



The asymptotic expansion for a class of non-linear singularly perturbed problems with optimal control

Han Xu*, Yinlai Jin

School of Science, Linyi University, Linyi, Shandong 276005, P. R. China.

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Abstract

In this article, we discuss a class of three-dimensional non-linear singularly perturbed systems with optimal control. Firstly, we confirm the existence of heteroclinic orbits connecting two equilibrium points about their associated systems by necessary conditions of optimal control and functional theory. Secondly, we study the asymptotic solutions of the singularly perturbed optimal control problems by the methods of boundary layer functions and prove the existence of the smooth solutions and the uniform validity of the asymptotic expansion. Finally, we cite an example to illustrate the result. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Contrast-structure problems [1, 7, 11] have been focus of mathematical study in a long time. Contrast structures are relevant to homoclinic orbits and heteroclinic orbits about corresponding associated systems [4, 5, 6, 13]. We often classify contrast structures as step-type contrast structures and spider-type contrast structures in [9]. Because contrast structures can express the instantaneous transformation more accurately, we can often use them as the models of the collision of cars and the transfer law of neurons.

Recently, Contrast structures with optimal control have been attached great importance. In [8], Ni and Dmitriev study a kind of linear singularly perturbed problems with optimal control

$$\begin{cases} J(u) = \int_0^T F(u, y, t) dt \rightarrow \min_u, \\ \mu \frac{dy}{dt} = a(t) + b(t)u, \\ y(0, \mu) = y^0, y(T, \mu) = y^T. \end{cases}$$

*Corresponding author

Email addresses: xuhan@lyu.edu.cn (Han Xu), jinyinlai@sina.com (Yinlai Jin)

Wu, Ni and Lu study a kind of nonlinear problems with step-type contrast structures in [12] as follows:

$$\begin{cases} J(u) = \int_0^T F(u, y, t)dt \rightarrow \min_u, \\ \mu \frac{dy}{dt} = g(u, y, t), \\ y(0, \mu) = v^0, y(T, \mu) = v^T. \end{cases}$$

In this paper, we will study the contrast structures for a class of high-dimensional non-linear singularly perturbed systems

$$\begin{cases} J(u) = \int_0^T G(u, v, w, t)dt \rightarrow \min_u, \\ \mu \frac{dv}{dt} = F_1(u, v, w, t), \\ \mu \frac{dw}{dt} = F_2(u, v, w, t), \\ v(0, \mu) = v^0, \\ v(T, \mu) = v^T, \\ w(0, \mu) = w^0, \\ w(T, \mu) = w^T. \end{cases}$$

By the methods of boundary layer functions [10, 14] and optimal control theory [2, 3], we study the asymptotic solution of contrast structures. By necessary conditions of Euler Equations, we can confirm the existence of the heteroclinic orbits connecting two equilibrium points. The results of the paper are new and supplement the previous ones.

Assume that $z = \begin{pmatrix} v \\ w \end{pmatrix}$, $z^0 = \begin{pmatrix} v^0 \\ w^0 \end{pmatrix}$, $z^T = \begin{pmatrix} v^T \\ w^T \end{pmatrix}$, $\begin{pmatrix} F_1(u, v, w, t) \\ F_2(u, v, w, t) \end{pmatrix} = A(t, z, u) + B(t)u$, so the above system is equivalent to the following system

$$\begin{cases} J(u) = \int_0^T G(u, z, t)dt \rightarrow \min_u, \\ \mu \frac{dz}{dt} = A(u, z, t) + B(t)u, \\ z(0, \mu) = z^0, \\ z(T, \mu) = z^T, \end{cases} \tag{1.1}$$

where $A(t, z, u)$ is a two-dimensional column vector and $B(t)$ is a known second-order reversible matrix. We give the following conditions for ease of discussion.

Definition 1.1. The vector functions $A(u, z, t)$ and $G(u, z, t)$ are sufficiently smooth in the field of $D = \{(u, z, t) \mid \|z\| \leq C, u \in R, 0 \leq t \leq T\}$, where C is a positive constant.

