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On common best proximity points for generalized $\alpha - \psi$-proximal contractions

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Abstract

We establish some common best proximity point results for generalized $\alpha - \psi$-proximal contractive non-self mappings. We provide some concrete examples. We also derive some consequences on some best proximity results on a metric space endowed with a graph.

Keywords: Common best proximity point, common fixed point, $\alpha - \psi$-proximal contraction.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. Clearly, if $A \cap T(A) = \emptyset$, the fixed point equation $Tx = x$ has no solution. In this case, we have $d(x, Tx) > 0$ for all $x \in A$. Also, $d(A, B) \leq d(x, Tx)$ for all $x \in A$. So, the aim of best proximity theory is to find $x \in A$ such that $d(x, Tx)$ is minimum and so to guarantee the existence a best proximal point of T, named $x \in X$ that is, $d(A, B) = d(x, Tx)$. In 2005, Eldred and Veeramani [4] gave existence and convergence of best proximity points in the setting of a uniformly convex Banach space. Al-Thagafi and Shahzad [1] studied convergence and existence results of best proximity points for cyclic φ-contraction maps. In 2011, Sadiq Basha [17] stated some best proximity point theorems for proximal contractions. For some other results on

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best proximity points, see for example [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 21, 22, 23, 24, 25]. We recall some notations and definitions, which will be used in the sequel. Let A and B be two nonempty subsets of a metric space (X, d).

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : \ a \in A, \ b \in B\}, \\ A_0 &= \{a \in A : d(a,b) = d(A,B), \text{ for some } b \in B\}, \\ B_0 &= \{b \in B : d(a,b) = d(A,B), \text{ for some } a \in A\}. \end{aligned}$$

Definition 1.1. Let (X, d) be a metric space. Consider A and B two nonempty subsets of X. An element $a \in X$ is said to be a common best proximity point of the mappings $S, T : A \to B$ if

$$d(a, Sa) = d(a, Ta) = d(A, B).$$

It is clear that a common fixed point coincides with a common best proximity point if $d(A, B) = 0$. In 2013, Zhang et al. [26] introduced the concept of a weak (P)-property.

Definition 1.2 ([26])**.** Let A and B be nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have the weak (P)-property if and only if

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

In 2012, Samet et al. [20] are the first who introduced the concept of α-admissible mappings. This nice concept was generalized and extended in many directions. Now, as in [7], we introduce the concept of an α-proximal admissible pair of non-self mappings.

Definition 1.3. Let A and B be nonempty subsets of a metric space (X, d) and $\alpha : X \times X \to [0, \infty)$. A pair of non-self mappings $S, T : A \to B$ is named α-proximal admissible if

$$\begin{cases} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Sx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Rightarrow \min\{\alpha(u_1, u_2), \alpha(u_2, u_1)\} \geq 1$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $d(A, B) = 0$, the pair (S, T) is α-proximal admissible implies that the pair (S, T) is α-admissible [2]. Now, let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:
$(\psi_1)\psi$ is nondecreasing,
$(\psi_2) \sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \geq 0$, where ψ^n is the nth iterate of ψ.

Clearly, if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. In the following, we give some generalized α-proximal contractions.

Definition 1.4. Let A and B two nonempty subsets of a metric space (X, d). Take $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$. Consider a pair of non-self mappings $S, T : A \to B$.
i) (S, T) is called a generalized $\alpha - \psi$-proximal contraction pair if

$$d(Sx, Ty) \leq \psi(M(x, y)) \tag{1.1}$$

for all $x, y \in A$ satisfying $\alpha(x, y) \geq 1$, where

$$M(x,y) = \max\{d(x,y), \frac{d(x,Sx)+d(y,Ty)-2d(A,B)}{2}, \frac{d(y,Sx)+d(x,Ty)-2d(A,B)}{2}\}. \tag{1.2}$$

ii) (S, T) is called a generalized $\alpha - \psi$-proximal contraction pair of the first kind if

$$\begin{cases} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Sx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{cases} \Rightarrow d(u_1, u_2) \leq \psi(M(x_1, x_2)),$$

where $x_1, x_2, u_1, u_2 \in A$.

In this paper, we establish some existence results on common best proximity points for $\alpha - \psi$-proximal contractive pairs of non-self mappings. We will support the obtained theorems by some concrete examples. Some corollaries and consequences are also provided.

