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Calculations on topological degrees of semi-closed 1-set-contractive operators in M-PN-spaces and applications

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Abstract

The aim of the paper is to study some calculating problems of topological degrees of semi-closed 1-set-contractive operators in M-PN-spaces. Under some weak and natural conditions, several calculation results are obtained. Finally, in order to verify the validity of our results, a support example is given at the end of the paper.

Keywords: Topological degree, M-PN-space, semi-closed 1-set-contractive operator, fixed point.
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1. Introduction

Since Menger in 1942 introduced the concept of probabilistic metric spaces, many results based on probabilistic metric spaces have been gotten. For example, Schweizer and Sklar [8] described detailedly the topological structure of probabilistic metric space; the authors in [9] summarized the current development of probabilistic metric spaces; and in 2001, the authors in [2] established the topological degree theory of completely continuous operators in M-PN-spaces and obtained some important properties as well as some fixed point theorems under the condition that t-norm $\triangle$ satisfies $\triangle(t, t) \geq t$ for all $t \in [0, 1]$. The topological degrees of k-set-contractive operators, condensing operators and the A-proper degree in M-PN-spaces were widely studied (see [4, 7, 12] and the references therein). However, the topological degree for 1-set-contractive

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field in PN-spaces had not been discussed before 2006. In [11], the authors filled that gap. Of course, all the established topological degree theories provide an important theoretical tool for the future investigations of some nonlinear problems, especially for the existence of fixed points of nonlinear operators on M-PN-spaces (see [6, 13]). And since then, the calculation problems of topological degrees about nonlinear operators on M-PN-spaces have naturally become one of the main considerable topics in this field. This work aims to deal with such a topic. Under some weak and natural conditions, several calculation problems on topological degrees of semi-closed 1-set contractive operators on M-PN-spaces are studied and some new results are obtained. A part of the presented results generalize some known conclusions. Before introducing the main results in this paper, let us recall some basic concepts.

Denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ the sets of real numbers, non-negative real numbers and positive integers, respectively. A function $f : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous and satisfies the following conditions:

$$\inf_{t\in\mathbb{R}} f(t) = 0, \quad \sup_{t\in\mathbb{R}} f(t) = 1.$$

Use $\mathbb{D}$ to denote the collection of distribution functions. Define a specific distribution function $H(t)$ as follows:

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \le 0. \end{cases}$$

A mapping $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (a t-norm for short) if for any $a, b, c, d \in [0, 1]$, the following conditions are satisfied:

(1) $\triangle(a, 1) = a$;

(2) $\triangle(a, b) = \triangle(b, a)$;

(3) $a \ge b,\ c \ge d \Rightarrow \triangle(a, c) \ge \triangle(b, d)$;

(4) $(\triangle(a, \triangle(b, c)) = \triangle(\triangle(a, b), c)$.

Next, we review the conception of Menger PN spaces.

Definition 1.1 ([10])**.** An Menger probabilistic normed linear space (Menger PN space or M-PN-space in brief) is an ordered triple $(E, \mathcal{F}, \triangle)$, where E is a real normed linear space (the zero element of E is denoted by θ), $\triangle$ is a t-norm, and $\mathcal{F}$ is a mapping from E into $\mathbb{D}$ (in the following, we replace $F(x)$ with f_x, and use $f_x(t)$ to denote the value of the distribution function f_x at $t \in \mathbb{R}$), and f_x satisfies the following conditions:

(PN-1) $f_x(0) = 0$, $\forall x \in E$;

(PN-2) $f_x(t) = H(t)$, $\forall t \in \mathbb{R}$ if and only if $x = \theta$;

(PN-3) for any $\alpha \in\mathbb{R}$ and $\alpha \neq 0$, we have $f_{\alpha x}(t) = f_x(\frac{t}{|\alpha|})$;

(PN-4) for any $x, y \in E$ and all $t_1, t_2 \in \mathbb{R}^+$, we have $f_{x+y}(t_1 + t_2) \ge \triangle(f_x(t_1), f_y(t_2))$.

For convenience, the conception of probabilistically bounded set in the following definition is needed.