Definition 1.2. The degenerate equation $A(u, z, t) + B(t)u = 0$ has an isolated solution $\bar{u}(t) = \bar{u}(\bar{z}, t)$ in D .

Let $\mu = 0$, we have

$$J(\bar{u}) = \int_0^T g(\bar{z}, t)dt \rightarrow \min_{\bar{z}}$$

where $g(\bar{z}, t) = G(\bar{u}, \bar{z}, t)$.

We give the following conditions to study (1.1).

(A1) There exist two discontinuous vector functions $\bar{z}_1 = \alpha_1(t)$ and $\bar{z}_2 = \alpha_2(t)$, satisfying the following conditions

$$\min g(\bar{z}, t) = \begin{cases} g(\alpha_1(t), t), & 0 \leq t \leq t_0, \\ g(\alpha_2(t), t), & t_0 \leq t \leq T; \end{cases}$$

(A2) The transit point t_0 is determined by the equations $g(\alpha_1(t), t) = g(\alpha_2(t), t)$ and satisfies the following conditions

$$\frac{d}{dt}g(\alpha_1(t_0), t_0) \neq \frac{d}{dt}g(\alpha_2(t_0), t_0).$$

(A3) $g_u(\alpha_1(t), t) = 0, g_{uu}(\alpha_1(t), t) > 0$, when $0 \leq t \leq t_0$.

(A4) $g_u(\alpha_2(t), t) = 0, g_{uu}(\alpha_2(t), t) > 0$, when $t_0 \leq t \leq T$.

By assumptions (A1)-(A4), we can confirm that

$$\bar{u} = \begin{cases} \beta_1(t) = u(\alpha_1(t), t), & 0 \leq t \leq t_0, \\ \beta_2(t) = u(\alpha_2(t), t), & t_0 \leq t \leq T. \end{cases}$$

The widen function of (1.1) is

$$J'(u) = \int_0^T [G(u, z, t) + \mu^{-1}\lambda^T A(u, z, t) + \mu^{-1}\lambda^T B(t)u - \lambda^T z']dt. \tag{1.2}$$

Assuming that $H(u, z, \lambda, t) = G(u, z, t) + \lambda^T \mu^{-1} A(u, z, t) + \lambda^T \mu^{-1} B(t)u$, the function $J'(u)$ can be transformed into

$$J'(u) = \int_0^T [H(u, z, \lambda, t) - \lambda^T z']dt, \tag{1.3}$$

where $H(u, z, \lambda, t)$ is the Hamiltonian function and λ is an undetermined two-dimensional Lagrange multiplier. By the Euler Equation of $J'(u)$, we can obtain the following necessary optimality conditions

$$\begin{cases} \frac{\partial H}{\partial \lambda} = \dot{z}, \\ \frac{\partial H}{\partial \lambda} + \dot{\lambda} = 0, \\ \frac{\partial H}{\partial z} = 0. \end{cases} \tag{1.4}$$

The system (1.4) is equal to the following system

$$\begin{cases} \dot{\lambda} = -G_z(u, z, t) - \mu^{-1}A_z^T(u, z, t)\lambda, \\ G_u(u, z, t) + \mu^{-1}C^{-1}(t)\lambda = 0, \\ \mu\dot{z} = A(u, z, t) + B(t)u, \end{cases} \tag{1.5}$$

where $C^{-1}(t) = B(t) + A_u(u, z, t)$ and $A_z^T(u, z, t)$ is the transposed matrix of $A_z(u, z, t)$. From the second equation of (1.5), we can solve $\lambda = -\mu C(t)G_u$, then substituting it into the first and third equations, we can get the following equations

$$\begin{cases} \mu\dot{u} = H_1(u, z, t) + \mu H_2(u, z, t), \\ \mu\dot{z} = A(u, z, t) + B(t)u, \\ z(0, \mu) = z^0, z(T, \mu) = z^T, \end{cases} \tag{1.6}$$

where

$$H_1(u, z, t) = (C_u G_u + C G_{uu})^{-1}[-(C_z G_u + C G_{uz})(A + B(t)u) + G_z - A_z^T C(t)G_u],$$

$$H_2(u, z, t) = (C_u G_{ut} + C G_{uu})^{-1}(C G_{ut} + C_t G_u).$$

2. The Existence of Asymptotic Solution

We will proof the existence of step-type asymptotic solution of system (1.6). Assuming that $x = \begin{pmatrix} u \\ z \end{pmatrix}$,

