{n+1}, gy_{n+2})] \\ &+ k_{11} [S_2(gx_n, gx_n, gx_{n+1}) + S_2(gx_{n+1}, gx_{n+1}, gx_{n+2})] \\ &+ k_{12} [S_2(gy_n, gy_n, gy_{n+1}) + S_2(gy_{n+1}, gy_{n+1}, gy_{n+2})] \end{aligned}$$

$$
\begin{aligned}
&= \left(k_1 + \frac{3}{2}k_3 + k_5 + k_7 + k_9 + k_{11}\right) S_2(gx_n, gx_n, gx_{n+1}) \\
&\quad + \left(k_2 + \frac{3}{2}k_4 + k_6 + k_8 + k_{10} + k_{12}\right) S_2(gy_n, gy_n, gy_{n+1}) \\
&\quad + \left(\frac{3}{2}k_3 + k_9 + k_{11}\right) S_2(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\
&\quad + \left(\frac{3}{2}k_4 + k_{10} + k_{12}\right) S_2(gy_{n+1}, gy_{n+1}, gy_{n+2}) \qquad (2.5)\\
&\leq \left(k_1 + \frac{3}{2}k_3 + k_5 + k_7 + k_9 + k_{11}\right) S_1(gx_n, gx_n, gx_{n+1}) \\
&\quad + \left(k_2 + \frac{3}{2}k_4 + k_6 + k_8 + k_{10} + k_{12}\right) S_1(gy_n, gy_n, gy_{n+1}) \\
&\quad + \left(\frac{3}{2}k_3 + k_9 + k_{11}\right) S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\
&\quad + \left(\frac{3}{2}k_4 + k_{10} + k_{12}\right) S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}).
\end{aligned}
$$

We can similarly prove the following result

$$
\begin{aligned}
S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) &\leq \left(k_1 + \frac{3}{2}k_3 + k_5 + k_7 + k_9 + k_{11}\right) S_1(gy_n, gy_n, gy_{n+1}) \\
&\quad + \left(k_2 + \frac{3}{2}k_4 + k_6 + k_8 + k_{10} + k_{12}\right) S_1(gx_n, gx_n, gx_{n+1}) \\
&\quad + \left(\frac{3}{2}k_3 + k_9 + k_{11}\right) S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) \\
&\quad + \left(\frac{3}{2}k_4 + k_{10} + k_{12}\right) S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}).
\end{aligned} \tag{2.6}
$$

It follows from (2.5) and (2.6) that

$$
\begin{aligned}
&S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) \\
&\quad \leq \left(k_1 + \frac{3}{2}k_3 + k_5 + k_7 + k_9 + k_{11} + k_2 + \frac{3}{2}k_4 + k_6 + k_8 + k_{10} + k_{12}\right) [S_1(gx_n, gx_n, gx_{n+1}) \\
&\quad\quad + S_1(gy_n, gy_n, gy_{n+1})] \\
&\quad\quad + \left(\frac{3}{2}k_3 + k_9 + k_{11} + \frac{3}{2}k_4 + k_{10} + k_{12}\right) [S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_1(gy_{n+1}, gy_{n+1}, gy_{n+2})] \\
&\quad = \left(\sum_{i=1}^{12} k_i + \frac{k_3}{2} + \frac{k_4}{2}\right) [S_1(gx_n, gx_n, gx_{n+1}) + S_1(gy_n, gy_n, gy_{n+1})] \\
&\quad\quad + \left(\frac{3}{2}k_3 + \frac{3}{2}k_4 + k_9 + k_{10} + k_{11} + k_{12}\right) [S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_1(gy_{n+1}, gy_{n+1}, gy_{n+2})].
\end{aligned} \tag{2.7}
$$

The above inequality (2.7) implies that

$$
\begin{aligned}
&S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) \\
&\quad \leq \frac{\left(\sum_{i=1}^{12} k_i + \frac{k_3}{2} + \frac{k_4}{2}\right) [S_1(gx_n, gx_n, gx_{n+1}) + S_1(gy_n, gy_n, gy_{n+1})]}{1 - \left(\frac{3}{2}k_3 + \frac{3}{2}k_4 + k_9 + k_{10} + k_{11} + k_{12}\right)}.
\end{aligned} \tag{2.8}
$$

Now by taking

$$k = \frac{\left(\sum_{i=1}^{12} k_i + \frac{k_3}{2} + \frac{k_4}{2}\right)}{1 - \left(\frac{3}{2}k_3 + \frac{3}{2}k_4 + k_9 + k_{10} + k_{11} + k_{12}\right)},$$

combining this with (2.2), we get $0 \leq k < 1$. Then (2.8) becomes

$$\begin{aligned} S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) &+ S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) \\ &\leq k[S_1(gx_n, gx_n, gx_{n+1}) + S_1(gy_n, gy_n, gy_{n+1})]. \end{aligned} \tag{2.9}$$

Applying the above inequality (2.9) n times, we obtain

$$\begin{aligned} S_1(gx_n, gx_n, gx_{n+1}) &+ S_1(gy_n, gy_n, gy_{n+1} \\ &\leq k[S_1(gx_{n-1}, gx_{n-1}, gx_n) + S_1(gy_{n-1}, gy_{n-1}, gy_n)] \\ &\leq k^2[S_1(gx_{n-2}, gx_{n-2}, gx_{n-1}) + S_1(gy_{n-2}, gy_{n-2}, gy_{n-1})] \\ &\leq \cdots \leq k^n[S_1(gx_0, gx_0, gx_1) + S_1(gy_0, gy_0, gy_1)]. \end{aligned} \tag{2.10}$$

Next we shall show $\{gx_n\}, \{gy_n\}$ are S-Cauchy sequences in gX. Indeed, for any $m, n \in \mathbb{N}$, $m > n$, from Lemma 1.4 and (2.10), we have

$$\begin{aligned} &S_1(gx_n, gx_n, gx_m) + S_1(gy_n, gy_n, gy_m) \\ &\quad\leq[2S_1(gx_n, gx_n, gx_{n+1}) + S_1(gx_{n+1}, gx_{n+1}, gx_m)] \\ &\qquad + [2S_1(gy_n, gy_n, gy_{n+1}) + S_1(gy_{n+1}, gy_{n+1}, gy_m)] \\ &\quad\leq[2S_1(gx_n, gx_n, gx_{n+1}) + 2S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S_1(gx_{n+2}, gx_{n+2}, gx_m)] \\ &\qquad + [2S_1(gy_n, gy_n, gy_{n+1}) + 2S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) + S_1(gy_{n+2}, gy_{n+2}, gy_m)] \\ &\quad\leq[2S_1(gx_n, gx_n, gx_{n+1}) + 2S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + \cdots + 2S_1(gx_{m-1}, gx_{m-1}, gx_m) \\ &\qquad + S_1(gx_m, gx_m, gx_m)] + [2S_1(gy_n, gy_n, gy_{n+1}) + 2S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) \\ &\qquad + \cdots + 2S_1(gy_{m-1}, gy_{m-1}, gy_m) + S_1(gy_m, gy_m, gy_m)] \\ &\quad=[2S_1(gx_n, gx_n, gx_{n+1}) + 2S_1(gx_{n+1}, gx_{n+1}, gx_{n+2}) + \cdots + 2S_1(gx_{m-1}, gx_{m-1}, gx_m)] \\ &\qquad + [2S_1(gy_n, gy_n, gy_{n+1}) + 2S_1(gy_{n+1}, gy_{n+1}, gy_{n+2}) + \cdots + 2S_1(gy_{m-1}, gy_{m-1}, gy_m)] \\ &\quad=2\left\{[S_1(gx_n, gx_n, gx_{n+1}) + S_1(gy_n, gy_n, gy_{n+1})] + [S_1(gx_{n+1}, gx_{n+1}, gx_{n+2})\right. \\ &\qquad \left.+S_1(gy_{n+1}, gy_{n+1}, gy_{n+2})] + \cdots + [S_1(gx_{m-1}, gx_{m-1}, gx_m) + S_1(gy_{m-1}, gy_{m-1}, gy_m)]\right\} \\ &\quad\leq 2(k^n + k^{n+1} + \cdots + k^{m-1})[S_1(gx_0, gx_0, gx_1) + S_1(gy_0, gy_0, gy_1)] \\ &\quad=2k^n \frac{1 - k^{m-n}}{1 - k}[S_1(gx_0, gx_0, gx_1) + S_1(gy_0, gy_0, gy_1)] \\ &\quad\leq\frac{2k^n}{1 - k}[S_1(gx_0, gx_0, gx_1) + S_1(gy_0, gy_0, gy_1)]. \end{aligned}$$

