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Optimality conditions for pessimistic trilevel optimization problem with middle-level problem being pessimistic

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Abstract

This paper mainly studies the optimality conditions for a class of pessimistic trilevel optimization problem, of which middle-level is a pessimistic problem. We firstly translate this problem into an auxiliary pessimistic bilevel optimization problem, by applying KKT approach for the lower level problem. Then we obtain a necessary optimality condition via the differential calculus of Mordukhovich. Finally, we obtain an existence theorem of optimal solution by direct method.

Keywords: Pessimistic trilevel optimization, bilevel programming, optimality conditions.
2010 MSC: 90C33, 90C30.

1. Introduction

Multi-level optimization problem specially trilevel and bilevel optimization problems are active research areas in mathematical programming at present [3, 10, 11, 15, 17, 18, 20, 23]. Trilevel optimization model which can be described as the following (1.1)-(1.3)

$$\begin{aligned} &\min_{x}\ f_1(x,y,z) \\ &s.t.\ \ g_1(x)\le 0, (y,z)\in\psi(x), \end{aligned} \tag{1.1}$$

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where $\psi(x)$ is the solution set of the following problem

$$\begin{aligned} &\min_{y}\ f_2(x,y,z) \\ &s.t.\ \ g_2(x,y,z) \le 0, z \in \psi(x,y), \end{aligned} \tag{1.2}$$

where $\psi(x,y)$ is the solution set of the following problem

$$\begin{aligned} &\min_{z} f_3(x,y,z) \\ &s.t.\ \ g_3(x,y,z) \le 0, \end{aligned} \tag{1.3}$$

here $x \in R^n$, $y \in R^m$, and $z \in R^p$; $f_i : R^n \times R^m \times R^p \to R,\ i = 1,2,3$; $g_1 : R^n \to R^{q_1}$, $g_i : R^n \times R^m \times R^p \to R^{q_i},\ i = 2,3$; $f_1(x,y,z)$, $f_2(x,y,z)$, $f_3(x,y,z)$ are the top-level, middle-level and lower-level objective functions, respectively. This model has a framework to deal with decision processes involving three decision makers with hierarchical nested structure. The top-level decision maker has the first choice, then the middle-level decision maker reacts optimal solution to the top maker's selection, finally the lower-level decision maker reacts optimal solution to the top and middle maker's sections. But sometimes the $\psi(x)$ and $\psi(x,y)$ are not singletons, so the lower-level decision maker may reacts a solution that the least one the middle-level want to get, to the middle-level. In a similar way the middle-level decision maker may reacts a solution that the least one top-level want to get to the top-level this is what "min_y" and "min_x" stands for. This model is an appropriate tool to solve the optimization problems in several areas such as electric system [1], transportation [6], supply chain management [21], and so on.

The model are called pessimistic trilevel optimization problem, if the lower-level (middle-level) reacts a solution that the least one the middle-level (top-level) want to get to the middle-level (top-level). In this paper, we consider the following pessimistic trilevel optimization problem with middle-level problem being pessimistic (PTOPM), in which the lower-level decision maker may reacts a solution that middle-level most want to get to the middle-level, but the middle-level decision maker reacts a solution that the least one top-level want to get to the top-level:

$$\begin{aligned} &\min_{x} \left\{ \max_{y,z} f_1(x,y,z) \right\} \\ &s.t.\ \ g_1(x) \le 0, (y,z) \in \psi(x), \end{aligned} \tag{1.4}$$

where $\psi(x)$ is the solution set of the following problem and suppose that $\psi(x)$ is not a singleton.

$$\begin{aligned} &\min_{y,z} f_2(x,y,z) \\ &s.t.\ \ g_2(x,y,z) \le 0, z \in \psi(x,y), \end{aligned} \tag{1.5}$$

where $\psi(x,y)$ is the solution set of the following problem

$$\begin{aligned} &\min_{z} f_3(x,y,z) \\ &s.t.\ \ g_3(x,y,z) \le 0. \end{aligned} \tag{1.6}$$

This model has many applications. For example in cloud market pricing [19, 22], the Software-as-a-Service (SaaS), the Platform-as-a-Service (PaaS), and the Infrastructure-as-a-Service (IaaS) are the top-level decision maker, middle-level decision maker, and lower-level decision maker respectively. All of them wish to optimization their profit, but every levels' price influence the setting of other levels' price. If the PaaS is able to persuade the IaaS to select an optimal solution which is the best one from the PaaS's point of view. But the SaaS is unfortunately. PaaS selects an optimal solution which is the worst one from the SaaS's point of view. Then it can be described as a PTOPM (1.4)-(1.6).

Although PTOPM has many applications, but to the best of our knowledge, there are few results on the optimality conditions and algorithms for the PTOPM (1.4)-(1.6). Since optimality conditions are essential to the design of algorithm and the convergence analysis, thus in this paper, we will discuss the optimality conditions for PTOPM.

For describing this model better, we introduce some definitions and hypotheses:

(1) Feasible set for the lower-level for each fixed (x, y):

$$K(x, y) = \{z \in R^p : g_3(x, y, z) \leq 0\}.$$

(2) Inequality constraint set for the top-level:

$$X = \{x \in R^n : g_1(x) \leq 0\}.$$

(3) Inequality constraint set for the middle-level:

$$Y = \{y \in R^m : \exists (x, z),\ \text{s.t. } g_2(x, y, z) \leq 0\}.$$

(4) The upper and middle level's decision space:

$$Q(X, Y) = \left\{(x, y) \in R^{n+m} : \exists z,\ \text{s.t. } g_1(x) \leq 0, g_i(x, y, z) \leq 0, i = 2, 3\right\}.$$

(5) The solution set of the lower-level optimization problem (1.6) for fixed $(x, y) \in Q(X, Y)$:

$$\psi(x, y) = \{z \in R^p : z \in \arg\min \{f_3(x, y, z) : g_3(x, y, z) \leq 0\}\}.$$

(6) The solution set of the middle-level optimization problem for fixed $x \in X$:

$$\psi(x) = \left\{(y, z) : (y, z) \in R^{m+p} \in \arg\min \{f_2(x, y, z) : g_2(x, y, z) \leq 0, z \in \psi(x, y)\}\right\}.$$

The rest of this paper is organized as follows. In Section 2, we recall some important results about variational analysis. In Section 3, we firstly translate the PTOPM into a pessimistic bilevel optimization problem by KKT approach, and the relationships between the two problems are discussed. Then we get a necessary optimality condition for the PTOPM (1.4)-(1.6) via the pessimistic bilevel optimization problem. In Section 4, we get an existence theorem of optimal solution.

2. Preliminaries

In this section, we mainly recall some basic definitions and results about variational analysis, which are needed in our main results.

Definition 2.1 ([13])**.** Given a point $\bar{z}$, $\limsup_{z\to\bar{z}} \Xi(z)$ is said to be the Kuratowski-Painlevée outer upper limit of a set-valued mapping $\Xi : R^n \longrightarrow 2^{R^m}$ at $\bar{z}$, if

$$\limsup_{z\to\bar{z}} \Xi(z) := \{v \in R^m : \exists z_k \to \bar{z}, v_k \to v \textit{ with } v_k \in \Xi(z_k) \textit{ as } k \to \infty\}.$$

Its graph $gph\Xi$ is denoted as follows:

$$gph\Xi := \{(u, v) \in R^n \times R^m : v \in \Xi(u)\}.$$

Definition 2.2 ([9])**.** Given a set-valued mapping $\Xi : R^n \longrightarrow 2^{R^m}$ and a point $\bar{z}$ with $\Xi(\bar{z}) \neq \emptyset$, we say that Ξ is inner semicompact at $\bar{z}$ if and only if for every sequence $z_k \to \bar{z}$ with $\Xi(z_k) \neq \emptyset$ there is a sequence of $y_k \in \Xi(z_k)$ that contains a convergent subsequence as $k \to \infty$.