Definition 1.2 ([3])**.** Let $(E, \mathcal{F})$ be a PN-space and A a nonempty subset of E. Then the function

$$D_A(t) = \sup_{s<t} \inf_{x,y\in A} F_{x-y}(s),\ t \in \mathbb{R}$$

is called the probabilistic diameter of A. If $\sup_{t>0} D_A(t) = 1$, then A is said to be probabilistically bounded; if $0 < \sup_{t>0} D_A(t) < 1$, then A is said to be probabilistically semi-bounded; if $\sup_{t>0} D_A(t) = 0$, then A is said to be probabilistically unbounded.

Based on the notion of probabilistically bounded set, Bocşan and Constantin [1] introduced the concept of Kuratowski's function which is a technical tool for others to establish the topological degree of k-set contractive operator in M-PN-spaces.

Definition 1.3 ([1])**.** Let $(E, \mathcal{F}, \triangle)$ be an M-PN-space. A Kuratowski's function for a probabilistically bounded subset A of E is the function α_A on $\mathbb{R}$ defined by

$$\alpha_A(t) = \sup\{\varepsilon : \varepsilon > 0, \text{ there is a finite cover of } \mathcal{A} \text{ such that } D_B(t) \geq \varepsilon \text{ for all } B \in \mathcal{A}\}.$$

Let p be a point in an M-PN-space $(E, \mathcal{F}, \triangle)$. An (ε, λ)-neighborhood of p with $\varepsilon > 0$ and $\lambda > 0$ is $N_p(\varepsilon, \lambda) = \{q \in E : F_{p-q}(\varepsilon) > 1 - \lambda\}$.

In the sequel, for simplicity, we denote by τ the topology induced by the family $N_p(\varepsilon, \lambda)$ of (ε, λ)-neighborhoods.

The Kuratowski's function has the following important properties.

Lemma 1.4 ([1])**.** *Let $(E, \mathcal{F})$ be a PN-space and let A and B be probabilistically bounded subsets of E. Then the following properties hold:*

(i) $\alpha_A(t) \geq D_A(t), \; t \in \mathbb{R}$;
(ii) $A \subset B \Rightarrow D_A(t) \geq D_B(t), \; t \in \mathbb{R}$;
(iii) $\alpha_{(A\cup B)}(t) = \min\{\alpha_A(t), \alpha_B(t)\}, \; t \in \mathbb{R}$;
(iv) $\alpha_A(t) = \alpha_A(t)$ *for* $t \in \mathbb{R}$, *where* $\overline{A}$ *is the closure of* A *under the* τ*-topology on* E;
(v) A *is relatively compact if and only if* $\alpha_{\overline{A}}(t) = H(t)$ *for all* $t \in \mathbb{R}$;
(vi) *If* $(E, \mathcal{F}, \triangle)$ *is an M-PN-space with* $\triangle = \min$, *then* $\alpha_A = \alpha_{\overline{co}A}$.

Under the preparation of Kuratowski's functions, Chang et al. [2] naturally introduced the conception of k-set-contractive operator as follows.

Definition 1.5 ([2])**.** Let $(X, \mathcal{F})$ be a PN-space and let $T : D(T) \subset X \to X$ be a mapping and A a probabilistically bounded subset of $D(T)$. If there exists $k > 0$ such that

$$\alpha_{TA}(t) \geq \alpha_A(\frac{t}{k})$$

for all $t \in \mathbb{R}$, then T is called a k-set-contractive operator.

Inspired by the works of [2] and [5], Wu and Zhu [11] introduced the conception of semi-closed 1-set-contractive operator in M-PN-spaces.

Definition 1.6 ([11])**.** Let $(X, \mathcal{F})$ be a PN-space, τ the topology induced by the family of (ε, λ)- neighborhoods on E and $T : \Omega \to X$ a 1-set-contractive operator. Then T is called a semi-closed 1-set-contractive operator if $I - T$ is a τ-closed operator.