2. Main results

The first main result is

Theorem 2.1. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Let $S, T : A \to B$ be a generalized $\alpha - \psi$-proximal contraction pair. Assume that*

(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;
(ii) (S, T) is an α-proximal admissible pair;
(iii) there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Sx_0) = d(A,B) \quad \textit{and} \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1;$$

(iv) S and T are continuous.

Then there exists $u \in A$ such that $d(u, Su) = d(u, Tu) = d(A, B)$, that is, u is a common best proximity point of S and T.

Proof. By assumption (iii), there exist x_0 and $x_1 \in A_0$ such that

$$d(x_1, Sx_0) = d(A,B) \quad \text{and} \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1. \tag{2.1}$$

From condition (i), we have $Tx_1 \in B_0$, so there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A,B). \tag{2.2}$$

By (2.1), (2.2) and from the fact that (S, T) is an α-proximal admissible pair

$$\min\{\alpha(x_1, x_2), \alpha(x_2, x_1)\} \geq 1.$$

Again, from condition (i), we have $Sx_2 \in B_0$, so there exists $x_3 \in A_0$ such that

$$d(x_3, Sx_2) = d(A,B).$$

Similarly, we have

$$\min\{\alpha(x_2, x_3), \alpha(x_3, x_2)\} \geq 1.$$

Repeating the above strategy, by induction, we construct a sequence $\{x_n\}$ in A_0 such that

$$\min\{\alpha(x_n, x_{n+1}), \alpha(x_{n+1}, x_n)\} \geq 1 \quad \text{for all } n \geq 0 \tag{2.3}$$

and

$$d(x_{2n+1}, Sx_{2n}) = d(x_{2n+2}, Tx_{2n+1}) = d(A, B) \quad \text{for all } n \geq 0. \tag{2.4}$$

From condition (i), the pair (A, B) satisfies the weak (P)-property, so

$$d(x_{2n+1}, x_{2n+2}) \leq d(Sx_{2n}, Tx_{2n+1}) \quad \text{for all } n \geq 0. \tag{2.5}$$

Similarly,

$$d(x_{2n}, x_{2n+1}) \leq d(Sx_{2n}, Tx_{2n-1}) \quad \text{for all } n \geq 1.$$

The pair (S, T) is a generalized $\alpha - \psi$-proximal contraction, so for all $n \geq 1$, using (2.3), ($\alpha(x_{2n}, x_{2n+1}) \geq 1$), (2.5) and (1.1),

$$d(x_{2n+1}, x_{2n+2}) \leq d(Sx_{2n}, Tx_{2n+1}) \leq \psi(M(x_{2n}, x_{2n+1})),$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) = \max\{&d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1}) - 2d(A, B)}{2}, \\ &\frac{d(x_{2n+1}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1}) - 2d(A, B)}{2}\}. \end{aligned}$$

By a triangular inequality, using (2.4), we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &\leq \max\{d(x_{2n}, x_{2n+1}), \\ &\quad \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Sx_{2n}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, Tx_{2n+1}) - 2d(A, B)}{2}, \\ &\quad \frac{1}{2}(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))\}, \\ &= \max\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\}, \\ &\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Taking in consideration that ψ is a nondecreasing function, we get

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}). \tag{2.6}$$

A similar reasoning shows that

$$d(x_{2n}, x_{2n+1}) \leq \psi(\max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\}).$$

If $d(x_{2n_0}, x_{2n_0+1}) = 0$ for some n_0, then from (2.6),

$$d(x_{2n_0+1}, x_{2n_0+2}) \leq \psi(\max\{0, d(x_{2n+1}, x_{2n+2})\}) = \psi(d(x_{2n_0+1}, x_{2n_0+2})),$$

which necessarily yields that $d(x_{2n_0+1}, x_{2n_0+2}) = 0$. So, we have $x_{2n_0} = x_{2n_0+1} = x_{2n_0+2}$ and from (2.4), we obtain $d(x_{2n_0}, Sx_{2n_0}) = d(x_{2n_0}, Tx_{2n_0}) = d(A, B)$, that is x_{2n_0} is a common best proximity point of S and T. Similarly, if $d(x_{2n_0+1}, x_{2n_0+2}) = 0$ for some n_0, then we get that x_{2n_0+1} is a common best proximity point of S and T and the proof is completed.