$f(x, \bar{t}) = \begin{pmatrix} H_1(x, \bar{t}) + \mu H_2(x, \bar{t}) \\ A(x, \bar{t}) + B(\bar{t})u \end{pmatrix}$, So the system (1.6) is equivalent to the following system

$$\mu \frac{dx}{dt} = f(x, \mu, \bar{t}). \tag{2.1}$$

The associated system of (2.1) is

$$\frac{d\tilde{x}}{d\tau} = f(\tilde{x}, 0, \bar{t}). \tag{2.2}$$

Lemma 2.1. *The degenerate system of (1.6) has two solo solutions $(\alpha_1(t), \beta_1(t), \gamma_1(t))$ and $(\alpha_2(t), \beta_2(t), \gamma_2(t))$.*

The system (2.2) has two equilibrium points $M_l(\alpha_l(t), \beta_l(t), \gamma_l(t))$, $l = 1, 2$. Assuming that $A(\bar{t}) = D_{\tilde{x}}f(\tilde{x}, 0, \bar{t})|_{M_l}$, $l = 1, 2$, so the characteristic roots of $A(\bar{t})$ have sixteen possible signs. There might exist a heteroclinic orbit in three conditions as follows:

- (1) $M_1(-, -, +, +)$, $M_2(-, -, +, +)$;
- (2) $M_1(-, -, -, +)$, $M_2(-, -, -, +)$;
- (3) $M_1(-, +, +, +)$, $M_2(-, +, +, +)$.

We will discuss the first condition, the rest conditions can be discussed similarly.

Lemma 2.2. *The system (2.2) has two hyperbolic saddle points $M_l(\alpha_l(t), \beta_l(t), \gamma_l(t))$, $l = 1, 2$.*

We can confirm that there exists a heteroclinic orbit connecting M_1 with M_2 .

The manifold crossing $M_1(\alpha_1(t), \beta_1(t), \gamma_1(t))$ is $\Phi_l(\tilde{x}^{(-)}, \bar{t}) = \Phi_l(\alpha_1(t), \beta_1(t), \gamma_1(t), \bar{t})$. The manifold crossing $M_2(\alpha_2(t), \beta_2(t), \gamma_2(t))$ is $\Phi_l(\tilde{x}^{(+)}, \bar{t}) = \Phi_l(\alpha_2(t), \beta_2(t), \gamma_2(t), \bar{t})$. So we can reach the necessary conditions about the existence of a heteroclinic orbit as follows:

$$\Phi_l(\alpha_1(t), \beta_1(t), \gamma_1(t), \bar{t}) = \Phi_l(\alpha_2(t), \beta_2(t), \gamma_2(t), \bar{t}). \tag{2.3}$$

To discuss (2.2), the following condition is given:

(A5) The associate system (2.2) has two manifolds $\Phi_l(\tilde{x}^{(\mp)}, \bar{t}) = C_l$.

By assumption (A5) and the equation (2.3), we can determine that

$$\begin{cases} \tilde{v}^{(-)}(\tau) = \Psi_1^{(-)}(\tilde{u}^{(-)}, \bar{t}), \tilde{w}^{(-)}(\tau) = \Psi_2^{(-)}(\tilde{u}^{(-)}, \bar{t}), \\ \tilde{v}^{(+)}(\tau) = \Psi_1^{(+)}(\tilde{u}^{(+)}, \bar{t}), \tilde{w}^{(+)}(\tau) = \Psi_2^{(+)}(\tilde{u}^{(+)}, \bar{t}). \end{cases} \tag{2.4}$$

Let

$$H_1(\bar{t}) = \tilde{v}^{(-)}(\bar{t}) - \tilde{v}^{(+)}(\bar{t}), H_2(\bar{t}) = \tilde{w}^{(-)}(\bar{t}) - \tilde{w}^{(+)}(\bar{t}), \tag{2.5}$$

we can obtain the following hypothesis:

(A6) The system (2.5) has a solution of $\bar{t} = t_0$ and $\frac{\partial H_i(\bar{t})}{\partial \bar{t}}|_{\bar{t}=t_0} \neq 0$.