Suppose that $(E, \mathcal{F}, \triangle)$ is an M-PN-space and the t-norm $\triangle$ is continuous and satisfies $\triangle(t, t) \geq t$ for all $t \in [0, 1]$. Assume that Ω is a nonempty open subset of E, $T : \overline{\Omega} \to E$ is a semi-closed 1-set-contractive operator, $S = I - T$ and $p \notin S(\partial\Omega)$. The topological degree $deg(S, \Omega, p)$ of S in M-PN-spaces was defined in [11] (see [11] for more details). The topological degree $deg(S, \Omega, p)$ defined in [11] is just for the case of $p = \theta$. By the similar method, we can easily define the topological degree $deg(S, \Omega, p)$ for any $p \notin S(\partial\Omega)$, and $deg(S, \Omega, p)$ has also the following classical properties.

(i) (Normalization) $deg(I, \Omega, p) = 1, \; \forall p \in \Omega$;
(ii) (Solution property) if $deg(S, \Omega, p) \neq 0$, then $S(x) = p$ has at least one solution in Ω;
(iii) (Additivity) suppose that Ω_1 and Ω_2 are two disjoint open subsets of Ω, and $p \notin S(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then $deg(S, \Omega, p) = deg(S, \Omega_1, p) + deg(S, \Omega_2, p)$;
(iv) (Homotopy invariance) suppose that $H(t, x)$ is a semi-closed 1-set-contractive operator on $[0, 1] \times \overline{\Omega}$ and $p \notin (I - H(t, \cdot))(\partial\Omega)$ for all $t \in [0, 1]$, then $deg(I - H(t, \cdot), \Omega, p)$ is independent of $t \in [0, 1]$.

2. Main results

In this section, we always suppose that $(E, \mathcal{F}, \triangle)$ is an Menger PN space and the t-norm $\triangle$ is continuous and satisfies $\triangle(t,t) \geq t$ for all $t \in [0,1]$, and Ω is a nonempty open subset of E.

Theorem 2.1. *Suppose that $F : \overline{\Omega} \to E$ and $F_1 : \overline{\Omega} \to E$ are two semi-closed 1-set-contractive operators. Let $S = I - F$, $S_1 = I - F_1$. If $p \in E \setminus (S(\partial\Omega) \cup S_1(\partial\Omega))$ and there exists $\lambda \in [0,1]$ such that*

$$f_{F_1(x)-F(x)}(s) \geq f_{x-\lambda F_1(x)-(1-\lambda)F(x)-p}(s) \ \ \forall s > 0, \forall x \in \partial\Omega,$$

then $deg(S_1, \Omega, p) = deg(S, \Omega, p)$.

Proof. Let $H_t(x) = F_1(x) + t[F(x) - F_1(x)]$, $t \in [0,1]$, $x \in \overline{\Omega}$, then H is continuous and $H([0,1] \times \overline{\Omega})$ is probabilistically bounded.

(1) We prove that $I - H$ is τ-closed. Let S be a τ-closed subset of $[0,1] \times \overline{\Omega}$ of the form $S = M \times N$ with M a closed subset of $[0,1]$ and N a τ-closed subset of $\overline{\Omega}$. Suppose that $y_n \in (I - H)(M \times N)$ such that $y_n \to y_0$ as $n \to \infty$. We now prove that $y_0 \in (I - H)(M \times N)$. Since $I(t,x) = x$ for $x \in \overline{\Omega}$ and $t \in [0,1]$, and $y_n = (I - H)(t_n, x_n)$ with $t_n, x_n \in M \times N$, we have $x_n - t_nF(x_n) - (1 - t_n)F_1(x_n) \to y_0$. Taking a subsequence if necessary, we may assume that $t_n \to t_0$ as $n \to \infty$. Clearly, $t_0 \in M$ and $0 \leq t_0 \leq 1$, and

$$\begin{aligned}[I - t_0F - (1 - t_0)F_1](x_n) = x_n - t_nF(x_n) - (1 - t_n)F_1(x_n) - t_0F(x_n) - (1 - t_0)F_1(x_n) \\ + t_nF(x_n) + (1 - t_n)F_1(x_n).\end{aligned}$$

Hence, $[I - t_0F - (1 - t_0)F_1](x_n) \to y_0$, as $n \to \infty$. By the definition of semi-closed 1-set-contractive operator, it is easy to prove that $t_0F + (1 - t_0)F_1$ is a semi-closed 1-set-contractive operator, so $y_0 \in (t_0F + (1 - t_0)F_1)N$. Thus there exists $x_0 \in N$ such that $y_0 = x_0 - [t_0F + (1 - t_0)F_1](x_0)$. Therefore $y_0 \in (I - H)(M \times N)$, which means that $I - H$ is a τ-closed operator.