Now, we suppose that

$$d(x_n, x_{n+1}) > 0, \quad \text{for all } n \geq 0.$$

Suppose that $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$ for some n, then from (2.6) and since $\psi(t) < t$ for all $t > 0$, we obtain

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction. Then, we have

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1})), \quad \text{for all } n \geq 0.$$

Similarly,

$$d(x_{2n}, x_{2n+1}) \leq \psi(d(x_{2n-1}, x_{2n})), \quad \text{for all } n \geq 1.$$

We deduce

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \quad \text{for all } n \geq 1.$$

Therefore,

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \quad \text{for all } n \geq 0.$$

Then, for all $k \geq 0$

$$d(x_n, x_{n+k}) \leq \sum_{m=n}^{n+k-1} d(x_m, x_{m+1}) \leq \sum_{m=n}^{\infty} d(x_m, x_{m+1}) \leq \sum_{m=n}^{\infty} \psi^m(d(x_0, x_1)).$$

Since $\psi \in \Psi$, it follows that $\{x_n\}$ is a Cauchy sequence in A, which is a closed subset of the complete metric space (X, d), then there exists $u \in A$ such that $x_n \to u$ as $n \to \infty$. The mapping S is continuous at u, so $\lim_{n\to\infty} Sx_{2n} = Su$. Moreover the continuity of the metric function d implies that $\lim_{n\to\infty} d(x_{2n+1}, Sx_{2n}) = d(u, Su)$. Then, by (2.4), we obtain $d(A, B) = d(u, Su)$. Similarly, by the continuity of T we have $d(A, B) = d(u, Tu)$. So, u is a common best proximity point of S and T.

□

In the next result, we replace the continuity hypothesis by the following condition on A.

(H) If$\{x_n\}$ is a sequence in A such that $\min\{\alpha(x_n, x_{n+1}), \alpha(x_{n+1}, x_n)\} \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\min\{\alpha(u, x_{n(k)}), \alpha(x_{n(k)}, u)\} \geq 1$ for all k.

Theorem 2.2. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Let $S, T : A \to B$ be a generalized $\alpha - \psi$-proximal contraction pair. Assume that*

(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;
(ii) (S, T) is an α-proximal admissible pair;
(iii) there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Sx_0) = d(A, B) \quad \text{and} \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1;$$

(iv) (H) holds.

Then, there exists $u \in A$ such that $d(u, Su) = d(u, Tu) = d(A, B)$.

Proof. Following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ in A_0 such that (2.3) and (2.4) hold. Also, $\{x_n\}$ is a Cauchy sequence in the subset A, which is closed in the complete metric space (X, d), then there exists $u \in A$ such that $x_n \to u$ as $n \to \infty$. By hypothesis (H), we have $\alpha(u, x_{2n(k)+1}) \geq 1$.

On the other hand, by a triangular inequality, using (1.1) and (2.4), we get

$$\begin{aligned} d(u, Su) &\leq d(u, x_{2n(k)+2}) + d(x_{2n(k)+2}, Tx_{2n(k)+1}) + d(Tx_{2n(k)+1}, Su) \\ &\leq d(u, x_{2n(k)+2}) + d(A, B) + \psi(M(u, x_{2n(k)+1})), \end{aligned}$$

where

$$\begin{aligned} M(u, x_{2n(k)+1}) = \max\{&d(u, x_{2n(k)+1}), \frac{d(u, Su) + d(x_{2n(k)+1}, Tx_{2n(k)+1}) - 2d(A, B)}{2}, \\ &\frac{d(x_{2n(k)+1}, Su) + d(u, Tx_{2n(k)+1}) - 2d(A, B)}{2}\}. \end{aligned}$$

Again, by a triangular inequality, using (2.4), we have

$$M(u,x_{2n(k)+1}) \leq \max\{d(u,x_{2n(k)+1}), \frac{d(u,Su)+d(x_{2n(k)+1},x_{2n(k)+2})-d(A,B)}{2}, \frac{d(x_{2n(k)+1},u)+d(u,Su)+d(u,x_{2n(k)+2})-d(A,B)}{2}\}.$$

We get $d(u,Su)-d(A,B)\geq 0$. Suppose that $d(u,Su)-d(A,B)>0$. We know that

$$\lim_{k\to\infty} d(u,x_{2n(k)+1}) = \lim_{k\to\infty} d(x_{2n(k)+1},x_{2n(k)+2}) = \lim_{k\to\infty} d(u,x_{2n(k)+2}) = 0.$$