By assumption (A6), we can determine the solution of (2.4) and the existence of the heteroclinic orbit connecting M_1 with M_2 . Then the system (2.2) has a contrast-structure solution. Finally, we can confirm the system (1.1) has a interior-layer solution.

3. The Construction of Asymptotic Solution

Assuming the transit point is t^* , $t^* \in (0, 1)$,

$$t^* = t_0 + \mu t_1 + \dots + \mu^k t_k + \dots, \tag{3.1}$$

where t_i is undetermined coefficient. Supposing that the asymptotic solution of (1.1) is

$$\begin{cases} u^{(-)}(t, \mu) = \sum_{i=0}^{\infty} \mu^i (\bar{u}_i(t) + \Pi_i u(\tau_0) + Q_i^{(-)} u(\tau)), \\ z^{(-)}(t, \mu) = \sum_{i=0}^{\infty} \mu^i (\bar{z}_i(t) + \Pi_i z(\tau_0) + Q_i^{(-)} z(\tau)), \end{cases} \tag{3.2}$$

$$\begin{cases} u^{(+)}(t, \mu) = \sum_{i=0}^{\infty} \mu^i (\bar{u}_i(t) + Q_i^{(+)} u(\tau)) + R_i u(\tau_1), \\ z^{(+)}(t, \mu) = \sum_{i=0}^{\infty} \mu^i (\bar{z}_i(t) + Q_i^{(+)} z(\tau)) + R_i z(\tau_1), \end{cases} \tag{3.3}$$

where $\tau_0 = t/\mu$, $\tau = (t - t^*)/\mu$, $\tau_1 = (t - T)/\mu$ and $\bar{z}_k(t)$ are coefficients of the regular terms. Respectively, $\Pi_i z(\tau_0)$ and $R_i z(\tau_1)$ are coefficients of boundary layer terms at $t = 0$ and $t = T$, $Q_i^{(\mp)} z(\tau)$ are the left and right coefficients of the internal transition terms at $t = t^*$. We can obtain

$$\min_u J(u) = \min_u J(u_0) + \sum_{i=1}^n \mu^i \min_{u_i} \tilde{J}_i(u_i) + \dots,$$

where $\tilde{J}_i(u_i) = J_i(u_i, \tilde{u}_{i-1}, \dots, \tilde{u}_0)$, $\tilde{u}_k = \arg(\min \tilde{J}_k(u_k))$, and $k = 0, 1, \dots, i - 1$.

Substituting (3.2) and (3.3) into (1.1) and separating the terms on t , τ_0 , τ and τ_1 by boundary layer function methods [1, 11], then equating the terms with the same power of μ , we can obtain a series of optimal control problems to determine the coefficients in (3.2), (3.3).

The zeroth-order coefficients of the regular terms satisfy

$$\begin{cases} J(\bar{u}_0) = \int_0^T G(\bar{z}_0, \bar{u}_0, t) dt \rightarrow \min_{\bar{u}_0}, \\ F(\bar{u}_0, \bar{z}_0, t) = 0. \end{cases} \tag{3.4}$$

We can solve by (3.4)

$$\bar{u}_0(t) = \begin{cases} \alpha_1(t), & 0 \leq t \leq t_0, \\ \alpha_2(t), & t_0 \leq t \leq T, \end{cases} \quad \bar{v}_0(t) = \begin{cases} \beta_1(t), & 0 \leq t \leq t_0, \\ \beta_2(t), & t_0 \leq t \leq T, \end{cases} \quad \bar{w}_0(t) = \begin{cases} \gamma_1(t), & 0 \leq t \leq t_0, \\ \gamma_2(t), & t_0 \leq t \leq T. \end{cases}$$

