Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2900–2913
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Common fixed point results involving contractive condition of integral type in complex valued metric spaces

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Communicated by M. Eslamian

Abstract

By using the Closed Range Property of the involved pairs (in short CLR property), common fixed point results for two pairs of weakly compatible mappings satisfying contractive condition of integral type in complex valued metric spaces are established, which are new even in ordinary metric spaces. We furnish suitable illustrative examples.

Keywords: Complex valued metric spaces, common fixed points, weakly compatible mappings, $(E.A)$ property, (CLR) property.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Fixed point theory is one of the most fruitful and applicable topics of nonlinear analysis, which is very old but still a young area of research. Banach contraction principle [7] is indeed the most popular result of metric fixed point theory. This principle has fruitful application in several domains such as: Ordinary differential equations, Partial differential equations, Random differential equations, Integral equations, Economics, Wild life and several others. Owing to its importance, especially due sound and natural applications, this principle

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Received 2015-12-17

has been extended and generalized in numerous spaces namely: 2-metric spaces, D-metric spaces, G-metric spaces, Partial metric spaces, b-metric spaces, rectangular metric spaces and several others.

Recently, Azam et al. [6] introduced the notion of complex valued metric spaces, which are relatively more general than ordinary metric spaces, and studied fixed point theorems for mappings satisfying a rational type inequality. The authors in [2, 3, 8, 10, 12, 14, 15, 16, 20, 23] continue the study of fixed point in complex valued metric spaces. Verma and Pathak [24] adopted the concepts of (E.A) and (CLR) properties in complex valued metric spaces and utilize the same to prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying a contractive condition of maximum type. Manro et al. [18] proved common fixed point theorems satisfying integral type contractive condition using the (E.A) and (CLR) properties in complex valued metric space which generalize the noted theorem of Branciari [9]. In recent years, the theorem of Branciari [9] had also been generalized to two pairs of weakly compatible mappings, by several authors in metric spaces, which include [4, 5, 11, 14, 17, 21] and some others.

Our aim is to prove common fixed point theorems for two pairs of weakly compatible mappings satisfying contractive condition of integral type in complex valued metric spaces. Furthermore, some common fixed point theorems for two pairs of weakly compatible mappings satisfying integral type contractive condition of maximum type are also studied. Besides, our results also extend the corresponding results of [18] proved in complex valued setting.

To prove our results, we need to recall some basic definitions and results which can also be found in [6]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

Consequently, one can say that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(1) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$;
(2) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$;
(3) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$;
(4) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(1), (2)$ and (3) is satisfied, and we write $z_1 \prec z_2$ if only (3) is satisfied. Notice that

(i) $a, b \in R$ and $a \leq b \Rightarrow az \precsim bz$ for all $z \in \mathbb{C}$;
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$;
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1.1 ([24])**.** The "max" function for the partial order relation "$\precsim$" is defined as follows:

1. $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$;
2. if $z_1 \precsim \max\{z_2, z_3\}$, then $z_1 \precsim z_2$ or $z_1 \precsim z_3$;
3. $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

Azam et al. [6] defined the complex valued metric space (X, d) in the following way:

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies

1. $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$, for all $x, y \in X$;
3. $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and the pair (X, d) is called a complex valued metric space.

Example 1.3 ([10])**.** Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|,$$

where $0 \leq k \leq \frac{\pi}{2}$. Then (X, d) is a complex valued metric space.

Definition 1.4 ([6])**.** Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then x is called the limit of $\{x_n\}$ if for every $c \in \mathbb{C}$ with $0 \prec c$ there is an $n_0 \in N$ such that $d(x_n, x) \prec c$ for all $n > n_0$, and we write $\lim_{n\to\infty} x_n = x$.

Lemma 1.5 ([6])**.** *Any sequence $\{x_n\}$ in complex valued metric space (X, d) converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Definition 1.6 ([22])**.** Let S and T be selfmaps defined on a nonempty set X. Then

(i) $x \in X$ is said to be fixed point of T if $Tx = x$;
(ii) $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$;
(iii) $x \in X$ is said to be a common fixed point of S and T if $Sx = Tx = x$.