Definition 2.3 ([2])**.** Let P be a nonempty subset of R^n. A set-valued mapping $\Xi : P \to 2^{R^m}$ is said to be

(i) lower semicontinuous (shortly, lsc) at $\bar{z} \in P$ if for each open set $V \subset R^m$ with $\Xi(\bar{z}) \cap V \neq \emptyset$, there exists $\delta > 0$ such that

$$\Xi(z) \cap V \neq \emptyset, \quad \forall z \in B(\bar{z}, \delta).$$

Ξ is lower semicontinuous if it is lower semicontinuous at all $\bar{z} \in P$.

(ii) compact-valued if, the images $\Xi(\nu)$ of all points $\nu \in P$ are compact.

Definition 2.4 ([13, 16])**.** For an extended real-valued function $\psi : R^n \to \bar{R}$, $\hat{\partial}\psi(\bar{z})$ is said to be the Fréchet subdifferential of ψ at a point $\bar{z}$ of it's domain if

$$\hat{\partial}\psi(\bar{z}) = \left\{ v \in R^n : \liminf_{z \to \bar{z}} \frac{\psi(z) - \psi(\bar{z}) - \langle v, z - \bar{z} \rangle}{\|z - \bar{z}\|} \geq 0 \right\},$$

given a point $\bar{z}$, $\partial\psi(\bar{z})$ is said to be the basic/Mordukovich subdifferential of ψ at $\bar{z}$ if

$$\partial\psi(\bar{z}) = \limsup_{z \to \bar{z}} \hat{\partial}\psi(z).$$

If ψ is convex, $\psi(\bar{z}) \neq \emptyset$, then $\partial\psi(\bar{z})$ reduces to the subdifferential in the sense of convex analysis:

$$\partial\psi(\bar{z}) = \{ v \in R^n : \psi(z) - \psi(\bar{z}) \geq \langle v, z - \bar{z} \rangle, \forall z \in R^n \},$$

the two subdifferentials coincide in this case.

$\partial\psi(\bar{z})$ is nonempty and compact when ψ is local Lipschitz continuous, its convex hull is the Clark subdifferential $\bar{\partial}\psi(\bar{z})$:

$$\bar{\partial}\psi(\bar{z}) = \mathrm{co}\partial\psi(\bar{z}), \tag{2.1}$$

here,"co" stands for the convex hull of the set in question. Via this link between the basic and Clark subdifferential, we have the following convex hull property which plays an important role in this paper:

$$\mathrm{co}\partial(-\psi)(\bar{z}) = -\mathrm{co}\partial\psi(\bar{z}). \tag{2.2}$$

Definition 2.5 ([16])**.** Let Ω be a nonempty subset of a finite dimensional space Z, given $z \in \Omega$, the cone

$$\hat{N}(z; \Omega) = \left\{ \xi : \langle \xi, z' - z \rangle \leq o(\|z' - z\|) \ \forall z' \in \Omega \right\}$$

is called regularity normal cone. The cone

$$N(z; \Omega) = \left\{ \xi : \exists \xi_k \to \xi, z_k \to z (z_k \in \Omega) : \xi_k \in \hat{N}(z_k; \Omega) \right\}$$

is called the limiting (Mordukhovich) normal cone to Ω at point z.

Proposition 2.6 ([16])**.** *Let $X \subset R^n$ and $D \subset R^m$ be two closed sets, $F : R^n \to R^m$ be a continuously differentiable mapping. Here $F(x) = (f_1(x), \cdots, f_m(x))$. Let $C = \{x \in X : F(x) \in D\}$, at any $\bar{x} \in C$ one has*

$$\hat{N}(\bar{x}; C) \supset \left\{ \sum_{i=1}^{m} y_i \nabla f_i(\bar{x}) + z : y \in \hat{N}(F(\bar{x}); D), z \in \hat{N}(\bar{x}; X) \right\},$$

where $y = (y_1, y_2, \cdots, y_m)$. On the other hand, one has

$$N(\bar{x}; C) \subset \left\{ \sum_{i=1}^{m} y_i \nabla f_i(\bar{x}) + z : y \in N(F(\bar{x}); D), z \in N(\bar{x}; X) \right\},$$

at any $\bar{x}$ satisfying the constraint qualification that, the only vector $y \in N(F(\bar{x}); D)$ for which

$$-\sum_{i=1}^{m} y_i \nabla f_i(\bar{x}) \in N(\bar{x}; X),$$

is $y = (0, \cdots, 0)$.

3. Necessary optimality condition

Next we give the definition of optimal solution of PTOPM (1.4)-(1.6). It can be described as the following problem (3.1)-(3.2)

$$\min_{x} \{\varphi_{pm}(x) \mid x \in X\}, \tag{3.1}$$

where

$$\varphi_{pm}(x) := \max_{y,z} \{f_1(x, y, z) \mid (y, z) \in \psi(x)\}. \tag{3.2}$$

Definition 3.1. A point $(x^*, y^*, z^*) \in R^n \times R^m \times R^q$ is called a local pessimistic solution for problem (3.1)-(3.2) if $x^* \in X$, $(y^*, z^*) \in \psi(x^*)$ with

$$f_1(x^*, y^*, z^*) \geq f_1(x^*, y, z) \quad \forall (y, z) \in \psi(x^*), \tag{3.3}$$

and there exists an open neighborhood $U(x^*, \delta), \delta > 0$, with

$$\varphi_{pm}(x^*) \leq \varphi_{pm}(x) \quad \forall x \in X \cap U(x^*, \delta). \tag{3.4}$$

It is called a global pessimistic solution if $\delta = \infty$ can be selected.

While designing algorithm for liner trilevel optimization problem (LTOP), Bard [5] translated LTOP into a bilevel optimization problem, by replacing the lower level problem with it's KKT conditions. The results show that this is an effective methods. In this part, we will firstly translate PTOPM (1.4)-(1.6) into a bilevel optimization problem by applying KKT approach. Then we will discuss the relationships between the two problems and wish it to be useful for designing algorithm for PTOPM. But in this subsection, we need to assume the lower-level problem to be convex for every parametric valued. If the lower level problem is not convex for fixed parametric valued, the set of feasible solution is enlarged by adding local optimal as well as stationary solutions of the lower-level problem to it. Since we know that, the KKT conditions of lower-level problem is not always sufficient and necessary. So we suppose that the following Slater's constraint qualification holds.