The zero-order interior layer term $Q_0^{(\pm)} z(\tau)$ satisfies

$$\begin{cases} Q_0^{(\mp)} J = \int_{-\infty(0)}^{0(+\infty)} \Delta_0^{(\mp)} G(\alpha_{1,2}(t_0) + Q_0^{(\mp)} u, \bar{z}_0(t_0) + Q_0^{(\mp)} z, t_0) d\tau \rightarrow \min_{Q_0^{(\mp)} u}, \\ \frac{dQ_0^{(\mp)} v}{d\tau} = F_1(\alpha_{1,2}(t_0) + Q_0^{(\mp)} u, \beta_{1,2}(t_0) + Q_0^{(\mp)} v, \gamma_{1,2}(t_0) + Q_0^{(\mp)} w, t_0), \\ \frac{dQ_0^{(\mp)} w}{d\tau} = F_2(\alpha_{1,2}(t_0) + Q_0^{(\mp)} u, \beta_{1,2}(t_0) + Q_0^{(\mp)} v, \gamma_{1,2}(t_0) + Q_0^{(\mp)} w, t_0), \\ Q_0^{(\mp)} v(0) = \varphi(t_0) - \beta_{1,2}(t_0), \quad Q_0^{(\mp)} w(0) = \psi(t_0) - \gamma_{1,2}(t_0), \\ Q_0^{(\mp)} v(\mp\infty) = Q_0^{(\mp)} w(\mp\infty) = 0, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} \Delta_0^{(\mp)} G(\alpha_{1,2}(t_0) + Q_0^{(\mp)} u, \bar{z}_0(t_0) + Q_0^{(\mp)} z, t_0) = & G(\alpha_{1,2}(t_0) + Q_0^{(\mp)} u, \beta_{1,2}(t_0) + Q_0^{(\mp)} v, \gamma_{1,2}(t_0) \\ & + Q_0^{(\mp)} w, t_0) - G(\alpha_{1,2}(t_0), \beta_{1,2}(t_0), \gamma_{1,2}(t_0), t_0). \end{aligned}$$

Let $\tilde{u}^{(\mp)} = \alpha_{1,2}(t_0) + Q_0^{(\mp)}u$, $\tilde{v}^{(\mp)} = \beta_{1,2}(t_0) + Q_0^{(\mp)}v$, $\tilde{w}^{(\mp)} = \gamma_{1,2}(t_0) + Q_0^{(\mp)}w$, so (3.5) can be changed into

$$\begin{cases} Q_0^{(\mp)}J = \int_{-\infty(0)}^{0(+\infty)} \Delta_0^{(\mp)}\tilde{G}(\tilde{u}^{(\mp)}, \tilde{v}^{(\mp)}, \tilde{w}^{(\mp)}, t_0)d\tau \rightarrow \min_{\tilde{u}^{(\pm)}}, \\ \frac{d\tilde{v}^{(\mp)}}{d\tau} = F_1(\tilde{u}^{(\mp)}, \tilde{v}^{(\mp)}, \tilde{w}^{(\mp)}, t_0), \\ \frac{d\tilde{w}^{(\mp)}}{d\tau} = F_2(\tilde{u}^{(\mp)}, \tilde{v}^{(\mp)}, \tilde{w}^{(\mp)}, t_0), \\ \tilde{v}^{(\mp)}(0) = \varphi(t_0), \tilde{w}^{(\mp)}(0) = \psi(t_0), \tilde{v}^{(\mp)}(\mp\infty) = \varphi(t_0), \tilde{w}^{(\mp)}(\mp\infty) = \psi(t_0). \end{cases} \tag{3.6}$$

By(2.3) and (2.5), we can determine the solution of (3.6) is the heteroclinic orbit connecting M_1 and M_2 . Meanwhile, t_0 , $\tilde{v}^{(\mp)}(\tau) = \Psi_1^{(\mp)}(\tilde{u}^{(\mp)}, \bar{t})$ and $\tilde{w}^{(\mp)}(\tau) = \Psi_2^{(\mp)}(\tilde{u}^{(\mp)}, \bar{t})$ can be determined. $Q_0^{(\mp)}z(\tau)$ can also be determined. Substituting $Q_0^{(\mp)}z(\tau)$ into (3.5), we can solve $Q_0^{(\mp)}u(\tau)$. We can also determine that $Q_0^{(\mp)}z(\tau)$ and $Q_0^{(\mp)}u(\tau)$ decay exponentially as $\tau \rightarrow \mp\infty$.