Jungck [13] introduced the concept of weakly compatible maps in ordinary metric spaces, while Bhatt et al. [8] defined the same concept in the complex valued metric spaces in the following way.

Definition 1.7. Let X be a complex valued metric space. Then a pair of self-mapping $S, T : X \to X$ are said to be weakly compatible if they commute at their coincidence points i.e., $x \in X$ with $Sx = Tx$ implies that $STx = TSx$.

Aamri and Moutawakil [1] generalized the notion of noncompatible mappings to (E.A) property in ordinary metric spaces, while Verma and Pathak [24] adapted the same concept for complex valued metric space in the following way.

Definition 1.8. Let $T, S : X \to X$ be two selfmaps on a complex-valued metric space (X, d). Then the pair (T, S) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x \ \text{ for some } \ x \in X.$$

Sintunavarat and Kumam [22] introduced the notion of (CLR) property in ordinary metric spaces. Similarly Verma and Pathak [24] defined this notion in a complex valued metric space in the following way.

Definition 1.9. Let $T, S : X \to X$ be two selfmaps on a complex-valued metric space (X, d). Then T and S are said to satisfy the common limit range property of with respect to S (denoted by (CLR$_S$)) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Sx \ \text{ for some } \ x \in X.$$

Lemma 1.10 ([19])**.** *If $\{a_n\}$ is a sequence in $[0, \infty)$, then $\lim_{n\to\infty} \int_0^{a_n} \phi(s)ds = 0$ if and only if $a_n \to 0$, as $n \to \infty$.*

2. Main results

From [9], let $\Phi = \{\phi : \phi : [0, \infty[\to [0, \infty[$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty[$, nonnegative, nondecreasing and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)\, dt > 0\}$.

Now, let $\mathbb{C}_+ = \{z \in \mathbb{C} : z \succsim 0\}$. Then for any $z_1, z_2 \in \mathbb{C}_+$, define

$$[z_1, z_2] = \{r(s) \in \mathbb{C} : r(s) = z_1 + s(z_2 - z_1) \text{ for some } s \in [0, 1]\}, \tag{2.1}$$

$$(z_1, z_2] = \{r(s) \in \mathbb{C} : r(s) = z_1 + s(z_2 - z_1) \text{ for some } s \in (0, 1]\}. \tag{2.2}$$

A set $P = \{z_1 = w_0, w_1, w_2, \ldots, w_n = z_2\}$ is a partition of $[z_1, z_2]$ if and only if the sets $\{[w_{i-1}, w_i)\}_{i=1}^{n}$ are pairwise disjoint and their union along with z_2 is $[z_1, z_2]$.

Let $\zeta : [z_1, z_2] \to \mathbb{C}$ be defined by

$$\zeta(x, y) = (\phi_1(x), \phi_2(y)),$$

where $(x, y) \in [z_1, z_2]$ and $\phi_1, \phi_2 \in \Phi$. Now, for a given partition $\hat{P}$ of $[z_1, z_2]$, we define the lower summation by

$$S_L\left(\zeta, \hat{P}\right) = \sum_{n=0}^{n-1} (\phi_1(x_i), \phi_2(y_i)) \left|(x_{i+1}, y_{i+1}) - (x_i, y_i)\right|,$$

and the upper summation by

$$S_U\left(\zeta, \hat{P}\right) = \sum_{n=0}^{n-1} \zeta(\phi_1(x_{i+1}), \phi_2(y_i)) \left|(x_{i+1}, y_{i+1}) - (x_i, y_i)\right|.$$