Definition 3.2. We say that the Slater's constraint qualification (Slater's CQ) is satisfied for $K(x, y)$ at $(x, y) \in R^n \times R^m$, if there exists $\bar{z} \in R^p$ such that

$$g_3^i(x, y, \bar{z}) < 0, \ i = 1, 2, \cdots, q_3.$$

KKT translation can be shown as the following auxiliary pessimistic bilevel optimization problem

$$\begin{aligned} &\min_{x} \left\{ \max_{y,z,\lambda} f_1(x, y, z) \right\} \\ s.t. \ \ & g_1(x) \leq 0, (y, z, \lambda) \in \psi_{kkt}(x), \end{aligned} \tag{3.5}$$

where $\psi_{kkt}(x)$ is the solution set of the following parametric MPEC problem

$$\begin{aligned} &\min_{y,z,\lambda} f_2(x, y, z) \\ s.t. \ \ & g_2(x, y, z) \leq 0, \\ & \nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^\top \lambda = 0, \\ & g_3(x, y, z)^\top \lambda = 0, \\ & \lambda \geq 0, g_3(x, y, z) \leq 0. \end{aligned} \tag{3.6}$$

Problem (3.5)-(3.6) is equal to the following form:

$$\min_{x} \{\varphi_{pmk}(x) \mid x \in X\}, \tag{3.7}$$

where

$$\varphi_{pmk}(x) = \max_{y,z,\lambda} \{f_1(x,y,z) \mid (y,z,\lambda) \in \psi_{kkt}(x)\}. \tag{3.8}$$

Since problem (3.1)-(3.2) may be not equal to problem (3.5)-(3.6), we need to discuss the relationship between the two problems.

Theorem 3.3.

(i) Assume that $f_3(x,y,\cdot)$, $g_3^i(x,y,\cdot)$, $i = 1,2,\cdots,q_3$ are convex continuously differentiable functions on $K(x,y)$, and the inequality constraint set for the middle-level Y is closed, and Slater's CQ for the lower-level problem holds at all $(x,y) \in X \times Y$. If $(\bar{x},\bar{y},\bar{z})$ is a local pessimistic solution of problem (3.1)-(3.2)*. Then for each*

$$\bar{\lambda} \in \Lambda(\bar{x},\bar{y},\bar{z}) := \left\{ \lambda \in R^{q_3} : \begin{array}{c} \nabla_z f_3(\bar{x},\bar{y},\bar{z}) + \nabla_z g_3(\bar{x},\bar{y},\bar{z})^\top \lambda = 0, \\ \lambda \geq 0, g_3(\bar{x},\bar{y},\bar{z})^\top \lambda = 0, \end{array} \right\},$$

$(\bar{x},\bar{y},\bar{z},\bar{\lambda})$ is a local pessimistic solution of problem (3.5)-(3.6)*.*

(ii) Assume that $f_3(x,y,\cdot)$, $g_3^i(x,y,\cdot)$, $i = 1,2,\cdots,q_3$ are convex continuously differentiable functions on $K(x,y)$, and the inequality constraint set for the middle-level Y is a closed set, and Slater's CQ for the lower-level problem holds at all $(x,y) \in X \times Y$. If $(\bar{x},\bar{y},\bar{z},\bar{\lambda})$ is a local (global) pessimistic solution of problem (3.5)-(3.6)*, then $(\bar{x},\bar{y},\bar{z})$ is a local (global) pessimistic solution of problem* (3.1)-(3.2)*.*

Proof. We can obtain this theorem via Theorem 2.1 and Theorem 2.3 in [7] easily. □

For parametric MPEC problem (3.6), we denote it's feasible set as

$$S(x) := \left\{ (y,z,\lambda) : \begin{array}{c} g_2(x,y,z) \leq 0, g_3(x,y,z) \leq 0, g_3(x,y,z)^\top \lambda = 0, \\ \lambda \geq 0, \nabla_z f_3(x,y,z) + \nabla_z g_3(x,y,z)^\top \lambda = 0, \end{array} \right\},$$

then problem (3.6) is equal to

$$\varphi_{kkt}(x) := \min_{y,z,\lambda} \{f_2(x,y,z) : (y,z,\lambda) \in S(x)\}. \tag{3.9}$$

In this part we will consider the first-order optimality conditions for auxiliary problem (3.7)-(3.8). Since we know that, φ_{kkt} is non-differentiable, so we need to consider its subdifferential. Demple, Mordukhovich. et al. gave the calculating method of subdifferential of optimal valued function for parametric MPEC problem (see Theorem 3.2 in [8]). Next we will obtain the subdifferential of φ_{kkt} by the method in [8]. Firstly, we define some constraint qualification and Lagrange functions which are similar to [8].

Fixed a point $(\bar{x},\bar{y},\bar{z},\bar{\lambda}) \in gph\ S$ we have the following partition of the indices for the complementarity functions in $S(\bar{x})$

$$\alpha = \alpha(\bar{x},\bar{y},\bar{z},\bar{\lambda}) := \left\{ i : \bar{\lambda}_i = 0, g_3^i(\bar{x},\bar{y},\bar{z}) < 0 \right\},$$
$$\beta = \beta(\bar{x},\bar{y},\bar{z},\bar{\lambda}) := \left\{ i : \bar{\lambda}_i = 0, g_3^i(\bar{x},\bar{y},\bar{z}) = 0 \right\},$$
$$\gamma = \gamma(\bar{x},\bar{y},\bar{z},\bar{\lambda}) := \left\{ i : \bar{\lambda}_i > 0, g_3^i(\bar{x},\bar{y},\bar{z}) = 0 \right\}.$$

For $(y_k,z_k,\lambda_k) \in S(\bar{x})$, we define

$$\alpha_k = \alpha_k(\bar{x},y_k,z_k,\lambda_k) := \left\{ i : \lambda_{ki} = 0, g_3^i(\bar{x},y_k,z_k) < 0 \right\},$$

$$\beta_k = \beta_k(\bar{x},y_k,z_k,\lambda_k) := \left\{ i : \lambda_{ki} = 0, g_3^i(\bar{x},y_k,z_k) = 0 \right\},$$

$$\gamma_k = \gamma_k(\bar{x}, y_k, z_k, \lambda_k) := \left\{ i : \lambda_{ki} > 0, g_3^i(\bar{x}, y_k, z_k) = 0 \right\}.$$

The Lagrange-type functions, associated with the parametric problem in (3.9) is

$$\begin{aligned} L(x, y, z, \lambda, \eta^{g_2}, \eta^{fg\lambda}, \eta^{g_3}, \eta^{\lambda}) :=& f_2(x, y, z) + g_2(x, y, z)^\top \eta^{g_2} + (\nabla_z f_3(x, y, z) \\ &+ \nabla_z g_3(x, y, z)^\top \lambda)^\top \eta^{fg\lambda} + \lambda^\top \eta^{\lambda} - g_3(x, y, z)^\top \eta^{g_3}, \end{aligned}$$

where $\eta^{g_2} \in R^{q_2}$, $\eta^{fg\lambda} \in R^p$, $\eta^{g_3} \in R^{q_3}$, $\eta^{\lambda} \in R^{q_3}$. The singular Lagrange-type functions, associated with the parametric problem in (3.9) is

$$\begin{aligned} L_0(x, y, z, \lambda, \eta^{g_2}, \eta^{fg\lambda}, \eta^{g_3}, \eta^{\lambda}) :=& g_2(x, y, z)^\top \eta^{g_2} + \lambda^\top \eta^{\lambda} - g_3(x, y, z)^\top \eta^{g_3} \\ &+ (\nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^\top \lambda)^\top \eta^{fg\lambda}. \end{aligned}$$

For simplicity, we denote

$$\begin{aligned} L(x, y, z, \lambda) &= L(x, y, z, \lambda, \eta^{g_2}, \eta^{fg\lambda}, \eta^{g_3}, \eta^{\lambda}), \\ L_0(x, y, z, \lambda) &= L_0(x, y, z, \lambda, \eta^{g_2}, \eta^{fg\lambda}, \eta^{g_3}, \eta^{\lambda}). \end{aligned}$$

The derivative of $L(x, y, z, \lambda)$ and $L_0(x, y, z, \lambda)$ with respect to (x, y, z, λ) at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ are denoted as $\nabla L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$. Here "$\nabla$" stand for "$\nabla_{x,y,z,\lambda}$" similarly hereinafter. The partial derivative of $L(x, y, z, \lambda)$ with respect to x, y, z and λ at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ are denoted as $\nabla_x L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla_y L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla_z L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla_\lambda L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$. Similarly, the partial derivative of $L_0(x, y, z, \lambda)$ can be denoted as $\nabla_x L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla_y L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla_z L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$, $\nabla_\lambda L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$.