Then, there exists $N\in\mathbb{N}$ such that for all $k\geq N$,

$$M(u,x_{2n(k)+1}) \leq \frac{3}{4}[d(u,Su)-d(A,B)]$$

ψ in a nondecreasing function, so we obtain for all $k\geq N$,

$$d(u,Su)-d(A,B) \leq d(u,x_{2n(k)+2}) + \psi(\frac{3}{4}[d(u,Su)-d(A,B)]).$$

Having $\psi(t)<t$ for all $t>0$, then letting $k\to\infty$, we get

$$d(u,Su)-d(A,B) \leq \psi(\frac{3}{4}[d(u,Su)-d(A,B)]) < \frac{3}{4}[d(u,Su)-d(A,B)],$$

which is a contradiction. Hence, we find that $d(u,Su)-d(A,B)=0$, that is, $d(u,Su)=d(A,B)$. By a similar reasoning, we find that $d(u,Tu)=d(A,B)$. Thus, u is a common best proximity point of S and T. □

Now, we prove the uniqueness of such common best proximity point. Here, we need the following additional condition.

(U): For all $x,y\in CB(S,T)$, we have $\alpha(x,y)\geq 1$, where $CB(S,T)$ denotes the set of common best proximity points of S and T.

Theorem 2.3. *Adding condition* (U) *to the hypotheses of* Theorem 2.1 (resp. Theorem 2.2), *we obtain that* u *is the unique common best proximity point of* S *and* T.

Proof. We argue by contradiction, that is, there exist $u,v\in A$ such that $d(A,B)=d(u,Su)=d(u,Tu)=d(v,Sv)=d(v,Tv)$ with $u\neq v$. By assumption (U), we have $\alpha(u,v)\geq 1$. So, as the pair (A,B) satisfies the weak (P)-property, by (1.1), we have

$$0<d(u,v)\leq d(Su,Tv)\leq \psi(M(u,v)), \tag{2.7}$$

where

$$\begin{aligned} M(u,v) &= \max\{d(u,v), \frac{d(u,Su)+d(v,Tv)-2d(A,B)}{2}, \frac{d(v,Su)+d(u,Tv)-2d(A,B)}{2}\} \\ &= \max\{d(u,v), 0, \frac{d(v,Su)+d(u,Tv)-2d(A,B)}{2}\}. \end{aligned}$$

By a triangular inequality, we have

$$\begin{aligned} M(u,v) &\leq \max\{d(u,v), \frac{d(v,u)+d(u,Su)+d(u,v)+d(v,Tv)-d(A,B)}{2}\} \\ &= \max\{d(u,v), d(u,v)\} = d(u,v). \end{aligned} \tag{2.8}$$

Using (2.7), (2.8) and the fact that ψ in a nondecreasing function together with the property $\psi(t)<t$ for all $t>0$,

$$0<d(u,v)\leq \psi(d(u,v)) < d(u,v),$$

which is a contradiction. Hence, $u=v$. □

The following example illustrates Theorem 2.1.

Example 2.4. Let $X = [0, \infty) \times [0, \infty)$ be endowed with the metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Take $A = \{1\} \times [0, \infty)$ and $B = \{0\} \times [0, \infty)$. Remark that $d(A, B) = d((1, 0), (0, 0)) = 1$. Also, $A_0 = A$ and $B_0 = B$. Moreover, A is closed. Consider the mappings $S, T : A \to B$ as

$$S(1, x) = \begin{cases} (0, \frac{x}{2}) & \text{if } 0 \le x \le 1 \\ (0, x - \frac{1}{2}) & \text{if } x > 1, \end{cases} \quad \text{and} \quad T(1, x) = \begin{cases} (0, \frac{x}{2}) & \text{if } 0 \le x \le 1 \\ (0, 2x - \frac{3}{2}) & \text{if } x > 1. \end{cases}$$

We have $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$. Now, let $(1, x_1), (1, x_2) \in A$ and $(0, u_1), (0, u_2) \in B$ such that

$$\begin{cases} d((1, x_1), (0, u_1)) = d(A, B) = 1 \\ d((1, x_2), (0, u_2)) = d(A, B) = 1. \end{cases}$$

Necessarily, $(x_1 = u_1 \in [0, 1])$ and $(x_2 = u_2 \in [0, 1])$. In this case,

$$d((1, x_1), (1, x_2)) = d((0, u_1), (0, u_2)),$$

that is, the pair (A, B) has the weak (P)-property.