(2) Let A be a nonempty subset of $\overline{\Omega}$ such that $\alpha_A(s) \neq H(s)$. Then we have $\alpha H([0,1] \times A)(s) \geq \alpha_A(s)$ for $s \in \mathbb{R}$. In fact, for any $t \in [0,1]$,

$$H(t, A) = tF(A) + (1 - t)F_1(A) \subset co(F(A) \cup F_1(A)).$$

Since $H : [0,1] \times \overline{\Omega} \to X$ is probabilistically bounded and $H(\cdot, x) : [0,1] \to X$ is uniformly continuous with respect to $x \in \overline{\Omega}$, from the properties (iii), (iv) and (vi) of Lemma 1.4, for $s \in \mathbb{R}$, we have

$$\begin{aligned}\alpha_{H([0,1]\times A)}(s) = \min_{t\in[0,1]} \alpha_{H(t,A)}(s) \geq \alpha_{co(F(A)\cup F_1(A))}(t) \\ = \alpha_{F(A)\cup F_1(A)}(t) = \min\{\alpha_{F(A)}(t), \alpha_{F_1(A)}(t)\} \geq \alpha_A(t).\end{aligned}$$

(3) Put $h_t(x) = x - H_t(x)$. Next, we show that $p \notin h_t(\partial\Omega)$ for all $t \in [0,1]$. In fact, if there exists $x_0 \in \partial\Omega$ and $0 \leq t_0 \leq 1$ such that $p = h_{t_0}(x_0)$, then

$$x_0 - \lambda F_1(x_0) - (1 - \lambda)F(x_0) - (1 - \lambda - t_0)[F_1(x_0) - F(x_0)] = p.$$

So

$$x_0 - \lambda F_1(x_0) - (1 - \lambda)F(x_0) - p = (1 - \lambda - t_0)[F_1(x_0) - F(x_0)]. \tag{2.1}$$

By the given conditions, $t_0 \neq 0$ and $t_0 \neq 1$, so $0 < t_0 < 1$. Furthermore, ones can prove that $F_1(x_0) \neq F(x_0)$. Otherwise, $F_1(x_0) = F(x_0)$. Then from (2.1), it is easy to see that $x_0 - F(x_0) = p$, so $S(x_0) = p$ which is contrary to $p \notin S(\partial\Omega)$. This contradiction gives that $F_1(x_0) \neq F(x_0)$. Also it is not difficult to prove $0 < |1 - \lambda - t_0| < 1$. In fact, since $0 < t_0 < 1$, $-\lambda < 1 - \lambda - t_0 < 1 - \lambda$. It follows from $0 \leq \lambda \leq 1$ that $|1 - \lambda - t_0| < 1$. If $1 - \lambda - t_0 = 0$, then from (2.1) and the given assumptions, for any $s > 0$,

$$f_{F_1(x_0)-F(x_0)}(s) \geq f_{x_0-\lambda F_1(x_0)-(1-\lambda)F(x_0)-p}(s) = f_\theta(t) = 1,$$

so $F_1(x_0) = F(x_0)$, which contradicts $F_1(x_0) \neq F(x_0)$. So $1-\lambda-t_0 \neq 0$, and $0 < |1-\lambda-t_0| < 1$. Hence, from (2.1), it holds

$$f_{x_0-\lambda F_1(x_0)-(1-\lambda)F(x_0)-p}(s) = f_{F_1(x_0)-F(x_0)}\left(\frac{s}{|1-\lambda-t_0|}\right). \tag{2.2}$$

Then for any $s > 0$, by (2.2) and the given condition (2) as well as the non-decreasing property of distribution function, one can obtain that

$$\begin{aligned} f_{F_1(x_0)-F(x_0)}(s) &= f_{F_1(x_0)-F(x_0)}\left(\frac{s}{|1-\lambda-t_0|}\right) \\ &= f_{F_1(x_0)-F(x_0)}\left(\frac{s}{|1-\lambda-t_0|^2}\right) \\ &\;\vdots \\ &= f_{F_1(x_0)-F(x_0)}\left(\frac{s}{|1-\lambda-t_0|^n}\right),\ n = 1,2,3\cdots. \end{aligned}$$