It follows from the above inequality that

$$\lim_{n,m\to\infty}[S_1(gx_n, gx_n, gx_m) + S_1(gy_n, gy_n, gy_m)] = 0.$$

Which implies that

$$\lim_{n,m\to\infty} S_1(gx_n, gx_n, gx_m) = 0,$$
$$\lim_{n,m\to\infty} S_1(gy_n, gy_n, gy_m) = 0.$$

Hence we obtain that $\{gx_n\}, \{gy_n\}$ are S-Cauchy subsequences in gX. Since $g(X)$ is S_1-complete in X, therefore $\exists gx, gy \in gX$, which satisfy that $\{gx_n\}$ and $\{gy_n\}$ are S_1-convergent to gx and gy, respectively.

Next, we show that F and g have a coupled point of coincidence. By using $(S2)$, (2.3), Lemma 1.3, Lemma 1.5 (3),(5), Definition 1.6, Definition 1.7, Lemma 1.8 and Lemma 2.1, we can deduce

$$
\begin{aligned}
&S_1(gx_{n+1}, gx_{n+1}, F(x,y))\\
&=S_1(F(x_n,y_n),F(x_n,y_n),F(x,y))\\
&\le k_1S_2(gx_n,gx_n,gx)+k_2S_2(gy_n,gy_n,gy)+k_3S_2(gx_n,gx_n,F(x,y))\\
&\quad+k_4S_2(gy_n,gy_n,F(y,x))+k_5S_2(gx_n,F(x_n,y_n),gx)+k_6S_2(gy_n,F(y_n,x_n),gy)\\
&\quad+k_7S_2(F(x_n,y_n),gx_n,gx)+k_8S_2(F(y_n,x_n),gy_n,gy)+k_9S_2(gx_n,F(x_n,y_n),F(x,y))\\
&\quad+k_{10}S_2(gy_n,F(y_n,x_n),F(y,x))+k_{11}S_2(F(x_n,y_n),gx_n,F(x,y))+k_{12}S_2(F(y_n,x_n),gy_n,F(y,x))\\
&\quad+k_{13}S_2(F(x_n,y_n),F(x_n,y_n),gx)+k_{14}S_2(F(y_n,x_n),F(y_n,x_n),gy)\\
&=k_1S_2(gx_n,gx_n,gx)+k_2S_2(gy_n,gy_n,gy)+k_3S_2(gx_n,gx_n,F(x,y))\\
&\quad+k_4S_2(gy_n,gy_n,F(y,x))+k_5S_2(gx_n,gx_{n+1},gx)+k_6S_2(gy_n,gy_{n+1},gy)\\
&\quad+k_7S_2(gx_{n+1},gx_n,gx)+k_8S_2(gy_{n+1},gy_n,gy)+k_9S_2(gx_n,gx_{n+1},F(x,y))\\
&\quad+k_{10}S_2(gy_n,gy_{n+1},F(y,x))+k_{11}S_2(gx_{n+1},gx_n,F(x,y))+k_{12}S_2(gy_{n+1},gy_n,F(y,x))\\
&\quad+k_{13}S_2(gx_{n+1},gx_{n+1},gx)+k_8S_2(gy_{n+1},gy_{n+1},gy)\\
&\le k_1S_2(gx_n,gx_n,gx)+k_2S_2(gy_n,gy_n,gy)+k_3S_2(gx_n,gx_n,F(x,y))\\
&\quad+k_4S_2(gy_n,gy_n,F(y,x))+k_5\left[S_2(gx_n,gx_n,gx)+S_2(gx_{n+1},gx_{n+1},gx)\right]\\
&\quad+k_6\left[S_2(gy_n,gy_n,gy)+S_2(gy_{n+1},gy_{n+1},gy)\right]\\
&\quad+k_7\left[S_2(gx_n,gx_n,gx)+S_2(gx_{n+1},gx_{n+1},gx)\right]+k_8\left[S_2(gy_n,gy_n,gy)+S_2(gy_{n+1},gy_{n+1},gy)\right]\\
&\quad+k_9\left[S_2(gx_{n+1},gx_{n+1},gx_n)+S_2(gx_n,gx_n,F(x,y))\right]\\
&\quad+k_{10}\left[S_2(gy_{n+1},gy_{n+1},gy_n)+S_2(gy_n,gy_n,F(y,x))\right]\\
&\quad+k_{11}\left[S_2(gx_n,gx_n,gx_{n+1})+S_2(gx_{n+1},gx_{n+1},F(x,y))\right]\\
&\quad+k_{12}\left[S_2(gy_n,gy_n,gy_{n+1})+S_2(gy_{n+1},gy_{n+1},F(y,x))\right]\\
&\quad+k_{13}S_2(gx_{n+1},gx_{n+1},gx)+k_{14}S_2(gy_{n+1},gy_{n+1},gy)\\
&=(k_1+k_5+k_7)\,S_2(gx_n,gx_n,gx)+(k_2+k_6+k_8)\,S_2(gy_n,gy_n,gy)\\
&\quad+(k_3+k_9)\,S_2(gx_n,gx_n,F(x,y))+(k_4+k_{10})\,S_2(gy_n,gy_n,F(y,x))\\
&\quad+(k_5+k_7+k_{13})\,S_2(gx_{n+1},gx_{n+1},gx)+(k_6+k_8+k_{14})\,S_2(gy_{n+1},gy_{n+1},gy)\\
&\quad+k_{11}S_2(gx_{n+1},gx_{n+1},F(x,y))+k_{12}S_2(gy_{n+1},gy_{n+1},F(y,x))\\
&\quad+(k_9+k_{11})\,S_2(gx_n,gx_n,gx_{n+1})+(k_{10}+k_{12})\,S_2(gy_n,gy_n,gy_{n+1}).
\end{aligned}
\tag{2.11}
$$

Taking limit as $n\to\infty$ in (2.11), combining this with Lemma 1.8, we get

$$
\begin{aligned}
&S_1(gx,gx,F(x,y))\\
&\quad=\lim_{n\to\infty}S_1(gx_{n+1},gx_{n+1},F(x,y))\\
&\quad\le(k_1+k_5+k_7)\lim_{n\to\infty}S_2(gx_n,gx_n,gx)+(k_2+k_6+k_8)\lim_{n\to\infty}S_2(gy_n,gy_n,gy)\\
&\qquad+(k_3+k_9)\lim_{n\to\infty}S_2(gx_n,gx_n,F(x,y))+(k_4+k_{10})\lim_{n\to\infty}S_2(gy_n,gy_n,F(y,x))\\
&\qquad+(k_5+k_7+k_{13})\lim_{n\to\infty}S_2(gx_{n+1},gx_{n+1},gx)+(k_6!+k_8+k_{14})\lim_{n\to\infty}S_2(gy_{n+1},gy_{n+1},gy)\\
&\qquad+k_{11}\lim_{n\to\infty}S_2(gx_{n+1},gx_{n+1},F(x,y))+k_{12}\lim_{n\to\infty}S_2(gy_{n+1},gy_{n+1},F(y,x))\\
&\qquad+(k_9+k_{11})\lim_{n\to\infty}S_2(gx_n,gx_n,gx_{n+1})+(k_{10}+k_{12})\lim_{n\to\infty}S_2(gy_n,gy_n,gy_{n+1})\\
&\quad=(k_1+k_5+k_7)\cdot 0+(k_2+k_6+k_8)\cdot 0+(k_3+k_9)\cdot S_2(gx,gx,F(x,y))\\
&\qquad+(k_4+k_{10})\cdot S_2(gy,gy,F(y,x))+(k_5+k_7+k_{13})\cdot 0+(k_6+k_8+k_{14})\cdot 0\\
&\qquad+k_{11}S_2(gx,gx,F(x,y))+k_{12}S_2(gy,gy,F(y,x))+(k_9+k_{11})\cdot 0+(k_{10}+k_{12})\cdot 0
\end{aligned}
$$