Take $\psi(t) = \frac{1}{2}t$ for all $t \ge 0$. Define $\alpha : X \times X \to [0, \infty)$ as follows

$$\begin{cases} \alpha((x, y), (s, t)) = 1 & \text{if} \quad (x, y), (s, t) \in [0, 1] \times [0, 1] \\ \alpha((x, y), (s, t)) = 0 & \text{if not.} \end{cases}$$

Let $(1, x_1), (1, x_2), (1, u_1)$ and $(1, u_2)$ in A such that

$$\begin{cases} \alpha((1, x_1), (1, x_2)) \ge 1 \\ d((1, u_1), S(1, x_1)) = d(A, B) = 1 \\ d((1, u_2), T(1, x_2)) = d(A, B) = 1. \end{cases}$$

Then, necessarily, $(x_1, x_2) \in [0, 1] \times [0, 1]$. We also have $(u_1 = \frac{x_1}{2}$ and $u_2 = \frac{x_2}{2})$. So

$$\min\{\alpha((1, u_1), (1, u_2)), \alpha((1, u_2), (1, u_1))\} \ge 1,$$

that is, (S, T) is α-proximal admissible pair.

Let $(1, x)$ and $(1, y) \in A$ such that $\alpha((1, x), (1, y)) \ge 1$. Then, $x, y \in [0, 1]$. In this case, we have

$$\begin{aligned} d(S(1, x), T(1, y)) &= d((0, \frac{x}{2}), (0, \frac{y}{2})) \\ &= |\frac{x}{2} - \frac{y}{2}| = \psi(d((1, x), (1, y))) \le \psi(M((1, x), (1, y))). \end{aligned}$$

We deduce that (1.1) holds. Furthermore, S and T are continuous. Moreover, the condition (iii) of Theorem 2.1 is verified. Indeed, for $x_0 = (1, 1)$ and $x_1 = (1, \frac{1}{2})$, we have

$$d(x_1, Sx_0) = d((1, \frac{1}{2}), (0, \frac{1}{2})) = 1 = d(A, B) \quad \text{and} \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \ge 1.$$

Hence, all hypotheses of Theorem 2.1 are verified. So, the pair (S, T) admits a common best proximity point which is $u = (1, 0)$. It is also unique.

Theorem 2.5. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Let $S, T : A \to B$ be a generalized $\alpha - \psi$-proximal contraction pair of the first kind. Assume that*

(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$;

(ii) (S,T) *is an* α*-proximal admissible pair;*

(iii) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Sx_0) = d(A,B) \quad \textit{and} \quad \min\{\alpha(x_0,x_1), \alpha(x_1,x_0)\} \geq 1;$$

(iv) S *and* T *are continuous.*

Then, there exists $u \in A$ *such that* $d(u,Su) = d(u,Tu) = d(A,B)$.

Proof. Following the proof of Theorem 2.1, we construct a sequence $\{x_n\}$ in A_0 such that

$$\min\{\alpha(x_n,x_{n+1}), \alpha(x_{n+1},x_n)\} \geq 1 \quad \text{for all } n \geq 0 \tag{2.9}$$

and

$$d(x_{2n+1}, Sx_{2n}) = d(x_{2n+2}, Tx_{2n+1}) = d(A,B) \quad \text{for all } n \geq 0. \tag{2.10}$$

Since (S,T) is a generalized $\alpha - \psi$-proximal contraction pair of the first kind, from (2.9), (2.10) and (2.1), we have

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(M(x_{2n}, x_{2n+1}))$$

and

$$d(x_{2n+1}, x_{2n}) \leq \psi(M(x_{2n}, x_{2n-1})).$$

Then, following the proof of Theorem 2.1, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n) \quad \text{for all } n \geq 0.$$

Consequently, $\{x_n\}$ is a Cauchy sequence in the subset A, which is closed in the complete metric space (X,d), then there exists $u \in A$ such that $x_n \to u$ as $n \to \infty$. Also, using the continuity of S and T, we get that $d(u,Su) = d(u,Tu) = d(A,B)$. Thus, u is a common best proximity point of S and T.

□

Theorem 2.6. *Adding condition* (U) *to the hypotheses of* Theorem 2.5*, we obtain that* u *is the unique common best proximity point of* S *and* T.

Proof. We argue by contradiction, that is, there exist $u, v \in A$ such that $d(A,B) = d(u,Su) = d(u,Tu) = d(v,Sv) = d(v,Tv)$ with $u \neq v$. By assumption (U), we have $\alpha(u,v) \geq 1$. So, as the pair (S,T) is a generalized $\alpha - \psi$-proximal contraction of the first kind, then

$$0 < d(u,v) \leq \psi(M(u,v)) = \psi(d(u,v)) < d(u,v),$$

which is a contradiction. Hence, $u = v$. □

We provide the following example.