Then the integral $\int\limits_{z_1}^{z_2} \zeta d_C$, if exists, is defined by

$$\begin{aligned}\int\limits_{z_1}^{z_2} \zeta d_C &= \lim_{n\to\infty} \sum_{n=0}^{n-1} (\phi_1(x_i), \phi_2(y_i)) \left|(x_{i+1}, y_{i+1}) - (x_i, y_i)\right| \\ &= \lim_{n\to\infty} \sum_{n=0}^{n-1} \zeta(\phi_1(x_{i+1}), \phi_2(y_i)) \left|(x_{i+1}, y_{i+1}) - (x_i, y_i)\right|.\end{aligned}$$

For any $\zeta := (\phi_1, \phi_2) : [(a, b), (c, d)] \to \mathbb{C}$, define

$$\int\limits_{z_1=(a,b)}^{z_2=(c,d)} \zeta d_C = \left(\int\limits_{C_1} \phi_1(s) \left|z_2 - z_1\right| ds, \int\limits_{C_2} \phi_2(s) \left|z_2 - z_1\right| ds \right).$$

Using (2.1), we have

$$\int\limits_{z_1=(a,b)}^{z_2=(c,d)} \zeta d_C = \left(\int\limits_{C_1} \phi_1(s) \left|\acute{r}(s)\right| ds, \int\limits_{C_2} \phi_2(s) \left|\acute{r}(s)\right| ds \right).$$

Particularly for any $\zeta := (\phi_1, \phi_2) : [(0, 0), (a, b)] \to \mathbb{C}$ we have

$$\int\limits_{z_1=(0,0)}^{z_2=(a,b)} \zeta d_C = \left(\int\limits_{0}^{a} \phi_1(s) \left|\acute{r}(s)\right| ds, \int\limits_{0}^{b} \phi_2(s) \left|\acute{r}(s)\right| ds \right).$$

We denote the set of all complex integrable functions $\zeta : [z_1, z_2] \to \mathbb{C}$ by $\mathcal{L}^1([z_1, z_2], \mathbb{C})$.

Lemma 2.1. *Let $\zeta \in \mathcal{L}^1([z_1, z_2], \mathbb{C})$ and $\{z_n\}$ be a sequence in $\mathbb{C}_+$; then $\lim\limits_{n\to\infty} \int\limits_0^{z_n} \zeta(s) ds = (0, 0)$ if and only if $z_n \to (0, 0)$, as $n \to \infty$.*

Proof. From (2.1), we have $r(s) = (0, 0) + s(z_n - (0, 0)) \Rightarrow \acute{r}(s) = z_n$. Then

$$\lim_{n\to\infty} \int\limits_0^{z_n} \zeta(s) ds = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \left(\int\limits_0^{a_n} \phi_1(s) |z_n| ds, \int\limits_0^{b_n} \phi_2(s) |z_n| ds \right) = (0, 0).$$

$$\begin{aligned}
\lim_{n\to\infty}\int_0^{z_n}\zeta(s)\,ds=0 \;&\Leftrightarrow\; \lim_{n\to\infty}\int_0^{a_n}\phi_1(s)\,ds=0 \text{ and } \lim_{n\to\infty}\int_0^{b_n}\phi_2(s)\,ds=0\\
&\Leftrightarrow\; a_n\to 0 \text{ and } b_n\to 0,\ \text{ as } n\to\infty \text{ (by Lemma 1.10)}\\
&\Leftrightarrow\; a_n\to 0 \text{ and } b_n\to 0,\ \text{ as } n\to\infty\\
&\Leftrightarrow\; z_n\to(0,0),\ \text{ as } n\to\infty.
\end{aligned}$$

□

Definition 2.2. A complex valued function $\varphi:\mathbb{R}^n\to\mathbb{C}$ is measurable if both of its real and imaginary parts are measurable.

Now we extend our ideas to complex valued measurable functions. Let $E\subset\mathbb{R}^n$ be a measurable set. Suppose $f:E\to\mathbb{C}$. Split f into its real and imaginary parts so that $f=Re(f)+iIm(f)$. Then we define the Lebesgue integral of f by

$$\int_E f=\int_E Re(f)+i\int_E Im(f)=\left(\int_E Re(f),\int_E Im(f)\right),$$

provided that $Re(f)$ and $Im(f)$ are Lebesgue integrable. Denote the set of all such complex valued lebesgue integrable functions by $\mathcal{L}^1(E,\mathbb{C})$.