$$\begin{aligned}&= (k_3 + k_9 + k_{11}) \cdot S_2(gx, gx, F(x, y)) + (k_4 + k_{10} + k_{12}) \cdot S_2(gy, gy, F(y, x))\\ &\le (k_3 + k_9 + k_{11})\, S_1(gx, gx, F(x, y)) + (k_4 + k_{10} + k_{12})\, S_1(gy, gy, F(y, x)).\end{aligned}$$

This implies that

$$S_1(gx, gx, F(x, y)) \le (k_3 + k_9 + k_{11})\, S_1(gx, gx, F(x, y)) + (k_4 + k_{10} + k_{12})\, S_1(gy, gy, F(y, x)). \tag{2.12}$$

Similarly,

$$S_1(gy, gy, F(y, x)) \le (k_3 + k_9 + k_{11})\, S_1(gy, gy, F(y, x)) + (k_4 + k_{10} + k_{12})\, S_1(gx, gx, F(x, y)). \tag{2.13}$$

Combining (2.12) with (2.13) yields

$$\begin{aligned}&S_1(gx, gx, F(x, y)) + S_1(gy, gy, F(y, x))\\ &\qquad \le (k_3 + k_4 + k_9 + k_{10} + k_{11} + k_{12})\left[S_1(gx, gx, F(x, y)) + S_1(gy, gy, F(y, x))\right].\end{aligned} \tag{2.14}$$

Then by (2.2), we have $0 \le k_3 + k_4 + k_9 + k_{10} + k_{11} + k_{12} < 1$, thus it is easy to verify that

$$S_1(gx, gx, F(x, y)) + S_1(gy, gy, F(y, x)) = 0.$$

So that

$$S_1(gx, gx, F(x, y)) = 0, \;\; S_1(gy, gy, F(y, x)) = 0.$$

That is $gx = F(x, y)$ and $gy = F(y, x)$, we prove that (gx, gy) is a the coupled point of coincidence of the mappings g and F.

Now we prove the mappings g and F have a unique coupled point of coincidence. Assuming that $\exists (x^*, y^*) \in (X \times X)$ is also a coupled point of coincidence of the mappings g and F, thus $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$.

It follows from (2.1) that

$$\begin{aligned}S_1(gx, gx, gx^*) =& S_1(F(x, y), F(x, y), F(x^*, y^*))\\ \le& k_1 S_2(gx, gx, gx^*) + k_2 S_2(gy, gy, gy^*) + k_3 S_2(gx, gx, F(x^*, y^*)) + k_4 S_2(gy, gy, F(y^*, x^*))\\ &+ k_5 S_2(gx, F(x, y), gx^*) + k_6 S_2(gy, F(y, x), gy^*) + k_7 S_2(F(x, y), gx, gx^*)\\ &+ k_8 S_2(F(y, x), gy, gy^*) + k_9 S_2(gx, F(x, y), F(x^*, y^*)) + k_{10} S_2(gy, F(y, x), F(y^*, x^*))\\ &+ k_{11} S_2(F(x, y), gx, F(x^*, y^*)) + k_{12} S_2(F(y, x), gy, F(y^*, x^*))\\ &+ k_{13} S_2(F(x, y), F(x, y), gx^*) + k_{14} S_2(F(y, x), F(y, x), gy^*)\\ =& k_1 S_2(gx, gx, gx^*) + k_2 S_2(gy, gy, gy^*) + k_3 S_2(gx, gx, gx^*) + k_4 S_2(gy, gy, gy^*)\\ &+ k_5 S_2(gx, gx, gx^*) + k_6 S_2(gy, gy, gy^*) + k_7 S_2(gx, gx, gx^*) + k_8 S_2(gy, gy, gy^*)\\ &+ k_9 S_2(gx, gx, gx^*) + k_{10} S_2(gy, gy, gy^*) + k_{11} S_2(gx, gx, gx^*) + k_{12} S_2(gy, gy, gy^*)\\ &+ k_{13} S_2(gx, gx, gx^*) + k_{14} S_2(gy, gy, gy^*)\\ =& (k_1 + k_3 + k_5 + k_7 + k_9 + k_{11} + k_{13})\, S_2(gx, gx, gx^*)\\ &+ (k_2 + k_4 + k_6 + k_8 + k_{10} + k_{12} + k_{14})\, S_2(gy, gy, gy^*)\\ \le& (k_1 + k_3 + k_5 + k_7 + k_9 + k_{11} + k_{13})\, S_1(gx, gx, gx^*)\\ &+ (k_2 + k_4 + k_6 + k_8 + k_{10} + k_{12} + k_{14})\, S_1(gy, gy, gy^*).\end{aligned}$$

That is

$$\begin{aligned}S_1(gx, gx, gx^*) \le& (k_1 + k_3 + k_5 + k_7 + k_9 + k_{11} + k_{13})\, S_2(gx, gx, gx^*)\\ &+ (k_2 + k_4 + k_6 + k_8 + k_{10} + k_{12} + k_{14})\, S_1(gy, gy, gy^*),\end{aligned} \tag{2.15}$$

we can similarly prove the following result

$$\begin{aligned}S_1(gy, gy, gy^*) \le& (k_1 + k_3 + k_5 + k_7 + k_9 + k_{11} + k_{13})\, S_1(gy, gy, gy^*)\\ &+ (k_2 + k_4 + k_6 + k_8 + k_{10} + k_{12} + k_{14})\, S_1(gx, gx, gx^*).\end{aligned} \tag{2.16}$$

Combining (2.15) with (2.16), we have

$$S_1(gx,gx,gx^*)+S_1(gy,gy,gy^*)\leq\sum_{i=1}^{14}k_i[(gx,gx,gx^*)+S_1(gy,gy,gy^*)].$$

By (2.2), we have $0\leq\sum_{i=1}^{14}k_i<1$, then $S_1(gx,gx,gx^*)+S_1(gy,gy,gy^*)=0$, that is $S_1(gx,gx,gx^*)=S_1(gy,gy,gy^*)=0$, thus, $gx=gx^*$ and $gy=gy^*$, this implies the mappings g and F have a unique coupled point of coincidence.