Since $0 < |1-\lambda-t_0| < 1$ and $f_{F_1(x_0)-F(x_0)}$ is a distribution function, let $n \to \infty$ in the above formula, it is easy to see that $F_1(x_0) = F(x_0)$ which is in contradiction with $F_1(x_0) \neq F(x_0)$. So for any $t \in [0,1]$, $p \notin h_t(\partial\Omega)$. By the homotopy invariance of topological degree, $deg(S_1,\Omega,p) = deg(S,\Omega,p)$. □

Remark 2.2. In Theorem 2.1, if let $p = \theta$ and $\theta \in \Omega$, then by the normalization and solution property of topological degrees, F as well as F_1 has a fixed point in $\overline{\Omega}$.

Corollary 2.3. *Let $F : \overline{\Omega} \to E$ and $F_1 : \overline{\Omega} \to E$ be two semi-closed 1-set-contractive operators and $S = I - F$, $S_1 = I - F_1$. If $p \in E \setminus (S(\partial\Omega) \cup S_1(\partial\Omega))$ and*

$$f_{F_1(x)-F(x)}(s) \geq f_{x-F_1(x)-p}(s),\ \ \forall s > 0, \forall x \in \partial\Omega,$$

then $deg(S_1,\Omega,p) = deg(S,\Omega,p)$.

Proof. It only needs to put $\lambda = 1$ in Theorem 2.1. □

Corollary 2.4. *Let $\theta \in \Omega$ and $F : \overline{\Omega} \to E$ be a semi-closed 1-set-contractive operator. Suppose that there exists $0 \leq \lambda \leq 1$ such that*

$$x \neq F(x),\quad f_{F(x)}(s) \geq f_{x-\lambda F(x)}(s),\ \ \forall s > 0, \forall x \in \partial\Omega,$$

then $deg(I - F,\Omega,\theta) = 1$, so F has a fixed point in $\overline{\Omega}$.

Proof. It suffices to put $F_1 = \theta$ and $p = \theta$ in Theorem 2.1. □

Corollary 2.5. *Let $\theta \in \Omega$, $F : \overline{\Omega} \to E$ be a semi-closed 1-set-contractive operator. If*

$$f_{F(x)}(s) \geq f_x(s),\ \forall s > 0, \forall x \in \partial\Omega,$$

then F has a fixed point in $\overline{\Omega}$.

Proof. Without loss of generality, assume that $F(x) \neq x$ for all $x \in \partial\Omega$. Otherwise Corollary 2.5 is proved. For the rest, it only needs to let $\lambda = 0$ in Corollary 2.4. □

Remark 2.6. Corollaries 2.3 and 2.5 come from [2] (see Theorems 13.4.5 and 13.4.6 in [2]), thus Theorem 2.1 generalizes Theorems 13.4.5 and 13.4.6 in [2].

Theorem 2.7. *Let $F : \overline{\Omega} \to E$ and $F_1 : \overline{\Omega} \to E$ be two semi-closed 1-set-contractive operators and $S = I - F$, $S_1 = I - F_1$. If $p \in E \setminus (S(\partial\Omega) \cup S_1(\partial\Omega))$ and for every $x \in \partial\Omega$ and $\lambda \in (-1, 1)$,*

$$f_{F_1(x)+F(x)}(s) > f_{x+\lambda F_1(x)-p}(s), \ \ \forall s > 0, \tag{2.3}$$

then $deg(S_1, \Omega, p) = deg(S, \Omega, p)$.