We define $\Phi^*=\{\varphi:\mathbb{R}^n\to\mathbb{C}$ is a complex valued Lebesgue-integrable mapping (i.e., $\varphi\in\mathcal{L}^1(E,\mathbb{C})$), which is summable and nonvanishing on each measurable subset of $\mathbb{R}^n$, such that for each $\varepsilon\succ 0$, $\int_0^{\varepsilon}\varphi(t)\,dt\succ 0\}$.

The following remark and lemma are consequences of the above discussion.

Remark 2.3. Let $\varphi\in\Phi^*$, such that $Re(\varphi)$, $Im(\varphi)\in\Phi$ and let $\{z_n\}$ be a sequence in $\mathbb{C}_+$ converging to z; then $\lim_{n\to\infty}\int_0^{z_n}\varphi(s)\,ds=\int_0^{z}\varphi(s)\,ds$.

Lemma 2.4. *Let $\varphi\in\Phi^*$ such that $Re(\varphi)$, $Im(\varphi)\in\Phi$ and let $\{z_n\}$ be a sequence in $\mathbb{C}_+$; then $\lim_{n\to\infty}\int_0^{z_n}\varphi(s)\,ds=0$ if and only if $z_n\to(0,0)$, as $n\to\infty$.*

Now, we present our main results:

Theorem 2.5. *Let (X,d) be a complex valued metric space and $K,L,M,N:X\to X$ be four selfmappings satisfying the conditions:*

1. *either the pair (K,M) has (CLR_K) property or the pair (L,M) has (CLR_L) property;*
2. *for each x,y in X and $0\le\lambda<1$, one has*

$$\int_0^{d(Kx,Ly)}\varphi(t)dt\precsim\lambda\int_0^{d(Kx,Mx)+d(Ly,Ny)}\varphi(t)dt.$$

If $K(X)\subseteq N(X)$ and $L(X)\subseteq M(X)$, then the pairs (K,M) and (L,N) have a coincident point in X. Moreover if the pairs (K,M) and (L,N) are weakly compatible, then the mappings K,L,M and N have a unique common fixed point in X.

Proof. Assume that the pair (K,M) has (CLR$_K$) property. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}Kx_n=\lim_{n\to\infty}Mx_n=Kx \text{ for some } x\in X. \tag{2.3}$$

Since $K(X)\subseteq N(X)$, there exists a $u\in X$ such that $Kx=Nu$.

We assert that $Lu = Nu$. On contrary, let $Lu \neq Nu$. For this, using condition (2) of Theorem 2.5 with $x = x_n$ and $y = u$, it would follow that

$$\int_0^{d(Kx_n,Lu)} \varphi(t)dt \precsim \lambda \int_0^{d(Kx_n,Mx_n)+d(Lu,Nu)} \varphi(t)dt.$$

Taking limit as $n \to \infty$ and making use of (2.3), we would get

$$\int_0^{d(Kx,Lu)} \varphi(t)dt \precsim \lambda \int_0^{d(Lu,Kx)} \varphi(t)dt = \lambda \int_0^{d(Lu,Kx)} \varphi(t)dt \;\Rightarrow\; \left| \int_0^{d(Kx,Lu)} \varphi(t)dt \right| \leq \lambda \left| \int_0^{d(Lu,Kx)} \varphi(t)dt \right|,$$

which is not possible as $0 \leq \lambda < 1$. Thus $Kx = Lu$ and hence $Lu = Nu = Kx$. But $L(X) \subseteq M(X)$, so there exists a $v \in X$ such that $Lu = Mv$. Therefore

$$Lu = Nu = Mv = Kx. \tag{2.4}$$

Now, we claim that $Kv = Mv$. To substantiate our claim, let suppose that $Kv \neq Mv$. Then on setting $x = v$, $y = u$ in condition (2) of Theorem 2.5, we would have