Next, we prove that $gx=gy$ holds. Again, from (2.1) and Lemma 1.3, we have

$$\begin{aligned}
S_1(gx,&gx,gy)\\
=&S_1(F(x,y),F(x,y),F(y,x))\\
\leq&k_1S_2(gx,gx,gy)+k_2S_2(gy,gy,gx)+k_3S_2(gx,gx,F(y,x))+k_4S_2(gy,gy,F(x,y))\\
&+k_5S_2(gx,F(x,y),gy)+k_6S_2(gy,F(y,x),gx)+k_7S_2(F(x,y),gx,gy)+k_8S_2(F(y,x),gy,gx)\\
&+k_9S_2(gx,F(x,y),F(y,x))+k_{10}S_2(gy,F(y,x),F(x,y))+k_{11}S_2(F(x,y),gx,F(y,x))\\
&+k_{12}S_2(F(y,x),gy,F(x,y))+k_{13}S_2(F(x,y),F(x,y),gy)+k_{14}S_2(F(y,x),F(y,x),gx)\\
=&k_1S_2(gx,gx,gy)+k_2S_2(gy,gy,gx)+k_3S_2(gx,gx,gy)+k_4S_2(gy,gy,gx)\\
&+k_5S_2(gx,gx,gy)+k_6S_2(gy,gy,gx)+k_7S_2(gx,gx,gy)+k_8S_2(gy,gy,gx)\\
&+k_9S_2(gx,gx,gy)+k_{10}S_2(gy,gy,gx)+k_{11}S_2(gx,gx,gy)+k_{12}S_2(gy,gy,gx)\\
&+k_{13}S_2(gx,gx,gy)+k_{14}S_2(gy,gy,gx)\\
=&\sum_{i=1}^{14}k_iS_2(gx,gx,gy)\leq\sum_{i=1}^{14}k_iS_1(gx,gx,gy).
\end{aligned}$$

By (2.2), we get $0\leq\sum_{i=1}^{14}k_i<1$, that is $S_1(gx,gx,gy)=0$, thus $gx=gy$.

Finally, if the pair of mappings (g,F) is w-compatible, by taking $u=gx$, we get $gu=ggx=gF(x,y)=F(gx,gy)=F(u,u)$. Consequently, (gu,gu) is the a coupled point of coincidence of the mappings g and F. By virtue of the unique of coupled point of coincidence, we can easily obtain that $gu=gx=u$. Furthermore, we have $u=gu=F(u,u)$, that is (u,u) is a common coupled fixed point of mappings g and F. □

Example 2.2. Let $X=[0,\infty)$, define $S_1,S_2:X\times X\times X\to\mathbb{R}^+$ respectively by

$$S_1(x,y,z)=\mid x-z\mid+\mid y-z\mid,$$

$$S_2(x,y,z)=\frac{1}{2}\mid x-z\mid+\frac{1}{2}\mid y-z\mid.$$

Therefore, we conclude that S_1,S_2 are two S-metrics in X, furthermore $S_2(x,y,z)\leq S_1(x,y,z)\ \forall x,y,z\in X$.

Define two functions $F:X\times X\to X, g:X\to X$ respectively by

$$F(x,y)=\frac{1}{8}\ln(1+\mid x-y\mid),\ \ gx=2x\ \forall x,y\in X.$$

By the definition, we can easily obtain that $F(X\times X)\subseteq gX$. Next we show that the pair (F,g) is ω-compatible. In fact

$$\begin{cases}F(x,y)=gx;\\F(y,x)=gy.\end{cases}\Leftrightarrow\begin{cases}\frac{1}{8}\ln(1+\mid x-y\mid)=2x;\\\frac{1}{8}\ln(1+\mid y-x\mid)=2y.\end{cases}\Leftrightarrow x=y=0.$$

This shows that $(g0,g0)$ is the unique of coupled point of coincidence point of mappings F and g. Apparently, we get $F(g0,g0)=g(F(0,0))=0$, therefore the pair (F,g) is $\omega-$ compatible.

In next step, we will prove that the condition (2.1) holds in Theorem 2.1. In fact, $\forall(x,y),(u,v),(z,w)\in X\times X$, we have

$$
\begin{aligned}
&S_1(F(x,y),F(u,v),F(z,w))\\
&\quad=\left|\frac{1}{8}\ln(1+\mid x-y\mid)-\frac{1}{8}\ln(1+\mid z-w\mid)\right|+\left|\frac{1}{8}\ln(1+\mid u-v\mid)-\frac{1}{8}\ln(1+\mid z-w\mid)\right|\\
&\quad=\left|\frac{1}{8}\ln\left(\frac{1+\mid x-y\mid}{1+\mid z-w\mid}\right)\right|+\left|\frac{1}{8}\ln\left(\frac{1+\mid u-v\mid}{1+\mid z-w\mid}\right)\right|\\
&\quad=\left|\frac{1}{8}\ln\left(\frac{1+\mid x-y\mid-\mid z-w\mid+\mid z-w\mid}{1+\mid z-w\mid}\right)\right|+\left|\frac{1}{8}\ln\left(\frac{1+\mid u-v\mid-\mid z-w\mid+\mid z-w\mid}{1+\mid z-w\mid}\right)\right|\\
&\quad=\left|\frac{1}{8}\ln\left(1+\frac{\mid x-y\mid-\mid z-w\mid}{1+\mid z-w\mid}\right)\right|+\left|\frac{1}{8}\ln\left(1+\frac{\mid u-v\mid-\mid z-w\mid}{1+\mid z-w\mid}\right)\right|\\
&\quad\leq\left|\frac{1}{8}\ln\left(1+\frac{\mid(x-y)-(z-w)\mid}{1+\mid z-w\mid}\right)\right|+\left|\frac{1}{8}\ln\left(1+\frac{\mid(u-v)-(z-w)\mid}{1+\mid z-w\mid}\right)\right|\\
&\quad=\left|\frac{1}{8}\ln\left(1+\frac{\mid(x-z)-(y-w)\mid}{1+\mid z-w\mid}\right)\right|+\left|\frac{1}{8}\ln\left(1+\frac{\mid(u-z)-(v-w)\mid}{1+\mid z-w\mid}\right)\right|\\
&\quad\leq\frac{1}{8}\ln\left(1+\frac{\mid x-z\mid+\mid y-w\mid}{1+\mid z-w\mid}\right)+\frac{1}{8}\ln\left(1+\frac{\mid u-z\mid+\mid v-w\mid}{1+\mid z-w\mid}\right)\\
&\quad\leq\frac{1}{8}\ln[1+(\mid x-z\mid+\mid y-w\mid)]+\frac{1}{8}\ln[1+(\mid u-z\mid+\mid v-w\mid)]\\
&\quad\leq\frac{1}{8}(\mid x-z\mid+\mid y-w\mid)+\frac{1}{8}(\mid u-z\mid+\mid v-w\mid)\\
&\quad=\frac{1}{16}(\mid 2x-2z\mid+\mid 2u-2z\mid)+\frac{1}{16}(\mid 2y-2w\mid+\mid 2v-2w\mid)\\
&\quad=\frac{1}{16}(\mid gx-gz\mid+\mid gu-gz\mid)+\frac{1}{16}(\mid gy-gw\mid+\mid gv-gw\mid)\\
&\quad=\frac{1}{8}\left(\frac{1}{2}\mid gx-gz\mid+\frac{1}{2}\mid gu-gz\mid\right)+\frac{1}{8}\left(\frac{1}{2}\mid gy-gw\mid+\frac{1}{2}\mid gv-gw\mid\right)\\
&\quad=\frac{1}{8}S_2(gx,gu,gz)+\frac{1}{8}S_2(gy,gv,gw)\\
&\quad\leq\frac{1}{8}S_2(gx,gu,gz)+\frac{1}{8}S_2(gy,gv,gw)+\frac{1}{48}S_2(gx,gu,F(s,t))\\
&\qquad+\frac{1}{48}S_2(gy,gv,F(t,s))+\frac{1}{16}S_2(gx,F(u,v),gs)+\frac{1}{16}S_2(gy,F(v,u),gt)\\
&\qquad+\frac{1}{16}S_2(F(x,y),gu,gs)+\frac{1}{16}S_2(F(y,x),gv,gt)+\frac{1}{32}S_2(gx,F(u,v),F(s,t))\\
&\qquad+\frac{1}{32}S_2(gy,F(v,u),F(t,s))+\frac{1}{32}S_2(F(x,y),gu,F(s,t))+\frac{1}{32}S_2(F(y,x),gv,F(t,s))\\
&\qquad+\frac{1}{32}S_2(F(x,y),F(u,v),gs)+\frac{1}{32}S_2(F(y,x),F(v,u),gt).
\end{aligned}
$$

By virtue of $0\leq\frac{1}{8}+\frac{1}{8}+3\left(\frac{1}{48}+\frac{1}{48}\right)+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+2\left(\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}\right)+\frac{1}{32}+\frac{1}{32}=\frac{15}{16}<1$, then the mappings F and g satisfy all the conditions appearing in Theorem 2.1, by the result of Theorem 2.1, we get F and g have a coupled common fixed point. In fact, $(0,0)$ is a unique coupled common fixed point of F and g, that is $F(0,0)=g0=0$.