Proof. Let $H_t(x) = tF(x) + (1-t)F_1(x)$ for all $x \in \overline{\Omega}$, $t \in [0, 1]$. Similar to the proof of Theorem 2.1, we can verify that $H_t : \overline{\Omega} \to E$ is a semi-closed 1-set-contractive operator. For every $t \in [0, 1]$ and $x \in \overline{\Omega}$, let $h_t(x) = x - H_t(x)$. It only needs to prove that $p \notin h_t(\partial\Omega)$ for all $t \in [0, 1]$. Otherwise, suppose that there exist $t_0 \in [0, 1]$ and $x_0 \in \partial\Omega$ such that $p = h_{t_0}(x_0)$, then

$$p = x_0 - H_{t_0}(x_0),$$

that is,

$$p = x_0 - t_0 F(x_0) - (1 - t_0)F_1(x_0). \tag{2.4}$$

By the given conditions, it is easy to see $t_0 \neq 0$ and $t_0 \neq 1$, so $t_0 \in (0, 1)$. Then it follows from (2.4) that

$$F(x_0) = \frac{1}{t_0}[x_0 - (1 - t_0)F_1(x_0) - p],$$

thus

$$F_1(x_0) + F(x_0) = \frac{1}{t_0}\left[x_0 + (2t_0 - 1)F_1(x_0) - p\right].$$

So for every $s > 0$, it holds that

$$\begin{aligned} f_{F_1(x_0)+F(x_0)}(s) &= f_{\frac{1}{t_0}[x_0+(2t_0-1)F_1(x_0)-p]}(s) \\ &= f_{x_0+(2t_0-1)F_1(x_0)-p}(s \cdot t_0) \\ &\leq f_{x_0+(2t_0-1)F_1(x_0)-p}(s). \end{aligned}$$

Let $\lambda = 2t_0 - 1$, since $0 < t_0 < 1$, then $-1 < 2t_0 - 1 < 1$. Hence there exists a contradiction with the given condition (2.3), this contradiction yields that $p \notin h_t(\partial\Omega)$ for all $t \in [0, 1]$. By the homotopy invariance of topological degree, $deg(S_1, \Omega, p) = deg(S, \Omega, p)$. □

Theorem 2.8. *Let $A : \overline{\Omega} \to E$ be a semi-closed 1-set-contractive operator and $\theta \in \Omega$. If for every $x \in \partial\Omega$,*

$$f_{A(x)-x}(s) + f_x(s) < f_{A(x)+x}(s) + f_{A(x)}(s), \ \ \forall s > 0, \tag{2.5}$$

then A has a fixed point in $\overline{\Omega}$.

Proof. Without loss of generality, suppose that A has no fixed point in $\partial\Omega$, otherwise Theorem 2.8 is proved.

For every $t \in [0, 1]$, let $H_t(x) = tA(x)$, then by a proof similar to that of Theorem 2.1, we can check that $H_t(x) : [0, 1] \times \overline{\Omega} :\to E$ is a semi-closed 1-set-contractive operator. Next, it needs to show that for every $t \in [0, 1]$ and $x \in \partial\Omega$, $x \neq H_t(x)$.

Suppose on contrary that there exists $t_0 \in [0, 1]$ and $x_0 \in \partial\Omega$ such that $x_0 = H_{t_0}(x_0)$, that is, $x_0 = t_0 A(x_0)$. Since $\theta \in \Omega$ and A has no fixed point in $\partial\Omega$, $t_0 \neq 0$ and $t_0 \neq 1$, that is $t_0 \in (0, 1)$. Thus

$$Ax_0 = \frac{x_0}{t_0}.$$

Noting the given condition (2.5), one has that

$$f_{\frac{x_0}{t_0}-x_0}(s) + f_{x_0}(s) < f_{\frac{x_0}{t_0}+x_0}(s) + f_{\frac{x_0}{t_0}}(s), \ \ \forall s > 0.$$

According to the non-decreasing property of distribution function, it holds that

$$f_{x_0}\left(\frac{st_0}{1-t_0}\right) + f_{x_0}(s) < f_{x_0}\left(\frac{st_0}{1+t_0}\right) + f_{x_0}(t_0 s), \ \ \forall s > 0. \tag{2.6}$$

However, by the definition of distribution function and $t_0 \in (0,1)$, it is easy to see that there exists a contradiction between (2.6) and the fact that f_{x_0} is a distribution function. So for every $t \in [0,1]$ and $x \in \partial\Omega$, $x \neq H_t(x)$. By the homotopy invariance of topological degree, $deg(I-A,\Omega,\theta) = deg(I,\Omega,\theta)$. Also from the normalization property of topological degree, one has that $deg(I,\Omega,\theta) = 1$, thus $deg(I-A,\Omega,\theta) = 1$. So A has a fixed point in Ω. In summary, A has a fixed point in $\overline{\Omega}$. □