We define the set of M-type multipliers associated with problem (3.6) by

$$\Lambda^{cm}(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = \Lambda^{cm},$$

$$\Lambda^{cm} = \left\{ (\eta^{g_2}, \eta^{fg\lambda}, \eta^{g_3}, \eta^{\lambda}) : \begin{array}{l} \eta^{g_2} \geq 0, \quad g_2(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_2} = 0, \\ \eta_i^{g_3} = 0, i \in \alpha, \quad \eta_i^{\lambda} = 0, i \in \gamma, \\ (\eta_i^{g_3} < 0 \wedge \eta_i^{\lambda} < 0) \vee \eta_i^{g_3} \eta_i^{\lambda} = 0, i \in \beta, \\ \nabla L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0 \end{array} \right\}.$$

We define the set $\Lambda^{cm}_{\bar{y},\bar{z},\bar{\lambda}}$ which can be obtained by replacing the gradients of f_2, g_2, λ, g_3, $\nabla_z f_3(x, y, z) + \nabla_z g_3(x, y, z)^\top \lambda$, in equality

$$\nabla f_2(\bar{x}, \bar{y}, \bar{z}) + \nabla g_2(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_2} + \nabla(\nabla_z f_3(\bar{x}, \bar{y}, \bar{z}) + \nabla_z g_3(\bar{x}, \bar{y}, \bar{z})^\top \bar{\lambda})^\top \eta^{fg\lambda} + \nabla \lambda^\top \eta^{\lambda} - \nabla g_3(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_3} = 0,$$

by their partial derivatives with respect to y, z, λ. That is

$$\Lambda^{cm}_{\bar{y},\bar{z},\bar{\lambda}} = \left\{ (\eta^{g_2}, \eta^{fg\lambda}, \eta^{g_3}, \eta^{\lambda}) : \begin{array}{l} \eta^{g_2} \geq 0, \quad g_2(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_2} = 0, \\ \eta_i^{g_3} = 0, i \in \alpha, \quad \eta_i^{\lambda} = 0, i \in \gamma, \\ (\eta_i^{g_3} < 0 \wedge \eta_i^{\lambda} < 0) \vee \eta_i^{g_3} \eta_i^{\lambda} = 0, i \in \beta, \\ \nabla_y L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \\ \nabla_z L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \\ \nabla_\lambda L(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \end{array} \right\}. \tag{3.10}$$

The following solution map for problem (3.6) which plays a significant role in the next theorem, given by

$$\psi_{kkt}(x) := \left\{ (y, z, \lambda) \in S(x) : f_2(x, y, z) \leq \varphi_{kkt}(x) \right\}.$$

To proceed in this part, we introduce the following two regularity conditions at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$. They ware firstly defined in [8].

$$\left\{ \begin{array}{c} \eta^{g_2} \geq 0, \quad g_2(\bar{x}, \bar{y}, \bar{z})^\top \eta^{g_2} = 0, \\ \eta_i^{g_3} = 0, i \in \alpha, \quad \eta_i^{\lambda} = 0, i \in \gamma, \\ (\eta_i^{g_3} < 0 \wedge \eta_i^{\lambda} < 0) \vee \eta_i^{g_3} \eta_i^{\lambda} = 0, i \in \beta, \\ \nabla L_0(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}) = 0, \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \eta^{g_2} = 0, \\ \eta^{\lambda} = 0, \\ \eta^{g_3} = 0, \\ \eta^{fg\lambda} = 0, \end{array} \right. \tag{3.11}$$

$$\left\{\begin{array}{c}\eta^{g_2} \geq 0, \quad g_2(\bar{x},\bar{y},\bar{z})^\top \eta^{g_2} = 0,\\ \eta_i^{g_3} = 0, i \in \alpha, \quad \eta_i^\lambda = 0, i \in \gamma,\\ (\eta_i^{g_3} < 0 \wedge \eta_i^\lambda < 0) \vee \eta_i^{g_3}\eta_i^\lambda = 0, i \in \beta,\\ \nabla_y L_0(\bar{x},\bar{y},\bar{z},\bar{\lambda}) = 0,\\ \nabla_z L_0(\bar{x},\bar{y},\bar{z},\bar{\lambda}) = 0,\\ \nabla_\lambda L_0(\bar{x},\bar{y},\bar{z},\bar{\lambda}) = 0,\end{array}\right\} \Rightarrow \nabla_x L_0(\bar{x},\bar{y},\bar{z},\bar{\lambda}) = 0. \tag{3.12}$$

The top-level regularity conditions are

$$\left.\begin{array}{r}\nabla g_1(\bar{x})^\top \varepsilon = 0\\ \varepsilon \geq 0, g_1(\bar{x})^\top \varepsilon = 0\end{array}\right\} \Rightarrow \varepsilon = 0, \tag{3.13}$$

where, $\varepsilon \in R^{q_1}$. This regularity conditions will be used to ensure Proposition 2.6 is correct when we apply it to the proof process of the main theorem.

Next we will calculate the subdifferential of the optimal value function φ_{kkt} at $\bar{x}$ by the method of Theorem 3.2 in [8]. Here we need to assume that the functions f_2, g_2 are continuously differentiable. and f_3, g_3 are twice continuously differentiable.

Theorem 3.4. *Assume that ψ_{kkt} is inner semicompact at $\bar{x}$, and regularity condition* (3.11) *holds at $(\bar{x},\bar{y},\bar{z},\bar{\lambda})$ for all $(\bar{y},\bar{z},\bar{\lambda}) \in \psi_{kkt}(\bar{x})$. Then we have the subdifferential upper estimate*

$$\begin{aligned}\partial\varphi_{kkt}(\bar{x}) \subset \bigcup_{(\bar{y},\bar{z},\bar{\lambda})\in\psi_{kkt}(\bar{x})} \bigcup_{(\eta^{g_2},\eta^{fg\lambda},\eta^{g_3})\in\Lambda^{cm}_{\bar{y},\bar{z},\bar{\lambda}}} &\Big\{\nabla_x f_2(\bar{x},\bar{y},\bar{z}) + \nabla_x g_2(\bar{x},\bar{y},\bar{z})^\top \eta^{g_2}\\ &+\nabla_{zx} f_3(\bar{x},\bar{y},\bar{z})^\top \eta^{fg\lambda} + (\nabla_{zx} g_3(\bar{x},\bar{y},\bar{z})^\top \bar{\lambda})^\top \eta^{fg\lambda} - \nabla_x g_3(\bar{x},\bar{y},\bar{z})^\top \eta^{g_3}\Big\}.\end{aligned} \tag{3.14}$$

If in addition regularity condition (3.12) *is satisfied at $(\bar{x},\bar{y},\bar{z},\bar{\lambda})$, for all $(\bar{y},\bar{z},\bar{\lambda}) \in \psi_{kkt}(\bar{x})$, then the value function φ_{kkt} is Lipschitz continuous around $\bar{x}$.*