Corollary 2.3. *Let X be a nonempty set, S_1,S_2 are two S-metrics on X such that $S_2(x,y,z)\leq S_1(x,y,z)$ $\forall x,y,z\in X$. Suppose that the mappings $F:X\times X\to X$ and $g:X\to X$ satisfy the following contractive condition*

$$S_1(F(x,y),F(u,v),F(s,t))\leq k_1S_2(gx,gu,gs)+k_2S_2(gy,gv,gt), \tag{2.17}$$

$\forall(x,y),(u,v),(s,t)\in X\times X$, where $k_1,k_2\geq 0$ and $0\leq k_1+k_2<1$. If $F(X\times X)\subseteq gX$ and gX is a S_1-complete subspace of (X,S_1), then F and g have a unique coupled point of coincidence $(gx,gy)\in X\times X$, which satisfy $gx=F(x,y)=gy=F(y,x)$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form $(u, u) \in X \times X$, satisfying $u = gu = F(u, u)$.

Proof. By taking $k_i = 0, i = 3, 4, \cdots, 14$ in Theorem 2.1, then Corollary 2.3 holds. □

Corollary 2.4. *Let X be a nonempty set, S_1, S_2 are two S-metrics on X such that $S_2(x, y, z) \leq S_1(x, y, z)$ $\forall x, y, z \in X$. Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition*

$$\begin{aligned} S_1(F(x,y), F(u,v), F(s,t)) \leq & a_1 S_2(gx, gu, F(s,t)) + a_2 S_2(gy, gv, F(t,s)) \\ & + a_3 S_2(gx, F(u,v), gs) + a_4 S_2(gy, F(v,u), gt) \\ & + a_5 S_2(F(x,y), gu, gs) + a_6 S_2(F(y,x), gv, gt), \end{aligned} \tag{2.18}$$

$\forall (x, y), (u, v), (s, t) \in X \times X$, where $a_i \geq 0$ $\forall i = 1, 2, \cdots, 6$ and $0 \leq 3(a_1 + a_2) + a_3 + a_4 + a_5 + a_6 < 1$. If $F(X \times X) \subseteq gX$ and gX is a S_1-complete subspace of (X, S_1), then F and g have a unique coupled point of coincidence $(gx, gy) \in X \times X$, which satisfy $gx = F(x, y) = gy = F(y, x)$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form $(u, u) \in X \times X$, satisfying $u = gu = F(u, u)$.

Proof. By taking $k_i = 0, i = 1, 2, 9, 10, 11, 12, 13, 14$ in Theorem 2.1, then Corollary 2.4 holds. □

Corollary 2.5. *Let X be a nonempty set, S_1, S_2 are two S-metrics on X such that $S_2(x, y, z) \leq S_1(x, y, z)$ $\forall x, y, z \in X$. Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition*

$$\begin{aligned} S_1(F(x,y), F(u,v), F(s,t)) \leq & c_1 S_2(gx, F(u,v), F(s,t)) + c_2 S_2(gy, F(v,u), F(t,s)) \\ & + c_3 S_2(F(x,y), gu, F(s,t)) + c_4 S_2(F(y,x), gv, F(t,s)) \\ & + c_5 S_2(F(x,y), F(u,v), gs) + c_6 S_2(F(y,x), F(v,u), gt), \end{aligned} \tag{2.19}$$

$\forall (x, y), (u, v), (s, t) \in X \times X$, where $c_i \geq 0$ $\forall i = 1, 2, \cdots, 6$ and $0 \leq 2(c_1 + c_2 + c_3 + c_4) + c_5 + c_6 < 1$. If $F(X \times X) \subseteq gX$ and gX is a S_1-complete subspace of (X, S_1), then F and g have a unique coupled point of coincidence $(gx, gy) \in X \times X$, which satisfy $gx = F(x, y) = gy = F(y, x)$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form $(u, u) \in X \times X$, satisfying $u = gu = F(u, u)$.

By taking $g = I$ in Corollary 2.3–2.5, we obtain the following result:

Corollary 2.6. *Let X be a nonempty set and S_1, S_2 are two S-metrics on X such that $S_2(x, y, z) \leq S_1(x, y, z)$ $\forall x, y, z \in X$. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$S_1(F(x,y), F(u,v), F(s,t)) \leq k_1 S_2(x, u, s) + k_2 S_2(y, v, t), \tag{2.20}$$

$\forall (x, y), (u, v), (s, t) \in X \times X$. Where $k_1, k_2 \geq 0$ and $0 \leq k_1 + k_2 < 1$. If (X, S_1) is a complete S_1-metric space, then F has a unique coupled fixed point of the form $(u, u) \in X \times X$, satisfying $u = F(u, u)$.

Corollary 2.7. *Let X be a nonempty set and S_1, S_2 are two S-metrics on X such that $S_2(x, y, z) \leq S_1(x, y, z)$ $\forall x, y, z \in X$. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$\begin{aligned} S_1(F(x,y), F(u,v), F(s,t)) \leq & a_1 S_2(x, u, F(s,t)) + a_2 S_2(y, v, F(t,s)) \\ & + a_3 S_2(x, F(u,v), s) + a_4 S_2(y, F(v,u), t) \\ & + a_5 S_2(F(x,y), u, s) + a_6 S_2(F(y,x), v, t), \end{aligned} \tag{2.21}$$

$\forall (x, y), (u, v), (s, t) \in X \times X$, where $a_i \geq 0$ $\forall i = 1, 2, \cdots, 6$ and $0 \leq 3(a_1 + a_2) + a_3 + a_4 + a_5 + a_6 < 1$. If (X, S_1) is a complete S_1-metric space, then F has a unique coupled fixed point of the form $(u, u) \in X \times X$, satisfying $u = F(u, u)$.

Corollary 2.8. *Let X be a nonempty set and S_1, S_2 are two S-metrics on X such that $S_2(x,y,z) \leq S_1(x,y,z)$ $\forall x,y,z \in X$. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition*

$$\begin{aligned} S_1(F(x,y),F(u,v),F(s,t)) \leq & c_1 S_2(x,F(u,v),F(s,t)) + c_2 S_2(y,F(v,u),F(t,s)) \\ & + c_3 S_2(F(x,y),u,F(s,t)) + c_4 S_2(F(y,x),v,F(t,s)) \\ & + c_5 S_2(F(x,y),F(u,v),s) + c_6 S_2(F(y,x),F(v,u),t), \end{aligned} \tag{2.22}$$

$\forall (x,y),(u,v),(s,t) \in X \times X$, where $c_i \geq 0$ $\forall i = 1,2,\cdots,6$ and $0 \leq 2(c_1+c_2+c_3+c_4)+c_5+c_6 < 1$. If (X,S_1) is a complete S_1-metric space, then F has a unique coupled fixed point of the form $(u,u) \in X \times X$, satisfying $u = F(u,u)$.

If by taking $S_2(x,y,z) = S_1(x,y,z)$ $\forall x,y,z \in X$ in Theorem 2.1 and Corollary 2.3-2.8, we can obtain some new results.

Example 2.9. Taking $X = \mathbb{R}$, we assume that two S-metrics $S_1, S_2 : X^3 \to \mathbb{R}^+$ defined on X, defined respectively by $S_1(x,y,z) = |x-z| + |y-z|, S_2(x,y,z) = \frac{1}{2}(|x-z|+|y-z|)$ $\forall x,y,z \in X$, and define the function $F : X \times X \to X$ by $F(x,y) = \frac{x}{8} + \frac{y}{4}$ $\forall x \in X$.