Example 2.7. Let $X = [0,\infty) \times [0,\infty)$ be endowed with the metric $d((x_1,x_2),(y_1,y_2)) = |x_1-y_1|+|x_2-y_2|$. Take $A = \{1\} \times [0,\infty)$ and $B = \{0\} \times [0,\infty)$. We mention that $d(A,B) = 1$, $A_0 = A$ and $B_0 = B$. Consider the mappings $S,T : A \to B$ as

$$S(1,x) = \begin{cases} (0, \frac{x^2+1}{4}) & \text{if } 0 \leq x \leq 1 \\ (0, x - \frac{1}{2}) & \text{if } x > 1, \end{cases} \quad \text{and} \quad T(1,x) = \begin{cases} (0, \frac{x^2+1}{4}) & \text{if } 0 \leq x \leq 1 \\ (0, 2x - \frac{3}{2}) & \text{if } x > 1. \end{cases}$$

We have $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$.

Take $\psi(t) = \frac{1}{2}t$ for all $t \geq 0$. Define $\alpha : X \times X \to [0, \infty)$ as follows

$$\begin{cases} \alpha((x,y),(s,t)) = 1 & \text{if} \quad (x,y),(s,t) \in [0,1] \times [0,1] \\ \alpha((x,y),(s,t)) = 0 & \text{if not.} \end{cases}$$

Let $(1, x_1), (1, x_2), (1, u_1)$ and $(1, u_2)$ in A such that

$$\begin{cases} \alpha((1,x_1),(1,x_2)) \geq 1 \\ d((1,u_1), S(1,x_1)) = d(A,B) = 1 \\ d((1,u_2), T(1,x_2)) = d(A,B) = 1. \end{cases}$$

Then, necessarily, $(x_1, x_2) \in [0,1] \times [0,1]$. Also, we have ($u_1 = \frac{1+x_1^2}{4}$ and $u_2 = \frac{1+x_2^2}{4}$). So

$$\min\{\alpha((1,u_1),(1,u_2)), \alpha((1,u_2),(1,u_1))\} \geq 1$$

that is, (S, T) is an α-proximal admissible pair.

Moreover,

$$\begin{aligned} d((1,u_1),(1,u_2)) &= d((1, \frac{1+x_1^2}{4}), (1, \frac{1+x_2^2}{4})) \\ &= |\frac{1+x_1^2}{4} - \frac{1+x_2^2}{4}| = |\frac{x_1^2}{4} - \frac{x_2^2}{4}| = \frac{1}{4}(x_1 + x_2)|x_1 - x_2| \\ &\leq \frac{1}{2}|x_1 - x_2| = \psi(d((1,x_1),(1,x_2))) \leq \psi(M((1,x_1),(1,x_2))). \end{aligned}$$

Then, (S, T) is a generalized $\alpha - \psi$-proximal contraction pair of the first kind. Furthermore, S and T are continuous. Moreover, the condition (iii) of Theorem 2.1 is verified. Indeed, for $x_0 = (1, 1)$ and $x_1 = (1, \frac{1}{2})$, we have

$$d(x_1, Sx_0) = d((1, \frac{1}{2}), (0, \frac{1}{2})) = 1 = d(A,B) \quad \text{and} \quad \min\{\alpha(x_0,x_1), \alpha(x_1,x_0)\} \geq 1.$$

Hence, all hypotheses of Theorem 2.5 are verified. So, the pair (S, T) admits a common best proximity point which is $u = (1, 2 - \sqrt{3})$. It is also unique.

3. Consequences

In this paragraph, we present some consequences of our obtained results.

3.1. Some classical best proximity point results

We have the following natural results.

Corollary 3.1. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Let $S, T : A \to B$ and $\alpha : X \times X \to [0, \infty)$ be given non-self mappings such that*

$$\alpha(x,y) d(Sx, Ty) \leq \psi(M(x,y)),$$

for all $x, y \in A$, where $\psi \in \Psi$ and $M(x, y)$ is defined by (1.2).

Also, assume that

(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property;
(ii) (S, T) is an α-proximal admissible pair;
(iii) there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Sx_0) = d(A,B) \quad \textit{and} \quad \min\{\alpha(x_0,x_1), \alpha(x_1,x_0)\} \geq 1;$$

(iv) S *and* T *are continuous or* (H) *holds.*

Then, there exists $u \in A$ *such that* $d(u, Su) = d(u, Tu) = d(A, B)$.