Proof. Combining the assumption and Theorem 3.2 in [8], it is easy to show that φ_{kkt} is Lipschitz continuous around $\bar{x}$, and the following (3.15) holds

$$\partial\varphi_{kkt}(\bar{x}) \subset \bigcup_{(\bar{y},\bar{z},\bar{\lambda})\in\psi_{kkt}(\bar{x})} \bigcup_{(\eta^{g_2},\eta^{fg\lambda},\eta^{g_3},\eta^\lambda)\in\Lambda^{cm}_{\bar{y},\bar{z},\bar{\lambda}}} \left\{\nabla_x L(\bar{x},\bar{y},\bar{z},\bar{\lambda})\right\}. \tag{3.15}$$

Due to

$$(\eta^{g_2},\eta^{fg\lambda},\eta^{g_3},\eta^\lambda) \in \Lambda^{cm}_{\bar{y},\bar{z},\bar{\lambda}},$$

we have

$$\nabla_\lambda L(\bar{x},\bar{y},\bar{z},\bar{\lambda}) = (\nabla_z g_3(\bar{x},\bar{y},\bar{z})^\top e)^\top \eta^{fg\lambda} + e^\top \eta^\lambda = 0, \tag{3.16}$$

further

$$\eta^\lambda = -e^\top(\nabla_z g_3(\bar{x},\bar{y},\bar{z})^\top e)^\top \eta^{fg\lambda}. \tag{3.17}$$

From (3.17) it follows that η^λ can be replaced by $\eta^{fg\lambda}$. According to (3.15) and (3.17), it is easy to get (3.14). □

Since we know that, the subdifferential of optimal valued function (3.8) is necessary for the fist-order necessary condition of PTOPM, so we will discuss it here. In order to apply Theorem 3.4 in [9], we consider the following form.

$$\begin{aligned}-\varphi_{pmk}(x) &= \min_{y,z,\lambda}\{-f_1(x,y,z) \mid (y,z,\lambda) \in \psi_{kkt}(x)\}\\ &= \min_{y,z,\lambda}\{-f_1(x,y,z) \mid f_2(x,y,z) - \varphi_{kkt}(x) \leq 0\}.\end{aligned} \tag{3.18}$$

The solution set of problem (3.18) is defined as:

$$\psi_{pmk}(x) = \arg\min_{y,z,\lambda} \{-f_1(x,y,z) \mid f_2(x,y,z) - \varphi_{kkt}(x) \le 0\}. \tag{3.19}$$

It is obvious that

$$\psi_{pmk}(x) \subseteq \psi_{kkt}(x).$$

Now we can calculate the subdifferential of $-\varphi_{pmk}$. Similar to theorem 3.4, since we will apply (2.2) to calculate co$\varphi_{pmk}(\bar{x})$, we need to guarantee the local Lipschitz continuity of φ_{pmk}.

Theorem 3.5. *Assume that all conditions of Theorem 3.4 hold. Moreover we assume that ψ_{pmk} is inner semicompact at $\bar{x}$ and ψ_{kkt} satisfies Aubin's property around $(\bar{x}, y, z, \lambda) \in gph\psi_{kkt}$, for all $(y, z, \lambda) \in \psi_{pmk}(\bar{x})$. Then φ_{pmk} is Lipschitz continuous around $\bar{x}$, further more the Mordukhovich subdifferential of $-\varphi_{pmk}$ is estimated as:*

$$\begin{aligned}\partial(-\varphi)_{pmk}(\bar{x}) \subseteq & \bigcup_{(y,z,\lambda)\in\psi_{pmk}(\bar{x})} \bigcup_{\gamma\in\Lambda(\bar{x},y,z,\lambda)} \Bigg\{-\nabla_x f_1(\bar{x},y,z) + \gamma\nabla_x f_2(\bar{x},y,z) - \gamma\sum_{k=1}^{n+1}\rho_k \{\nabla_x f_2(\bar{x},y_k,y_k) \\ & + \nabla_x g_2(\bar{x},y_k,y_k)^\top \eta_k^{g_2} + \nabla_{zx} f_3(\bar{x},y_k,y_k)^\top \eta_k^{fg\lambda} \\ & +(\nabla_{zx} g_3(\bar{x},y_k,y_k)^\top \lambda_k)^\top \eta_k^{fg\lambda} - \nabla_x g_3(\bar{x},y_k,y_k)^\top \eta_k^{g_3}\}\Bigg\},\end{aligned} \tag{3.20}$$

where, the $\Lambda(\bar{x}, y, z, \lambda)$ of Lagrange multipliers for problem (3.18) with the parameter $\bar{x}$, is defined as

$$\Lambda(\bar{x},y,z,\lambda) = \left\{0 \le \gamma \in R \mid \begin{array}{c} -\nabla_y f_1(\bar{x},y,z) + \gamma\nabla_y f_2(\bar{x},y,z) = 0, \\ -\nabla_z f_1(\bar{x},y,z) + \gamma\nabla_z f_2(\bar{x},y,z) = 0 \end{array}\right\}, \tag{3.21}$$

$\rho_k \in R$, $\Sigma_{k=1}^{n+1}\rho_k = 1$, $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$ for all k, $(\eta_k^{g_2}, \eta_k^{fg\lambda}, \eta_k^{g_3}) \in \Lambda^{cm}_{y_k,z_k,\lambda_k}$.

Proof. Due to all conditions of Theorem 3.4 are satisfied so we can obtain $\partial\varphi_{kkt}(\bar{x})$ easily. Since ψ_{kkt} is inner semicompact at $\bar{x}$ and ψ_{kkt} satisfies Aubin's property around $(\bar{x}, y, z, \lambda) \in gph\psi_{kkt}$, for all $(y, z, \lambda) \in \psi_{pmk}(\bar{x})$, from Theorem 3.4 in [9], it follows that

$$\partial(-\varphi_{pmk})(\bar{x}) \subseteq \bigcup_{(y,z,\lambda)\in\psi_{pmk}(\bar{x})} \bigcup_{\gamma\in\bar{\Lambda}(\bar{x},y,z,\lambda)} \{-\nabla_x f_1(\bar{x},y,z) + \gamma\nabla_x f_2(\bar{x},y,z) + \gamma v^*\}, \tag{3.22}$$

where, the $\bar{\Lambda}(\bar{x}, y, z, \lambda)$ of Lagrange multipliers for problem (3.18) with the parameter $\bar{x}$, is defined as

$$\bar{\Lambda}(\bar{x},y,z,\lambda) = \left\{\gamma \in R \mid \begin{array}{c} -\nabla_y f_1(\bar{x},y,z) + \gamma\nabla_y f_2(\bar{x},y,z) = 0 \\ -\nabla_z f_1(\bar{x},y,z) + \gamma\nabla_z f_2(\bar{x},y,z) = 0 \\ \gamma \ge 0, \gamma(f_2(\bar{x},y,z) - \varphi_{kkt}(\bar{x})) = 0 \end{array}\right\} \tag{3.23}$$

and $v^* \in \partial(-\varphi_{kkt})(\bar{x})$. Since, $(y, z, \lambda) \in \psi_{pmk}(\bar{x}) \subseteq \psi_{kkt}(\bar{x})$, so

$$f_2(\bar{x},y,z) - \varphi_{kkt}(\bar{x}) = 0,$$

combining this with (3.23), we can obtain (3.21).