By the definition, we get that (X,S_1) is S_1-complete, furthermore, $\forall x,y,u,v,s,t \in X$, we obtain

$$\begin{aligned} & S_1(F(x,y),F(u,v),F(s,t)) \\ & \quad = S_1\left(\frac{x}{8}+\frac{y}{4}, \frac{u}{8}+\frac{v}{4}, \frac{s}{8}+\frac{t}{2}\right) = \left|\left(\frac{x}{8}+\frac{y}{4}\right) - \left(\frac{s}{8}+\frac{t}{4}\right)\right| + \left|\left(\frac{u}{8}+\frac{v}{4}\right) - \left(\frac{s}{8}+\frac{t}{4}\right)\right| \\ & \quad = \left|\left(\frac{x}{8}-\frac{s}{8}\right) + \left(\frac{y}{4}-\frac{t}{4}\right)\right| + \left|\left(\frac{u}{8}-\frac{s}{8}\right) + \left(\frac{v}{4}-\frac{t}{4}\right)\right| \\ & \quad \leq \left|\frac{x}{8}-\frac{s}{8}\right| + \left|\frac{y}{4}-\frac{t}{4}\right| + \left|\frac{u}{8}-\frac{s}{8}\right| + \left|\frac{v}{4}-\frac{t}{4}\right| \\ & \quad = \frac{1}{8}(|x-s|+|u-s|) + \frac{1}{4}(|y-t|+|v-t|) \\ & \quad = \frac{1}{4}S_2(x,u,s) + \frac{1}{2}S_2(y,v,t). \end{aligned}$$

It is easily seen that the condition (2.20) of Corollary 2.4 holds, where $k_1 = \frac{1}{4}, k_2 = \frac{1}{2}, k_1 + k_2 = \frac{3}{4} \in [0,1)$. Then the following condition appearing in Corollary 2.4 are satisfied, so F has a unique coupled fixed point. In fact, $(0,0)$ is a unique coupled fixed point of F.

Theorem 2.10. *Let X be a nonempty set and S_1, S_2 are two S-metrics on X such that $S_2(x,y,z) \leq S_1(x,y,z)$ $\forall x,y,z \in X$. Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition*

$$\begin{aligned} & S_1(F(x,y),F(u,v),F(s,t)) \\ & \leq k \max \left\{ \begin{array}{c} S_2(gx,gu,gs), S_2(gy,gv,gt), \frac{1}{3}S_2(gx,gu,F(s,t)), \frac{1}{3}S_2(gy,gv,F(t,s)) \\ S_2(gx,F(u,v),gs), S_2(gy,F(v,u),gt), S_2(F(x,y),gu,gs), S_2(F(y,x),gv,gt) \\ \frac{1}{2}S_2(gx,F(u,v),F(s,t)), \frac{1}{2}S_2(gy,F(v,u),F(t,s)), \frac{1}{2}S_2(F(x,y),gu,F(s,t)) \\ \frac{1}{2}S_2(F(y,x),gv,F(t,s)), S_2(F(x,y),F(v,u),gs), S_2(F(y,x),F(v,u),gt) \end{array} \right\}, \end{aligned} \tag{2.23}$$

$\forall (x,y),(u,v),(s,t) \in X \times X$, where $0 \leq k < 1$. If $F(X \times X) \subseteq gX$ and gX is a complete subspace of (X,S_1), then F and g have a unique coupled point of coincidence $(gx,gy) \in X \times X$, satisfying $gx = F(x,y) = gy = F(y,x)$.

Moreover, if F and g are w-compatible, then the mappings F and g have a unique common coupled fixed point $(u,u) \in X \times X$, satisfying $u = gu = F(u,u)$.

Proof. By similar arguments as Theorem 2.1, we get the Theorem 2.10. □

Corollary 2.11. *Let (X,S) be a S-metric space and the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition*

$$\begin{aligned} & S(F(x,y),F(u,v),F(s,t)) \\ \leq\ & k\max\left\{\begin{array}{c} S(gx,gu,gs), S(gy,gv,gt), \frac{1}{3}S(gx,gu,F(s,t)), \frac{1}{3}S(gy,gv,F(t,s)) \\ S(gx,F(u,v),gs), S(gy,F(v,u),gt), S(F(x,y),gu,gs), S(F(y,x),gv,gt) \\ \frac{1}{2}S(gx,F(u,v),F(s,t)), \frac{1}{2}S(gy,F(v,u),F(t,s)), \frac{1}{2}S(F(x,y),gu,F(s,t)) \\ \frac{1}{2}S(F(y,x),gv,F(t,s)), S(F(x,y),F(v,u),gs), S(F(y,x),F(v,u),gt) \end{array}\right\}, \end{aligned} \tag{2.24}$$

$\forall (x,y),(u,v),(s,t) \in X \times X$, where $0 \leq k < 1$. If $F(X\times X) \subseteq gX$ and gX is a complete subspace of (X,S), then F and g have a unique coupled point of coincidence $(gx,gy) \in X\times X$, satisfying $gx = F(x,y) = gy = F(y,x)$.

Moreover, if F and g are w-compatible, then the mappings F and g have a unique common coupled fixed point $(u,u) \in X \times X$, satisfying $u = gu = F(u,u)$.

Proof. By taking $S_2(x,y,z) = S_1(x,y,z)$ in Theorem 2.10, we can obtain the Corollary 2.11. □

By taking $g = I$ in Theorem 2.10 and Corollary 2.11, we can obtain the following corollary:

Corollary 2.12. *Let X be a nonempty set and S_1, S_2 are two S-metrics on X such that $S_2(x,y,z) \leq S_1(x,y,z)$ $\forall x,y,z \in X$. Suppose that the mapping $F : X\times X \to X$ satisfies the following contractive condition*

$$\begin{aligned} & S_1(F(x,y),F(u,v),F(s,t)) \\ \leq\ & k\max\left\{\begin{array}{c} S_2(x,u,s), S_2(y,v,t), \frac{1}{3}S_2(x,u,F(s,t)), \frac{1}{3}S_2(y,v,F(t,s)) \\ S_2(x,F(u,v),s), S_2(y,F(v,u),t), S_2(F(x,y),u,s), S_2(F(y,x),v,t) \\ \frac{1}{2}S_2(x,F(u,v),F(s,t)), \frac{1}{2}S_2(y,F(v,u),F(t,s)), \frac{1}{2}S_2(F(x,y),u,F(s,t)) \\ \frac{1}{2}S_2(F(y,x),v,F(t,s)), S_2(F(x,y),F(v,u),s), S_2(F(y,x),F(v,u),t) \end{array}\right\}, \end{aligned} \tag{2.25}$$

$\forall (x,y),(u,v),(s,t) \in X\times X$, where $0 \leq k < 1$. If (X,S_1) is S-complete S-metric space, then F have a unique coupled fixed point of the form $(u,u) \in X\times X$, which satisfy $u = F(u,u)$.

Corollary 2.13. *Let (X,S) be a S-complete S-metric space and the mapping $F : X\times X \to X$ satisfies the following contractive condition*

$$\begin{aligned} & S(F(x,y),F(u,v),F(s,t)) \\ \leq\ & k\max\left\{\begin{array}{c} S(x,u,s), S(y,v,t), \frac{1}{3}S(x,u,F(s,t)), \frac{1}{3}S(y,v,F(t,s)) \\ S(x,F(u,v),s), S(y,F(v,u),t), S(F(x,y),u,s), S(F(y,x),v,t) \\ \frac{1}{2}S(x,F(u,v),F(s,t)), \frac{1}{2}S(y,F(v,u),F(t,s)), \frac{1}{2}S(F(x,y),u,F(s,t)) \\ \frac{1}{2}S(F(y,x),v,F(t,s)), S(F(x,y),F(v,u),s), S(F(y,x),F(v,u),t) \end{array}\right\}, \end{aligned} \tag{2.26}$$

$\forall (x,y),(u,v),(s,t) \in X\times X$, where $0 \leq k < 1$. Then F have a unique coupled fixed point of the form $(u,u) \in X\times X$, which satisfies $u = F(u,u)$.