The boundary functions $\Pi_0y(\tau_0)$ and $\Pi_0u(\tau_0)$ satisfy

$$\begin{cases} \Pi_0J = \int_0^{+\infty} \Delta_0G(\alpha_1(0) + \Pi_0u, \beta_1(0) + \Pi_0v, \gamma_{1,2}(0) + \Pi_0w, 0)d\tau \rightarrow \min_{\Pi_0u}, \\ \frac{d\Pi_0v}{d\tau_0} = F_1(\alpha_1(0) + \Pi_0u, \beta_1(0) + \Pi_0v, \gamma_1(0) + \Pi_0w, 0), \\ \frac{d\Pi_0w}{d\tau_0} = F_2(\alpha_1(0) + \Pi_0u, \beta_1(0) + \Pi_0v, \gamma_1(0) + \Pi_0w, 0), \\ \Pi_0v(0) = v^0 - \beta_1(0), \Pi_0w(0) = w^0 - \gamma_1(0), \Pi_0v(+\infty) = \Pi_0w(+\infty) = 0. \end{cases} \tag{3.7}$$

Assuming that $\tilde{u}^l = \alpha_1(0) + \Pi_0u$, $\tilde{v}^l = \beta_1(0) + \Pi_0v$, $\tilde{w}^l = \gamma_1(0) + \Pi_0w$, then (3.7) can be changed into

$$\begin{cases} \Pi_0J = \int_0^{+\infty} \Delta_0G(\tilde{u}^l, \tilde{v}^l, \tilde{w}^l, 0)d\tau \rightarrow \min_{\tilde{u}^l}, \\ \frac{d\tilde{v}^l}{d\tau_0} = F_1(\tilde{u}^l, \tilde{v}^l, \tilde{w}^l, 0), \\ \frac{d\tilde{w}^l}{d\tau_0} = F_2(\tilde{u}^l, \tilde{v}^l, \tilde{w}^l, 0), \\ \tilde{v}^l(0) = v^0, \tilde{w}^l(0) = w^0, \tilde{v}^l(+\infty) = \beta_1(0), \tilde{w}^l(+\infty) = \gamma_1(0). \end{cases} \tag{3.8}$$

The boundary functions $R_0y(\tau_1)$ and $R_0u(\tau_1)$ satisfy

$$\begin{cases} R_0J = \int_{-\infty}^0 \Delta_0G(\alpha_2(T) + R_0u, \beta_2(T) + R_0v, \gamma_2(T) + R_0w, T)d\tau_1 \rightarrow \min_{R_0u}, \\ \frac{dR_0v}{d\tau_1} = F_1(\alpha_2(T) + R_0u, \beta_2(T) + R_0v, \gamma_2(T) + R_0w, T), \\ \frac{dR_0w}{d\tau_1} = F_2(\alpha_2(T) + R_0u, \beta_2(T) + R_0v, \gamma_2(T) + R_0w, T), \\ R_0v(0) = v^T - \beta_2(T), R_0w(0) = w^T - \gamma_2(T), R_0v(-\infty) = R_0w(-\infty) = 0. \end{cases} \tag{3.9}$$

Assuming that $\tilde{u}^r = \alpha_2(T) + R_0u$, $\tilde{v}^r = \beta_2(T) + R_0v$, $\tilde{w}^r = \gamma_2(T) + R_0w$, then (3.9) can be changed into

$$\begin{cases} R_0J = \int_{-\infty}^0 \Delta_0f(\tilde{u}^r, \tilde{v}^r, \tilde{w}^r, T)d\tau \rightarrow \min_{\tilde{u}^r}, \\ \frac{d\tilde{v}^r}{d\tau_0} = F_1(\tilde{u}^r, \tilde{v}^r, \tilde{w}^r, T), \\ \frac{d\tilde{w}^r}{d\tau_0} = F_2(\tilde{u}^r, \tilde{v}^r, \tilde{w}^r, T), \\ \tilde{v}^r(0) = v^T, \tilde{w}^r(0) = w^T, \tilde{v}^r(-\infty) = \beta_2(T), \tilde{w}^r(-\infty) = \gamma_2(T). \end{cases} \tag{3.10}$$