According to (2.2) it follows that

$$\partial(-\varphi_{kkt})(\bar{x}) \subseteq \text{co}\partial(-\varphi_{kkt})(\bar{x}) = -\text{co}\partial\varphi_{kkt}(\bar{x}). \tag{3.24}$$

Taking $\nu \in \text{co}\partial\varphi_{kkt}(\bar{x})$ and applying Carathéodory's theorem [12], we can find $\rho_k \in R$, and $\nu_k \in R^n$ with $k = 1, \dots, n+1$ such that

$$\nu = \Sigma_{k=1}^{n+1}\rho_k\nu_k,\ \Sigma_{k=1}^{n+1}\rho_k = 1,\ \rho_k \ge 0,\ \nu_k \in \partial\varphi_{kkt}(\bar{x}),\ for\ k = 1, \dots, n+1. \tag{3.25}$$

From Theorem 3.4 we have $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$, such that

$$\begin{aligned}\nu_k =& \nabla_x f_2(\bar{x}, y_k, z_k) + \nabla_x g_2(\bar{x}, y_k, z_k)^\top \eta_k^{g_2} + \nabla_{zx} f_3(\bar{x}, y_k, z_k)^\top \eta_k^{fg\lambda} \\ &+ (\nabla_{zx} g_3(\bar{x}, y_k, z_k)^\top \lambda_k)^\top \eta_k^{fg\lambda} - \nabla_x g_3(\bar{x}, y_k, z_k)^\top \eta_k^{g_3},\end{aligned} \tag{3.26}$$

here, $(\eta_k^{g_2}, \eta_k^{fg\lambda}, \eta_k^{g_3}) \in \Lambda^{cm}_{y_k, z_k, \lambda_k}$.

Combining (3.22)-(3.26) we can get (3.20) easily.

□

We are now ready to state one of the main results of this paper, which providers necessary optimality conditions for the auxiliary problem (3.7).

Theorem 3.6. *Let $\bar{x}$ be a local optimal solution of problem (3.7), then there exists a point $(\bar{y}, \bar{z}, \bar{\lambda})$ which is a solution of parametric problem (3.8) for fixed point $\bar{x}$. Assume that f_1, f_2, g_2 are continuously and differentiable at $(\bar{x}, \bar{y}, \bar{z})$, g_1 is differentiable at $\bar{x}$, and f_3, g_3 are twice continuously differentiable functions at $(\bar{x}, \bar{y}, \bar{z})$. Suppose that all conditions of Theorem 3.4 hold. Moreover we suppose that ψ_{pmk} is inner semicompact at $\bar{x}$ and ψ_{kkt} satisfies Aubin's property around $(\bar{x}, y, z, \lambda) \in gph\psi_{kkt}$, for all $(y, z, \lambda) \in \psi_{pmk}(\bar{x})$. Then we can find $\rho_k \in R$ with $k = 1, \ldots, n+1$, $\Sigma_{k=1}^{n+1} \rho_k = 1$, $\rho_k \geq 0$. Also we can find $\gamma_l, \sigma_l \in R$, $\gamma_l \geq 0$ with $l = 1, \ldots, n+1$, $\Sigma_{l=1}^{n+1} \sigma_l = 1$, $\sigma_l \geq 0$, and $\eta_k^{g_2} \in R^{q_2}, \eta_k^{fg\lambda} \in R^p, \eta_k^{g_3} \in R^{q_3}$, $\beta^{g_1} \in R^{q_1}$, $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$, $(y_l, z_l, \lambda_l) \in \psi_{pmk}(\bar{x})$, such that the following conditions holds:*

$$\begin{aligned}&\nabla g_1(\bar{x})^\top \beta^{g_1} - \sum_{l=1}^{n+1} \sigma_l \left\{-\nabla_x f_1(\bar{x}, y_l, z_l) + \gamma_l \nabla_x f_2(\bar{x}, y_l, z_l)\right\} + \sum_{k=1}^{n+1} \rho_k \left\{\nabla_x f_2(\bar{x}, y_k, y_k)\right. \\ &+ \nabla_x g_2(\bar{x}, y_k, y_k)^\top \eta_k^{g_2} + \nabla_{zx} f_3(\bar{x}, y_k, y_k)^\top \eta_k^{fg\lambda} \\ &\left.+ (\nabla_{zx} g_3(\bar{x}, y_k, y_k)^\top \lambda_k)^\top \eta_k^{fg\lambda} - \nabla_x g_3(\bar{x}, y_k, y_k)^\top \eta_k^{g_3}\right\} = 0,\end{aligned} \tag{3.27}$$

$$\begin{aligned}&\nabla_y f_2(\bar{x}, y_k, z_k) + \nabla_y g_2(\bar{x}, y_k, z_k)^\top \eta_k^{g_2} + \nabla_{zy} f_3(\bar{x}, y_k, z_k)^\top \eta_k^{fg\lambda} \\ &+(\nabla_{zy} g_3(\bar{x}, y_k, z_k)^\top \lambda_k)^\top \eta_k^{fg\lambda} - \nabla_y g_3(\bar{x}, y_k, z_k)^\top \eta_k^{g_3} = 0,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&\nabla_z f_2(\bar{x}, y_k, z_k) + \nabla_z g_2(\bar{x}, y_k, z_k)^\top \eta_k^{g_2} + \nabla_{z^2} f_3(\bar{x}, y_k, z_k)^\top \eta_k^{fg\lambda} \\ &+(\nabla_{z^2} g_3(\bar{x}, y_k, z_k)^\top \lambda_k)^\top \eta_k^{fg\lambda} - \nabla_z g_3(\bar{x}, y_k, z_k)^\top \eta_k^{g_3} = 0,\end{aligned} \tag{3.29}$$

$$(\nabla_z g_3(\bar{x}, y_k, z_k)^\top e)^\top \eta_k^{fg\lambda} + e^\top \eta_k^\lambda = 0, \tag{3.30}$$

$$\eta_k^{g_2} \geq 0, \quad g_2(\bar{x}, y_k, z_k)^\top \eta_k^{g_2} = 0, \tag{3.31}$$

$$\eta_{ki}^{g_3} = 0, i \in \alpha_k, \quad \eta_{ki}^\lambda = 0, i \in \gamma_k, \tag{3.32}$$

$$(\eta_{ki}^{g_3} < 0 \wedge \eta_{ki}^\lambda < 0) \vee \eta_{ki}^{g_3} \eta_{ki}^\lambda = 0, i \in \beta_k, \tag{3.33}$$

$$\beta^{g_1} \geq 0, \quad g_1(\bar{x})^\top \beta^{g_1} = 0, \tag{3.34}$$

$$- \nabla_y f_1(\bar{x}, y_l, z_l) + \gamma_l \nabla_y f_2(\bar{x}, y_l, z_l) = 0, \tag{3.35}$$

$$- \nabla_z f_1(\bar{x}, y_l, z_l) + \gamma_l \nabla_z f_2(\bar{x}, y_l, z_l) = 0. \tag{3.36}$$

Proof. Since $\bar{x}$ is a local solution of problem (3.7), it follows from Proposition 5.3 in [14] that

$$0 \in \partial \varphi_{pmk}(\bar{x}) + N(\bar{x}, X). \tag{3.37}$$

Due to $\partial\varphi_{pmk}(\bar{x}) \subseteq \mathrm{co}\partial\varphi_{pmk}(\bar{x})$, we can obtain the following equality

$$0 \in \mathrm{co}\partial\varphi_{pmk}(\bar{x}) + N(\bar{x}, X). \tag{3.38}$$

From (2.2) we know that

$$\mathrm{co}\partial\varphi_{pmk}(\bar{x}) = -\mathrm{co}\partial(-\varphi_{pmk})(\bar{x}). \tag{3.39}$$