3. Application to integral equations

In this section, we wish to study the existence and uniqueness problem of solution for a class of nonlinear integral equations by using the obtained result.

Throughout this section, we assume that $X = C[0,1]$ is the set of all continuous functions defined on $[0,1]$. Define $S_1, S_2 : X^3 \to \mathbb{R}^+$ respectively by

$$S_1(x,y,z) = \sup_{p\in[0,1]} \mid x(p) - z(p) \mid + \sup_{p\in[0,1]} \mid y(p) - z(p) \mid \ \ \forall x,y,z \in X$$

and

$$S_2(x,y,z)=\frac{1}{4}\sup_{p\in[0,1]}\mid x(p)-z(p)\mid+\frac{1}{4}\sup_{p\in[0,1]}\mid y(p)-z(p)\mid\ \ \forall x,y,z\in X.$$

Then we get (X,S_1) and (X,S_2) are S-metric spaces.

Consider the following nonlinear quadratic integral equation set

$$\begin{cases} x(p)=h(p)+\int_0^1 k(p,q)(f_1(q,x(q))+f_2(q,y(q)))dq, p\in[0,1], \\ y(p)=h(p)+\int_0^1 k(p,q)(f_1(q,y(q))+f_2(q,x(q)))dq, p\in[0,1]\ \forall x,y\in X, \end{cases} \tag{3.1}$$

where $h:[0,1]\to\mathbb{R}$, $k:[0,1]\times[0,1]\to\mathbb{R}^+$ and $f_1,f_2:[0,1]\times\mathbb{R}\to\mathbb{R}$.

Next, we will analyze (3.1) under the following conditions:

(i) h,k,f_1 and f_2 are continuous functions.

(ii) There exist constants $\mu,\nu>0$ such that

$$\begin{cases} \mid f_1(p,x)-f_1(p,y)\mid\le\mu\mid x-y\mid, \\ \mid f_2(p,x)-f_2(p,y)\mid\le\nu\mid x-y\mid, \end{cases}$$

$\forall p\in[0,1],x,y\in\mathbb{R}$.

(iii) $4\max\{\mu,\nu\}\parallel k\parallel_\infty\le\frac{1}{16}$. Where $\parallel k\parallel_\infty=\sup\{k(p,q):p,q\in[0,1]\}$.

Theorem 3.1. *Under the conditions (i)-(iii), then integral equation* (3.1) *has a unique common solution in* $C[0,1]$.

Proof. First the operators $F:X^2\to X$ and $g:X\to X$ are defined respectively by

$$F(x,y)(p)=h(p)+\int_0^1 k(p,q)(f_1(q,x(q))+f_2(q,y(q)))dq, p\in[0,1]\ \forall x,y,z\in X$$

and

$$gx(p)=x(p)\ \forall x\in X.$$

Then we induced $F(X\times X)\subseteq gX$, F and g are ω-compatible and gX is a complete subspace of (X,S_1). From the definition of S_1, we can get

$$\begin{aligned} &S_1(F(x,y),F(u,v),F(s,t)) \\ &\quad=\sup_{p\in[0,1]}|F(x,y)(p)-F(s,t)(p)|+\sup_{p\in[0,1]}|F(u,v)(p)-F(s,t)(p)| \\ &\quad=\sup_{p\in[0,1]}\Big|[h(p)+\int_0^1 k(p,q)(f_1(q,x(q))+f_2(q,y(q)))dq] \\ &\qquad-[h(p)+\int_0^1 k(p,q)(f_1(q,s(q))+f_2(q,t(q)))dq]\Big| \\ &\qquad+\sup_{p\in[0,1]}\Big|[h(p)+\int_0^1 k(p,q)(f_1(q,u(q))+f_2(q,v(q)))dq] \\ &\qquad-[h(p)+\int_0^1 k(p,q)(f_1(q,s(q))+f_2(q,t(q)))dq]\Big| \\ &\quad=\sup_{p\in[0,1]}\Big|\int_0^1 k(p,q)(f_1(q,x(q))+f_2(q,y(q)))dq-\int_0^1 k(p,q)(f_1(q,s(q))+f_2(q,t(q)))dq\Big| \end{aligned}$$

$$
\begin{aligned}
&+ \sup_{p\in[0,1]} \left| \int_0^1 k(p,q)(f_1(q,u(q))+f_2(q,v(q)))dq - \int_0^1 k(p,q)(f_1(q,s(q))+f_2(q,t(q)))dq \right| \\
&= \sup_{p\in[0,1]} \left| \int_0^1 k(p,q)[(f_1(q,x(q))-f_1(q,s(q)))+(f_2(q,y(q))-f_2(q,t(q)))] \right| dq \\
&+ \sup_{p\in[0,1]} \left| \int_0^1 k(p,q)[(f_1(q,u(q))-f_1(q,s(q)))+(f_2(q,v(q))-f_2(q,t(q)))] \right| dq \\
&\leq \sup_{p\in[0,1]} \int_0^1 k(p,q) \left[\mid f_1(q,x(q))-f_1(q,s(q)) \mid + \mid f_2(q,y(q))-f_2(q,t(q)) \mid\right] dq \\
&+ \sup_{p\in[0,1]} \int_0^1 k(p,q) \left[\mid f_1(q,u(q))-f_1(q,s(q)) \mid + \mid f_2(q,v(q))-f_2(q,t(q)) \mid\right] dq.
\end{aligned} \tag{3.2}
$$

By the condition (ii), we get

$$
\begin{cases}
\mid f_1(p,x(q))-f_1(p,y(q)) \mid \leq \mu \mid x(q)-y(q) \mid, \\
\mid f_2(p,x(q))-f_2(p,y(q)) \mid \leq \nu \mid x(q)-y(q) \mid .
\end{cases}
$$

Then Inequality (3.2) becomes

$$
\begin{aligned}
S_1(F(x,y),F(u,v),F(s,t)) &\leq \sup_{p\in[0,1]} \int_0^1 k(p,q)(\mu \mid x(q)-s(q) \mid + \nu \mid y(q)-t(q) \mid)dq \\
&\quad + \sup_{p\in[0,1]} \int_0^1 k(p,q)(\mu \mid u(q)-s(q) \mid + \nu \mid v(q)-t(q) \mid)dq \\
&\leq \max\{\mu,\nu\} \sup_{p\in[0,1]} \int_0^1 k(p,q)(\mid x(q)-s(q) \mid + \mid y(q)-t(q) \mid)dq \\
&\quad + \max\{\mu,\nu\} \sup_{p\in[0,1]} \int_0^1 k(p,q)(\mid u(q)-s(q) \mid + \mid v(q)-t(q) \mid)dq.
\end{aligned} \tag{3.3}
$$

By using $Cauchy - Schwartz$ inequality, we have

$$
\begin{aligned}
&\int_0^1 k(p,q)(\mid x(q)-s(q) \mid + \mid y(q)-t(q) \mid)dq \\
&\qquad \leq \left(\int_0^1 k^2(p,q)dq\right)^{\frac{1}{2}} \left(\int_0^1 (\mid x(q)-s(q) \mid + \mid y(q)-t(q) \mid)^2 dq\right)^{\frac{1}{2}} \\
&\qquad \leq \parallel k \parallel_\infty \left(\sup_{p\in[0,1]} \mid x(q)-s(q) \mid + \sup_{p\in[0,1]} \mid y(q)-t(q) \mid\right).
\end{aligned} \tag{3.4}
$$