Definition 3.1. Supposing that initial values $\tilde{v}^l(0) = v^0$ and $\tilde{w}^l(0) = w^0$ are intersected with the one-dimensional stable manifold $W^s(\tilde{M}_1(0))$ near the equilibrium point \tilde{M}_1 . Meanwhile, the initial values $R_0v(0) = v^T - \beta_2(T)$ and $R_0w(0) = w^T - \gamma_2(T)$ are intersected with the one-dimensional unstable manifold $W^u(\tilde{M}_2(T))$.

Lemma 3.2. By Definition 3.1 and associated systems (3.8), (3.10), we can determine that boundary functions $\Pi_0u(\tau_0)$, $\Pi_0z(\tau_0)$, $R_0u(\tau_1)$ and $R_0z(\tau_1)$ satisfy the following inequality

$$\|\Pi_0u(\tau_0)\| \leq C_1e^{-k_1\tau_0}, \|\Pi_0z(\tau_0)\| \leq C_2e^{-k_2\tau_0}, \|R_0u(\tau_1)\| \leq D_1e^{k_3\tau_1}, \|R_0z(\tau_1)\| \leq D_2e^{k_4\tau_1},$$

where C_l, D_l and k_m are all positive constants, $l = 1, 2$ and $m = 1, 2, 3, 4$.

Theorem 3.3. the system (1.1) has a step-like solution

$$\mathbf{x}(t, \mu) = \begin{cases} \bar{\mathbf{x}}_0^{(-)}(t) + \Pi_0\mathbf{x}(\tau_0) + Q_0^{(-)}\mathbf{x}(\tau) + O(\mu), & 0 \leq t \leq t^*, \\ \bar{\mathbf{x}}_0^{(+)}(t) + Q_0^{(+)}\mathbf{x}(\tau) + R_0\mathbf{x}(\tau_1) + O(\mu), & t^* \leq t \leq 1. \end{cases}$$

4. Examples

$$\begin{cases} J(u) = \int_0^{2\pi} (\frac{1}{4}v^4 - \frac{1}{3}v^3\text{sin}t - v^2 + w^2 + 2v\text{sin}t - \frac{1}{4}t^2 + \frac{1}{2}u_1^2 + \frac{1}{2}u_2^2)dt \rightarrow \min_u, \\ \mu \frac{dv}{dt} = t + 2u_1, \\ \mu \frac{dw}{dt} = t + 2u_2, \\ v(0, \mu) = 0, w(0, \mu) = 1, v(2\pi, \mu) = 2\sqrt{2}, w(2\pi, \mu) = 1. \end{cases} \tag{4.1}$$

The system (4.1) satisfying Definition 1.1 and Definition 3.1 has the following degenerate solutions

$$\bar{v}_0(t) = \begin{cases} -\sqrt{2}, & 0 \leq t \leq \pi, \\ \sqrt{2}, & \pi \leq t \leq 2\pi, \end{cases} \quad \bar{w}_0(t) = \begin{cases} 0, & 0 \leq t \leq \pi, \\ 0, & \pi \leq t \leq 2\pi, \end{cases} \quad \bar{u}_0(t) = (-\frac{t}{2}, -\frac{t}{2})^T, \quad t_0 = \pi.$$

The zero-order approximation terms of interior layers are

$$\frac{dQ_0^{(\mp)}v}{d\tau} = 2 - (\mp\sqrt{2} + Q_0^{(\mp)}v)^2, \quad Q_0^{(\mp)}v(0) = \pm\sqrt{2}.$$

By calculation, we can obtain

$$Q_0^{(-)}v(\tau) = \frac{2\sqrt{2}e^{2\sqrt{2}\tau}}{1 + e^{2\sqrt{2}\tau}}, \quad Q_0^{(-)}u_1(\tau) = \frac{4e^{2\sqrt{2}\tau}}{(1 + e^{2\sqrt{2}\tau})^2},$$

$$Q_0^{(+)}v(\tau) = \frac{-2\sqrt{2}e^{2\sqrt{2}\tau}}{1 + e^{2\sqrt{2}\tau}}, \quad Q_0^{(+)}u_1(\tau) = \frac{4e^{2\sqrt{2}\tau}}{(1 + e^{2\sqrt{2}\tau})^2}.$$