Taking $\upsilon \in \mathrm{co}\partial(-\varphi_{pmk})(\bar{x})$, also by Carathéodory's Theorem [12], we can find $\sigma_l \in R$, and $\upsilon_l \in R^n$ with $l = 1, \ldots, n+1$ such that

$$\upsilon = \Sigma_{l=1}^{n+1}\sigma_l\upsilon_l,\ \Sigma_{l=1}^{n+1}\sigma_l = 1,\ \sigma_l \geq 0,\ \nu_l \in \partial(-\varphi_{pmk})(\bar{x}),\ for\ l = 1, \ldots, n+1. \tag{3.40}$$

Since all conditions of Theorem 3.5 hold, so according to Theorem 3.4 and Theorem 3.5 we have $(y_l, z_l, \lambda_l) \in \psi_{pmk}(\bar{x})$, such that

$$\begin{aligned}\upsilon_l = &-\nabla_x f_1(\bar{x}, y_l, z_l) + \gamma_l \nabla_x f_2(\bar{x}, y_l, z_l) - \gamma_l \sum_{k=1}^{n+1} \rho_k \Big\{\nabla_x f_2(\bar{x}, y_k, y_k) + \nabla_x g_2(\bar{x}, y_k, y_k)^\top \eta_k^{g_2} \\ &+ \nabla_{zx} f_3(\bar{x}, y_k, y_k)^\top \eta_k^{fg\lambda} + (\nabla_{zx} g_3(\bar{x}, y_k, y_k)^\top \lambda_k)^\top \eta_k^{fg\lambda} - \nabla_x g_3(\bar{x}, y_k, y_k)^\top \eta_k^{g_3}\Big\}\end{aligned} \tag{3.41}$$

here, $\gamma_l \in \Lambda(\bar{x}, y_l, z_l, \lambda_l)$, $(\eta_k^{g_2}, \eta_k^{fg\lambda}, \eta_k^{g_3}) \in \Lambda^{cm}_{y_k, z_k, \lambda_k}$, $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$.

Due to the top level regular condition (3.13) is satisfied at $\bar{x}$, applying Proposition 2.6, and through some calculations we know that there exists $\beta^{g_1} \in R^{q_1}$ such that,

$$N(\bar{x}, X) \subset \Big\{\nabla g_1(\bar{x})^\top \beta^{g_1} \mid \beta^{g_1} \geq 0, g_1(\bar{x})^\top \beta^{g_1} = 0\Big\}. \tag{3.42}$$

From (3.42) we can get (3.34). Combining $(\eta_k^{g_2}, \eta_k^{fg\lambda}, \eta_k^{g_3}) \in \Lambda^{cm}_{y_k, z_k, \lambda_k}$, $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$ with (3.10) we can obtain (3.28)-(3.33). According to (3.38), (3.39), (3.40), (3.41) and (3.42), through some simple calculating we can obtain (3.27) easily. Since $\gamma_l \in \Lambda(\bar{x}, y_l, z_l, \lambda_l)$ we can get (3.35)-(3.36). □

Next we will give two examples to illustrate the rationality of Theorem 3.6.

Example 3.7. we consider the following pessimistic trilevel optimization problem

$$\begin{array}{rl}\min\limits_{x} \max\limits_{y,z} & x_1^2 + y_1 y_2 + 5z \\ s.t. & 2 - x_1 \leq 0, \\ & x_1 - 5 \leq 0, \\ & 1 - x_2 \leq 0, \\ & x_2 - z \leq 0, \\ & (y, z) \in \psi(x),\end{array} \tag{3.43}$$

here $\psi(x)$ is the solution set of the following parametric optimization problem

$$\begin{array}{rl}\min\limits_{y,z} & y_1 + y_2 + z + 50x_1 \\ s.t. & y_1 + y_2 - 20 \leq 0, \\ & 10 - y_1 - y_2 \leq 0, \\ & z - y_2 \leq 0, \\ & 1 - y_1 \leq 0, \\ & z \in \psi(x, y),\end{array} \tag{3.44}$$

here $\psi(x, y)$ is the solution set of the following parametric optimization problem

$$\begin{array}{rl}\min\limits_{z} & z^2 + x_1 + y_1 \\ s.t. & 2 - z \leq 0, \\ & z - 8 \leq 0.\end{array} \tag{3.45}$$

Through some calculation, we can get the one of pessimistic optimal solutions $\bar{x} = (x_1, x_2) = (2, 1.5)$, $\bar{y} = (y_1, y_2) = (5, 5)$, $\bar{z} = 2$. KKT transformation can be shown as follows:

$$\begin{aligned} \min_{x} \max_{y,z} \quad & x_1^2 + y_1 y_2 + 5z \\ s.t. \quad & 2 - x_1 \leq 0, \\ & x_1 - 5 \leq 0, \\ & 1 - x_2 \leq 0, \\ & x_2 - z \leq 0, \\ & (y, z) \in \psi_{kkt}(x), \end{aligned} \tag{3.46}$$

here $\psi_{kkt}(x)$ is the solution set of the following parametric optimization problem

$$\begin{aligned} \min_{y,z,\lambda} \quad & y_1 + y_2 + z + 50x_1 \\ s.t. \quad & \lambda \geq 0, \\ & 2 - z \leq 0, \\ & z - 8 \leq 0, \\ & z - y_2 \leq 0, \\ & 1 - y_1 \leq 0, \\ & y_1 + y_2 - 20 \leq 0, \\ & 10 - y_1 - y_2 \leq 0, \\ & 2z - \lambda_1 + \lambda_2 = 0, \\ & \lambda_1(2 - z) + \lambda_2(z - 8) = 0. \end{aligned} \tag{3.47}$$

Through a series of calculation, we can get one of the pessimistic optimal solution $\bar{x} = (x_1, x_2) = (2, 1.5)$, $\bar{y} = (y_1, y_2) = (5, 5)$, $\bar{z} = 2$, $\bar{\lambda} = (\lambda_1, \lambda_2) = (4, 0)$. We can verify that all assumptions of Theorem 3.6 hold, and there exist $n = 0$, $\gamma = 5$, $\rho = 1$, $\sigma = 1$, $(\beta_1^{g_1}, \beta_2^{g_1}, \beta_3^{g_1}, \beta_4^{g_1}) = (4, 0, 0, 0)$, $(\eta_1^{g_2}, \eta_2^{g_2}, \eta_3^{g_2}, \eta_4^{g_2}) = (0, 1, 0, 0)$, $(\eta_1^{\lambda}, \eta_2^{\lambda}) = (0, 0)$, $(\eta_1^{g_3}, \eta_2^{g_3}) = (-2, 0)$, $\eta^{fg\lambda} = \frac{1}{2}$, $(y_k, z_k, \lambda_k) = (y_l, z_l, \lambda_l) = (\bar{y}, \bar{z}, \bar{\lambda})$, such that conditions (3.27)-(3.36) hold.