Similarly, we can prove that

$$
\begin{aligned}
&\int_0^1 k(p,q)(\mid u(q)-s(q) \mid + \mid v(q)-t(q) \mid)dq \\
&\qquad \leq \left(\int_0^1 k^2(p,q)dq\right)^{\frac{1}{2}} \left(\int_0^1 (\mid u(q)-s(q) \mid + \mid v(q)-t(q) \mid)^2 dq\right)^{\frac{1}{2}} \\
&\qquad \leq \parallel k \parallel_\infty \left(\sup_{p\in[0,1]} \mid u(q)-s(q) \mid + \sup_{p\in[0,1]} \mid v(q)-t(q) \mid\right).
\end{aligned} \tag{3.5}
$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$
\begin{aligned}
&S_1(F(x,y),F(u,v),F(s,t))\\
&\quad\le \max\{\mu,\nu\}\parallel k\parallel_\infty\left(\sup_{p\in[0,1]}\mid x(q)-s(q)\mid+\sup_{p\in[0,1]}\mid y(q)-t(q)\mid\right)\\
&\qquad+\max\{\mu,\nu\}\parallel k\parallel_\infty\left(\sup_{p\in[0,1]}\mid u(q)-s(q)\mid+\sup_{p\in[0,1]}\mid v(q)-t(q)\mid\right)\\
&\quad= \max\{\mu,\nu\}\parallel k\parallel_\infty\left(\sup_{p\in[0,1]}\mid x(q)-s(q)\mid+\sup_{p\in[0,1]}\mid u(q)-s(q)\mid\right)\\
&\qquad+\max\{\mu,\nu\}\parallel k\parallel_\infty\left(\sup_{p\in[0,1]}\mid y(q)-t(q)\mid+\sup_{p\in[0,1]}\mid v(q)-t(q)\mid\right)\\
&\quad=4\max\{\mu,\nu\}\parallel k\parallel_\infty\left(\frac{1}{4}\sup_{p\in[0,1]}\mid x(q)-s(q)\mid+\frac{1}{4}\sup_{p\in[0,1]}\mid u(q)-s(q)\mid\right)\\
&\qquad+4\max\{\mu,\nu\}\parallel k\parallel_\infty\left(\frac{1}{4}\sup_{p\in[0,1]}\mid y(q)-t(q)\mid+\frac{1}{4}\sup_{p\in[0,1]}\mid v(q)-t(q)\mid\right)\\
&\quad=4\max\{\mu,\nu\}\parallel k\parallel_\infty\left(\frac{1}{4}\sup_{p\in[0,1]}\mid gx(q)-gs(q)\mid+\frac{1}{4}\sup_{p\in[0,1]}\mid gu(q)-gs(q)\mid\right)\\
&\qquad+4\max\{\mu,\nu\}\parallel k\parallel_\infty\left(\frac{1}{4}\sup_{p\in[0,1]}\mid gy(q)-gt(q)\mid+\frac{1}{4}\sup_{p\in[0,1]}\mid gv(q)-gt(q)\mid\right)\\
&\quad=4\max\{\mu,\nu\}\parallel k\parallel_\infty S_2(gx,gu,gs)+4\max\{\mu,\nu\}\parallel k\parallel_\infty S_2(gy,gv,gt)\\
&\quad\le\frac{1}{16}S_2(gx,gu,gs)+\frac{1}{16}S_2(gy,gv,gt)\\
&\quad\le\frac{1}{16}S_2(gx,gu,gs)+\frac{1}{16}S_2(gy,gv,gt)+\frac{1}{42}S_2(gx,gu,F(s,t))\\
&\qquad+\frac{1}{42}S_2(gy,gv,F(t,s))+\frac{1}{14}S_2(gx,F(u,v),gs)+\frac{1}{14}S_2(gy,F(v,u),gt)\\
&\qquad+\frac{1}{14}S_2(F(x,y),gu,gs)+\frac{1}{14}S_2(F(y,x),gv,gt)+\frac{1}{42}S_2(gx,F(u,v),F(s,t))\\
&\qquad+\frac{1}{42}S_2(gy,F(v,u),F(t,s))+\frac{1}{14}S_2(F(x,y),gu,F(s,t))+\frac{1}{14}S_2(F(y,x),gv,F(t,s))\\
&\qquad+\frac{1}{14}S_2(F(x,y),F(v,u),gs)+\frac{1}{14}S_2(F(y,x),F(v,u),gt).
\end{aligned}
$$

Then it is obvious that F and g satisfy all the conditions appearing in Theorem 2.1. Consequently, it follows from the result of Theorem 2.1 that F and g have a unique common coupled fixed point (u,u), satisfying $F(u,u)=gu=u$. So (u,u) is the unique solution of integral equation (3.1). □

Example 3.2. Consider the following functional integral equation set:

$$
\begin{cases}
x(p)=\frac{p}{1+\sqrt{p}}+\int_0^1\frac{\sin(q\cdot\pi)}{8+p}\cdot[\frac{e^{-px(q)}}{9}+\frac{\sin p}{10}\cdot\frac{|y|}{1+|y(q)|}]dq;\\
y(p)=\frac{p}{1+\sqrt{p}}+\int_0^1\frac{\sin(q\cdot\pi)}{8+p}\cdot[\frac{e^{-py(q)}}{9}+\frac{\sin p}{10}\cdot\frac{|x|}{1+|x(q)|}]dq,
\end{cases}
\tag{3.6}
$$

where the operators $h:[0,1]\to\mathbb{R}$, $k:[0,1]\times[0,1]\to\mathbb{R}^+$ and $f_1,f_2:[0,1]\times\mathbb{R}\to\mathbb{R}$, defined respectively by

$$
h(p)=\frac{p}{1+\sqrt{p}},k(p,q)=\frac{\sin(q\cdot\pi)}{8+p},f_1(p,x)=\frac{e^{-px}}{9},f_2(p,x)=\frac{\sin p}{10}\cdot\frac{\mid x\mid}{1+\mid x\mid}.
$$

It can be easily seen that $h(p), k(p,q), f_1(p,x), f_2(p,x)$ are continuous functions. Next, since

$$| f_1(p,x) - f_1(p,y) | = \left| \frac{e^{-px}}{9} - \frac{e^{-py}}{9} \right| = \left| \frac{-p \cdot e^{-p\xi}}{9}(x-y) \right| \leq \frac{1}{9} |x-y|,$$

$$\begin{aligned} |f_2(p,x) - f_2(p,y)| &= \left| \frac{\sin p}{10} \cdot \frac{| x |}{1+ | x |} - \frac{\sin p}{10} \cdot \frac{| y |}{1+ | y |} \right| \\ &\leq \frac{1}{10} \left| \frac{| x |}{1+ | x |} - \frac{| y |}{1+ | y |} \right| \\ &= \frac{1}{10} \left| \left(1 - \frac{1}{1+ | x |}\right) - \left(1 - \frac{1}{1+ | y |}\right) \right| \\ &= \frac{1}{10} \left| \frac{1}{1+ | y |} - \frac{1}{1+ | x |} \right| \\ &= \frac{1}{10} \left| -\frac{1}{(1+\varepsilon)^2}(|x| - |y|) \right| \\ &\leq \frac{1}{10} ||x| - |y|| \leq \frac{1}{10}|x-y|, \end{aligned}$$

where $p \in [0,1], \xi$ exist between x and y and ε exist between $|x|$ and $|y|$.

Then we have $\mu = \frac{1}{9}, \nu = \frac{1}{10}, \| k \|_\infty = \sup\{k(p,q) : p,q \in [0,1]\} = \frac{1}{8}$, thus

$$4 \max\{\mu, \nu\} \| k \|_\infty = \frac{1}{18} < \frac{1}{16}.$$

Consequently, all the conditions of Theorem 3.1 are satisfied, Hence the integral equation set (3.6) has a unique solution in $C[0,1]$.

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