Corollary 3.2. *Let* A *and* B *be nonempty subsets of a complete metric space* (X, d) *such that* $A_0 \neq \emptyset$ *and* A *is closed. Let* $S, T : A \to B$ *and* $\alpha : X \times X \to [0, \infty)$ *be given non-self mappings. Suppose there exists* $k \in [0, 1)$ *such that*

$$d(Sx, Ty) \leq kM(x, y),$$

for all $x, y \in A$, *satisfying* $\alpha(x, y) \geq 1$, *where* $M(x, y)$ *is defined by* (1.2). *Also, assume that*

(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the weak* (P)*-property;*
(ii) (S, T) *is an* α*-proximal admissible pair;*
(iii) there exist elements x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Sx_0) = d(A, B) \quad \text{and} \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1;$$

(iv) S *and* T *are continuous or* (H) *holds.*

Then, there exists $u \in A$ *such that* $d(u, Su) = d(u, Tu) = d(A, B)$.

Proof. It suffices to take $\psi(t) = kt$ in Theorem 2.1 (resp. Theorem 2.2). □

Corollary 3.3. *Let* A *and* B *be nonempty subsets of a complete metric space* (X, d) *such that* $A_0 \neq \emptyset$ *and* A *is closed. Let* $S, T : A \to B$ *and* $\alpha : X \times X \to [0, \infty)$ *be given non-self mappings. Suppose there exists* $k \in [0, 1)$ *such that*

$$d(Sx, Ty) \leq kd(x, y),$$

for all $x, y \in A$. *Also, assume that*

(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the weak* (P)*-property;*
(ii) (S, T) *is an* α*-proximal admissible pair;*
(iii) there exist elements x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Sx_0) = d(A, B) \quad \text{and} \quad \min\{\alpha(x_0, x_1), \alpha(x_1, x_0)\} \geq 1;$$

(iv) S *and* T *are continuous or* (H) *holds.*

Then, there exists $u \in A$ *such that* $d(u, Su) = d(u, Tu) = d(A, B)$.

In the case $A = B$, we have the following common fixed point result.

Corollary 3.4. *Let* A *be nonempty closed subset of a complete metric space* (X, d). *Let* $S, T : A \to A$ *be a generalized* $\alpha - \psi$*-proximal contraction pair. Assume that*

(i) (S, T) *is an* α*-proximal admissible pair;*
(ii) there exists x_0 *in* A_0 *such that*

$$\min\{\alpha(x_0, Sx_0), \alpha(Sx_0, x_0)\} \geq 1;$$

(iii) S *and* T *are continuous or* (H) *holds.*

Then, the pair (S, T) *admits a common fixed point.*

Corollary 3.5. *Let* A *and* B *be nonempty subsets of a complete metric space* (X, d) *such that* $A_0 \neq \emptyset$ *and* A *is closed. Take* $\psi \in \Psi$. *Let* $S, T : A \to B$ *be given non-self mappings such that*

$$d(Sx, Ty) \leq \psi(M(x, y))$$

for all $x, y \in A$, *where* $M(x, y)$ *is defined by* (1.2).

Assume that $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property. Then, the pair (S, T) admits a unique common best proximity point.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.2. □

Corollary 3.6. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Take $\psi \in \Psi$. Let $T : A \to B$ be a given non-self mapping such that*

$$d(Tx, Ty) \leq \psi(M(x, y))$$

for all $x, y \in A$, where $M(x, y)$ is defined by (1.2).

Assume that $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak (P)-property. Then, T has a unique best proximity point.

Corollary 3.7. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Let $S, T : A \to B$ be given continuous non-self mappings such that*

$$\begin{cases} d(u_1, Sx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Rightarrow d(u_1, u_2) \leq \psi(M(x_1, x_2)),$$

where $x_1, x_2, u_1, u_2 \in A$, $M(x, y)$ is defined by (1.2) and $\psi \in \Psi$. Assume that $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$. Then, there exists $u \in A$ such that $d(u, Su) = d(u, Tu) = d(A, B)$.