Example 3.8. Considering the following problem.

$$\begin{aligned} \min_{x} \max_{y,z} \quad & x + 2z + y_1 y_2 \\ s.t. \quad & 3 - x \leq 0, \\ & x - 5 \leq 0, \\ & (y, z) \in \psi(x), \end{aligned} \tag{3.48}$$

here $\psi(x)$ is the solution set of the following parametric optimization problem

$$\begin{aligned} \min_{y,z} \quad & x + z + y_1 + y_2 \\ s.t. \quad & y_1 + y_2 - 10 \leq 0, \\ & 4 - y_1 - y_2 \leq 0, \\ & 1 - y_1 \leq 0, \\ & 1 - y_2 \leq 0, \\ & z \in \psi(x, y), \end{aligned} \tag{3.49}$$

here $\psi(x, y)$ is the solution set of the following parametric optimization problem

$$\begin{aligned} \min_{z} \quad & z + 3x - y_1 - y_2 \\ s.t. \quad & 3 - z \leq 0, \\ & z - 10 \leq 0. \end{aligned} \tag{3.50}$$

Through some calculation, we can get one of pessimistic optimal solutions $\bar{x} = 3$, $\bar{y} = (y_1, y_2) = (2, 2)$, $\bar{z} = 3$. KKT transformation can be shown as follows:

$$\begin{aligned} \min_{x} \max_{y,z,\lambda} \quad & x + 2z + y_1 y_2 \\ s.t. \quad & 3 - x \le 0, \\ & x - 5 \le 0, \\ & (y, z, \lambda) \in \psi_{kkt}(x), \end{aligned} \tag{3.51}$$

here $\psi_{kkt}(x)$ is the solution set of the following parametric optimization problem,

$$\begin{aligned} \min_{y,z,\lambda} \quad & y_1 + y_2 + z + x \\ s.t. \quad & \lambda \ge 0, \\ & y_1 + y_2 - 10 \le 0, \\ & 4 - y_1 - y_2 \le 0, \\ & 1 - y_1 \le 0, \\ & 1 - y_2 \le 0, \\ & 3 - z \le 0, \\ & z - 10 \le 0, \\ & 1 - \lambda_1 + \lambda_2 = 0, \\ & \lambda_1(3 - z) + \lambda_2(z - 10) = 0. \end{aligned} \tag{3.52}$$

Through a series of calculation, we can get one of the pessimistic optimal solution $\bar{x} = 3$, $\bar{y} = (y_1, y_2) = (2, 2)$, $\bar{z} = 3$, $\bar{\lambda} = (\lambda_1, \lambda_2) = (1, 0)$. We can verify that all assumptions of Theorem 3.6 hold, and there exist $n = 0$, $\gamma = 2$, $\rho = 1$, $\sigma = 1$, $(\beta_1^{g_1}, \beta_2^{g_1}) = (4, 0)$, $(\eta_1^{g_2}, \eta_2^{g_2}, \eta_3^{g_2}, \eta_3^{g_2}) = (0, 1, 0, 0)$, $(\eta_1^{\lambda}, \eta_2^{\lambda}) = (0, 0)$, $(\eta_1^{g_3}, \eta_2^{g_3}) = (-1, 0)$, $\eta^{fg\lambda} = 1$, $(y_k, z_k, \lambda_k) = (y_l, z_l, \lambda_l) = (\bar{y}, \bar{z}, \bar{\lambda})$, such that conditions (3.27)-(3.36) hold.

Now we discuss the necessary optimality condition of PTOPM (1.4)-(1.6) via problem (3.7).

Theorem 3.9. *Let $(\bar{x}, \bar{y}, \bar{z})$ be a local solution of PTOPM (1.4)-(1.6). Assume that $f_3(x, y, \cdot)$, $g_3^i(x, y, \cdot)$, $i = 1, 2, \cdots, q_3$ are convex on $K(x, y)$, and twice continuously differentiable functions around $(\bar{x}, \bar{y}, \bar{z})$, and f_1, f_2, g_2 are continuous and differentiable around $(\bar{x}, \bar{y}, \bar{z})$, g_1 is differentiable around $\bar{x}$, and the inequality constraint set for the middle-level Y is a closed set, and Slater's CQ for the lower-level problem holds at all $(x, y) \in X \times Y$. If $(\bar{x}, \bar{y}, \bar{z})$ is a local pessimistic solution of problem (3.1)-(3.2). Then for each $\bar{\lambda} \in \Lambda(\bar{x}, \bar{y}, \bar{z})$, $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda})$ is a local solution of problem (3.7)-(3.8). Suppose that all conditions of Theorem 3.4 hold. Moreover we suppose that ψ_{pmk} is inner semicompact at $\bar{x}$ and ψ_{kkt} satisfies Aubin's property around $(\bar{x}, y, z, \lambda) \in gph\psi_{kkt}$, for all $(y, z, \lambda) \in \psi_{pmk}(\bar{x})$. Then we can find $\rho_k \in R$ with $k = 1, \ldots, n+1$, $\Sigma_{k=1}^{n+1} \rho_k = 1$, $\rho_k \ge 0$. Also we can find $\gamma_l, \sigma_l \in R$, $\gamma_l \ge 0$ with $l = 1, \ldots, n+1$, $\Sigma_{l=1}^{n+1} \sigma_l = 1$, $\sigma_l \ge 0$, and $\eta_k^{g_2} \in R^{q_2}, \eta_k^{fg\lambda} \in R^p, \eta_k^{g_3} \in R^{q_3}$, $\beta^{g_1} \in R^{q_1}$, $(y_k, z_k, \lambda_k) \in \psi_{kkt}(\bar{x})$, $(y_l, z_l, \lambda_l) \in \psi_{pmk}(\bar{x})$, such that the conditions (3.27)-(3.36) hold.*

Proof. Combining (i) of Theorem 3.3 and Theorem 3.6 we can obtain this theorem easily. □

4. Existence theorem of optimal solution

In this section we will consider the existence theorem of optimal solution for PTOPM (1.4)-(1.6). Since auxiliary pessimistic bilevel optimization problem (3.7)-(3.8) is a bridge, we firstly need consider the sufficient optimality condition for problem (3.7)-(3.8).

Theorem 4.1. *Let X be a non-empty compact set, $S(x)$ is nonempty and compact for every $x \in X$. Suppose that, ψ_{kkt} is lower semicontinuous at all $x \in X$. Then, problem (3.7)-(3.8) has a global pessimistic solution.*

Proof. From the lower semicontinuity of the set-valued mapping ψ_{kkt}, we know that the optimal value function φ_{pmk} is lower semicontinuous [4]. Since X is a non-empty compact set, so function ψ_{kkt} can attain its minimum on X. □

Next we will get an existence theorem of pessimistic optimal solution for PTOPM (1.4)-(1.6) based on auxiliary pessimistic bilevel optimization problem (3.7)-(3.8).

Theorem 4.2. *Suppose that f_1, f_2, g_1 and g_2 are continuous functions. Assume that $f_3(x, y, \cdot)$, $g_3^i(x, y, \cdot)$, $i = 1, 2, \cdots, q_3$ are convex continuously differentiable functions on $K(x, y)$, and the inequality constraint set of the middle-level Y is a closed, and Slater's CQ for the lower-level problem holds at all $(x, y) \in X \times Y$. Moreover we assume that X is a non-empty compact set, $S(x)$ is nonempty and compact for every $x \in X$. ψ_{kkt} is lower semicontinuous at all $x \in X$. Then PTOPM* (1.4)-(1.6) *has global pessimistic optimal solution.*

Proof. According to condition (ii) of Theorem 3.3 and Theorem 4.1, we can obtain this theorem easily. □

5. Conclusions

In this paper, we mainly study the optimality conditions for PTOPM. Since the middle-level decision maker is able to influence the lower-level decision maker's choice. So the problem (1.5)-(1.6) is a parametric optimistic bilevel optimization problem. Thus we can translate it into a parametric MPEC problem (3.6) by KKT approach. Since PTOPM is a very complex problem. So the necessary optimal condition we obtained is also complicated, we will Simplify it in the future.

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