The zero-order approximation terms of left boundary layers are

$$\frac{dL_0v}{d\tau_0} = (-\sqrt{2} + L_0v)^2 - 2, \quad L_0v(0) = \sqrt{2}. \tag{4.2}$$

The solution of (4.2) is

$$L_0v(\tau) = \frac{2\sqrt{2}e^{-2\sqrt{2}\tau_0}}{1 + e^{-2\sqrt{2}\tau_0}}, \quad L_0u_1(\tau) = \frac{-4e^{-2\sqrt{2}\tau_0}}{(1 + e^{-2\sqrt{2}\tau_0})^2}.$$

The zero-order approximation terms of right boundary layers are

$$\frac{dR_0v}{d\tau_1} = (-\sqrt{2} + R_0v)^2 - 2, \quad R_0v(0) = \sqrt{2}. \quad (4.3)$$

The solution of (4.3) is

$$R_0v(\tau_1) = \frac{2\sqrt{2}}{3e^{-2\sqrt{2}\tau_1} - 1}, \quad R_0u_1(\tau_1) = \frac{12e^{-2\sqrt{2}\tau_1}}{3e^{-2\sqrt{2}\tau_1} - 1}.$$

We can also determine that

$$Q_0^{(\mp)}w(\tau) = 0, \quad Q_0^{(\mp)}u_2(\tau) = 0, \quad \Pi_0w(\tau_0) = e^{-\tau_0}, \quad \Pi_0u_2(\tau_0) = -e^{-\tau_0}, \quad R_0w(\tau_1) = e^{\tau_1}, \quad R_0u_2(\tau_1) = e^{\tau_1}.$$

So the zeroth asymptotic solution of (4.1) is

$$u_1(t, \mu) = \begin{cases} -\frac{t}{2} + \frac{4e^{2\sqrt{2}\tau}}{(1 + e^{2\sqrt{2}})^2} + \frac{-4e^{-2\sqrt{2}\tau_0}}{(1 + e^{-2\sqrt{2}\tau_0})^2}, & 0 \leq t \leq \pi, \\ -\frac{t}{2} + \frac{4e^{2\sqrt{2}\tau}}{(1 + e^{2\sqrt{2}})^2} + \frac{12e^{-2\sqrt{2}\tau_1}}{3e^{-2\sqrt{2}\tau_1} - 1}, & \pi \leq t \leq 2\pi. \end{cases}$$

$$v(t, \mu) = \begin{cases} -\sqrt{2} + \frac{2\sqrt{2}e^{-2\sqrt{2}\tau_0}}{1 + e^{-2\sqrt{2}\tau_0}} + \frac{2\sqrt{2}e^{2\sqrt{2}\tau}}{1 + e^{2\sqrt{2}}}, & 0 \leq t \leq \pi \\ -\sqrt{2} + \frac{-2\sqrt{2}e^{2\sqrt{2}\tau}}{1 + e^{2\sqrt{2}}} + \frac{2\sqrt{2}}{3e^{-2\sqrt{2}\tau_1} - 1}, & \pi \leq t \leq 2\pi. \end{cases}$$

$$w(t, \mu) = \begin{cases} e^{-\tau_0}, & 0 \leq t \leq \pi, \\ e^{\tau_1}, & \pi \leq t \leq 2\pi. \end{cases}$$

$$u_2(t, \mu) = \begin{cases} -e^{-\tau_0}, & 0 \leq t \leq t_0, \\ e^{\tau_1}, & \pi \leq t \leq 2\pi. \end{cases}$$

5. Conclusive Remarks

By the boundary layer function method and optimal control theory, we study the asymptotic solution of contrast structures. By necessary conditions of Euler Equations, we can confirm the existence of the heteroclinic orbits connecting two equilibrium points. Then we obtain the asymptotic solution of the system (1.1). In comparison with [8] and [12], the system we study is more general.

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