Proof. It suffices to take $\alpha(x, y) = 1$ in Theorem 2.5. □

Corollary 3.8. *Let A and B be nonempty subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and A is closed. Let $T : A \to B$ be a given continuous non-self mapping such that*

$$\begin{cases} d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Rightarrow d(u_1, u_2) \leq \psi(M(x_1, x_2)),$$

where $x_1, x_2, u_1, u_2 \in A$, $M(x, y)$ is defined by (1.2) and $\psi \in \Psi$. Assume that $T(A_0) \subseteq B_0$. Then, T has a best proximity point.

3.2. Some best proximity results on a metric space endowed with a graph

Let (X, d) be a metric space and $\Delta := \{(x, x) : x \in X\}$ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subset E(G)$, where $E(G)$ is the set of edges of the graph. Assume also that G has no parallel edges, and thus one can identify G with the pair $(V(G), E(G))$.

We need in the sequel the following hypothesis.

(H_G) If $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x), (x, x_{n(k)}) \in E(G)$ for all k.

Again, we introduce the following definition.

Definition 3.9. Let A and B be nonempty subsets of a metric space (X, d) endowed with a graph G. $S, T : A \to B$ is named a G-proximal pair of mappings if

$$\begin{cases} (x_1, x_2) \in E(G) \\ d(u_1, Sx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{cases} \Rightarrow (u_1, u_2), (u_2, u_1) \in E(G),$$

where $x_1, x_2, u_1, u_2 \in A$.

We have two best proximity point results on a metric space endowed with a graph.

Corollary 3.10. *Let A and B be nonempty subsets of a complete metric space (X,d) endowed with a graph G such that $A_0 \neq \emptyset$ and A is closed. Let $S,T : A \to B$ be given non-self mappings such that*

$$d(Sx,Ty) \leq \psi(M(x,y)) \tag{3.1}$$

for all $x,y \in A$ such that $(x,y) \in E(G)$, where $\psi \in \Psi$ and $M(x,y)$ is defined by (1.2). Suppose that

- *(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$ and (A,B) satisfies the weak (P)-property;*
- *(ii) (S,T) is a G-proximal pair;*
- *(iii) there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1,Sx_0) = d(A,B) \quad and \quad (x_0,x_1),(x_1,x_0) \in E(G);$$

- *(iv) S and T are continuous or (H_G) holds.*

Then, there exists $u \in A$ such that

$$d(u,Su) = d(u,Tu) = d(A,B).$$

Proof. It suffices to consider $\alpha : X \times X \to [0,\infty)$ such that

$$\alpha(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in E(G) \\ 0 & \text{if not.} \end{cases}$$

All hypotheses of Theorem 2.1 (resp. Theorem 2.2) are satisfied. This completes the proof. □

Similar to Corollary 3.10, we may state the following result.

Corollary 3.11. *Let A and B be nonempty subsets of a complete metric space (X,d) endowed with a graph G such that $A_0 \neq \emptyset$ and A is closed. Let $S,T : A \to B$ be a given non-self mappings such that*

$$\begin{cases} d(u_1,Sx_1) = d(A,B) \\ d(u_2,Tx_2) = d(A,B) \end{cases} \Rightarrow d(u_1,u_2) \leq \psi(M(x_1,x_2)),$$

where $x_1,x_2,u_1,u_2 \in A$ such that $(x_1,x_2) \in E(G)$, where $\psi \in \Psi$ and $M(x,y)$ is defined by (1.2). Suppose that

- *(i) $S(A_0) \subseteq B_0$, $T(A_0) \subseteq B_0$;*
- *(ii) (S,T) is a G-proximal pair;*
- *(iii) there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1,Sx_0) = d(A,B) \quad and \quad (x_0,x_1),(x_1,x_0) \in E(G);$$

- *(iv) S and T are continuous.*

Then, there exists $u \in A$ such that

$$d(u,Su) = d(u,Tu) = d(A,B).$$

3.3. More consequences

It is easy to see that more consequences can be derived from our results by taking:

- (a) $S = T$;
- (b) $A = B$;
- (c) $\alpha(x,y)$ in a proper way like in [9];
- (d) $\psi(t)$ in a proper way like in [9];
- (e) $M(x,y)$ in a proper way.

The case of (a) is clear. Notice that in case of (b), we derive several existing fixed point theorems. For example, by using (c) together with (b), we conclude a number of existing fixed point results in the frame of cyclic mappings as well as fixed point results in the context of partially ordered metric spaces. It is worth mentioning that (d) implies several famous results by choosing $\psi(t)$ in a proper way. The number of elements in $M(x, y)$ can be restricted and each restriction gives another corollary.

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