Corollary 2.9. *Let $A : \overline{\Omega} \to E$ be a semi-closed 1-set-contractive operator and $\theta \in \Omega$. Suppose that for every $x \in \partial\Omega$, one of the following conditions is satisfied:*

(H_1) $f_{A(x)-x}(s) + f_x(s) < f_{A(x)+x}(s), \ \ \forall s > 0$;
(H_2) $f_{A(x)-x}(s) + f_x(s) < f_{A(x)}(s), \ \ \forall s > 0$;

then A has a fixed point in $\overline{\Omega}$.

Proof. Suppose that the condition (H_1) or (H_2) is satisfied. According to the non-decreasing property of distribution functions, it can be easily seen that all the conditions of Theorem 2.8 are satisfied. So Corollary 2.9 holds. □

3. Applications

As an application, we use the results in Section 2 to study the existence of solutions for a system of integral equations in M-PN-spaces. For simplicity, we give only one example.

Let $[0,a]$ be a fixed real interval with $0 < a < \infty$, $\mathbb{R}$ be the Banach space consisting of real numbers in which the norm is defined as usual, and $C([0,a],\mathbb{R})$ be the Banach space of all real-valued continuous functions defined on $[0,a]$ with the norm defined by

$$\|x\| = \sup_{0\le t\le a} |x(t)|, \quad x \in C([0,a],\mathbb{R}).$$

The space $C([0,a],\mathbb{R})$ can be endowed with another norm

$$\|x\|_1 = \sup_{0\le t\le a} (e^{-Lt}|x(t)|),$$

where L is a given positive number. Obviously, the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.

Let $(C([0,a],\mathbb{R}),\mathcal{F},\min)$ be the induced M-PN-space, where $\mathcal{F}$ is the set of mappings from $C([0,a],\mathbb{R})$ into $\mathcal{D}$ defined by $f_x(t) = H(t - \|x\|_1), x \in C([0,a],\mathbb{R}), t \in \mathbb{R}$.

We now study the existence of solutions for the system of nonlinear Volterra integral equations

$$\begin{cases} x(t) = z(t) + \displaystyle\int_0^t K_1(t,s,x(s))ds, \\ y(t) = z(t) + \displaystyle\int_0^t K_2(t,s,y(s))ds, \end{cases} \tag{3.1}$$

where $z \in C([0,a],\mathbb{R})$ is a given function.

Theorem 3.1. *The system of integral equations (3.1) has a solution in $C([0,a],\mathbb{R})$ if the following conditions are fulfilled:*

(1) *for* $i = 1, 2$, $K_i(t, s, x(s)) \in C([0, a] \times [0, a] \times C([0, a], \mathbb{R}), \mathbb{R})$ *and* $K_i(t, s, \overline{N_\theta(\varepsilon, \lambda)})$ *are relatively compact subsets of* $\mathbb{R}$, *and*

$$\|K_i\| = \sup_{t,s\in[0,a],x\in C([0,a],\mathbb{R})} |K_i(t, s, x(s))| < \infty;$$

(2) *there exist constants* $r, s > 0$ *with* $0 < r + s < 1$ *such that for any* $x, y \in \overline{N_\theta(\varepsilon, \lambda)}$ *and* $t \in [0, a]$, *if* $x \neq y$, *then*

$$s(y(t) - x(t)) \le \int_0^t (K_1(t, s, x(s)) - K_2(t, s, y(s)))\, ds \le r(x(t) - y(t)),$$

and

$$\int_0^t (K_i(t, s, x(s)) - K_i(t, s, y(s)))\, ds \le r(x(t) - y(t)), i = 1, 2;$$

(3) *there exists* p *with* $0 < p < 1$ *such that for any* $x \in \partial(\overline{N_\theta(\varepsilon, \lambda)})$ *and* $t \in [0, a]$,

$$\int_0^t (K_1(t, s, x(s)) - K_2(t, s, x(s))ds \le x(t) - p\int_0^t K_1(t, s, x(s))ds - (1 - p)\int_0^t K_2(t, s, x(s))ds,$$

and for each $i \in \{1, 2\}$, $A_i(\overline{N_\theta(\varepsilon, \lambda)})$ *is probabilistically bounded, where* A_i *is defined by*

$$(A_i x)(t) = z(t) + \int_0^t K_i(t, s, x(s))ds.$$

Proof. For each $i \in \{1, 2\}$, K_i is continuous, so A_i is continuous. Due to the condition (1) we know that A_i is compact, hence $I - A_i$ is τ-closed for each $i \in \{1, 2\}$. From the given condition (2), it follows that for any $x, y \in \overline{N_\theta(\varepsilon, \lambda)}$ and $t \in \mathbb{R}$,

$$f_{A_i x - A_i y}(t) \ge f_{x-y}(t), i = 1, 2. \tag{3.2}$$

Next we show that for each $i \in \{1, 2\}$, A_i is 1-set contractive on $\overline{N_\theta(\varepsilon, \lambda)}$. In fact, for arbitrary probabilistically bounded subset G of $\overline{N_\theta(\varepsilon, \lambda)}$ and any $\varepsilon > 0$, there exists a partition

$$G \subset \bigcup_{l=1}^{n} G_l,$$

such that $D_{G_l}(t) > \alpha_G(t) - \varepsilon$ for $l = 1, 2, \cdots, n$ and any $t \in \mathbb{R}$. Clearly,

$$A_j(G) \subset \bigcup_{l=1}^{n} A_j(G_l),\ \ j = 1, 2.$$

By (3.2), we get that for $j = 1, 2$ and any $t \in \mathbb{R}$,

$$D_{A_j(G_l)}(t) \ge D_{G_l}(t) > \alpha_G(t) - \varepsilon$$

for $l = 1, 2, \cdots, n$. By virtue of Lemma 1.4, we have, for each $j \in \{1, 2\}$ and any $t \in \mathbb{R}$, $\alpha_{A_j(G)}(t) \ge \alpha_G(t)$ which shows that A_j is a semi-closed 1-set contractive operator on $\overline{N_\theta(\varepsilon, \lambda)}$.

From the given condition (3), for any $x \in \partial(\overline{N_\theta(\varepsilon, \lambda)})$,

$$(A_1 x)(t) - (A_2 x)(t) \le x(t) - p(A_1 x)(t) - (1 - p)(A_2 x)(t)$$

for any $t \in [0, a]$. Then we have

$$\|A_1 x - A_2 x\|_1 \le \|x - p(A_1 x) - (1 - p)(A_2 x)\|_1,$$

which yields that

$$H(t-(A_1x-A_2x)) \geq H(t-(x-p(A_1x)-(1-p)(A_2x))).$$

In this way, for any $x \in \partial(\overline{N_\theta(\varepsilon,\lambda)})$ and $t \in \mathbb{R}$,

$$f_{(A_1x-A_2x)}(t) \geq f_{x-p(A_1x)-(1-p)(A_2x)}(t).$$

Hence for each $i \in \{1,2\}$, $A_i : \overline{N_\theta(\varepsilon,\lambda)} \to (C([0,a],\mathbb{R}),\mathcal{F},\triangle)$ satisfies all the conditions of Theorem 2.1. Noting that $\theta \in N_\theta(\varepsilon,\lambda)$, by Remark 2.2, we get that A_1 and A_2 have fixed points in $\overline{N_\theta(\varepsilon,\lambda)}$. Denote by $x^*(t)$ and $y^*(t)$ the fixed points of A_1 and A_2, respectively.

In the sequel, we prove that $x^*(t) = y^*(t)$. In fact, if not, noting that

$$x^*(t) = z(t) + \int_0^t K_1(t,s,x^*(s))ds,$$

and

$$y^*(t) = z(t) + \int_0^t K_2(t,s,y^*(s))ds,$$

by the given condition (ii), it is clear that $x^*(t)-y^*(t) \leq r(x^*(t)-y^*(t))$ and $y^*(t)-x^*(t) \leq (r+s)(y^*(t)-x^*(t))$ for any $t \in [0,a]$. Notice that $0 < r, r+s < 1$, we get $x^*(t) = y^*(t)$, which is a solution of the system of integral equations (3.1). □

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