$$\int_0^{d(Kv,Lu)} \varphi(t)dt \precsim \lambda \int_0^{d(Kv,Mv)+d(Lu,Nu)} \varphi(t)dt,$$

which on using equation (2.4), would yield

$$\int_0^{d(Kv,Kx)} \varphi(t)dt \precsim \lambda \int_0^{d(Kx,Kv)} \varphi(t)dt \quad \Rightarrow \left| \int_0^{d(Kv,Kx)} \varphi(t)dt \right| \leq \lambda \left| \int_0^{d(Kx,Kv)} \varphi(t)dt \right|,$$

which is possible only if $\left| \int_0^{d(Kv,Kx)} \varphi(t)dt \right| = 0$, yielding thereby $Kv = Kx$. Therefore from (2.4), we have

$$Kv = Lu = Mv = Nu = Kx = z \text{ (say)}. \tag{2.5}$$

Now, using the weak compatibility of the pairs (K, M), (L, N) and (2.5), it follows that

$$Kv = Mv \quad \Rightarrow KMv = MKv \quad \Rightarrow \quad Kz = Mz \tag{2.6}$$

and

$$Nu = Lu \quad \Rightarrow LNu = NLu \quad \Rightarrow \quad Lz = Nz. \tag{2.7}$$

That is, z is a coincident point of each of the pairs (K, M) and (L, N) in X.

Next, we confirm that z is a common fixed point of K, L, M and N in X. For this, using condition (2) of Theorem 2.5 with $x = z$ and $y = u$, we have

$$\int_0^{d(Kz,Lu)} \varphi(t)dt \precsim \lambda \int_0^{d(Kz,Mz)+d(Lu,Nu)} \varphi(t)dt,$$

which, on using equation (2.5), gives

$$\int_0^{d(Kz,z)} \varphi(t)dt \precsim \lambda \int_0^{d(Kz,Kz)+d(z,z)} \varphi(t)dt = 0.$$

Thus, $d(Kz, z) = 0 \Rightarrow Kz = z$ and hence from equation (2.6), it follows that

$$Kz = Mz = z. \tag{2.8}$$

Similarly, setting $x = v$ and $y = z$ in condition (2) of Theorem 2.5, we get $Lz = z$, which in view of equation (2.7) gives

$$Lz = Nz = z. \tag{2.9}$$

Making use equations (2.8) and (2.9), we get

$$Kz = Lz = Mz = Nz = z.$$

That is z is a common fixed point of K, L, M and N in X.

To prove the uniqueness of the common fixed point, let $z^* \neq z$ be another fixed point of K, L, M and N, i.e., $Kz^* = Lz^* = Mz^* = Nz^* = z^*$. Using condition (2) of Theorem 2.5, we would have

$$\int_0^{d(z^*,z)} \varphi(t)dt = \int_0^{d(Kz^*,Lz)} \varphi(t)dt \precsim \lambda \int_0^{d(Kz^*,Mz^*)+d(Lz,Nz)} \varphi(t)dt,$$
$$\Rightarrow \quad \left| \int_0^{d(z^*,z)} \varphi(t)dt \right| \leq \lambda \left| \int_0^{d(z^*,z^*)+d(z,z)} \varphi(t)dt \right| = 0,$$

which is a contradiction. Thus $z = z^*$ and hence z is a unique common fixed point of K, L, M and N in X. □

By setting $N = M$ in Theorem 2.5, we get the following corollary involving three mappings.

Corollary 2.6. *Let (X, d) be a complex valued metric space and $K, L, M : X \to X$ be three self-mappings satisfying the conditions:*

1. *either the pair (K, M) has (CLR_K) property or the pair (L, M) has (CLR_L) property;*
2. *for each $x, y \in X$ and $0 \leq \lambda < 1$, one has*

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda \int_0^{d(Kx,Mx)+d(Ly,My)} \varphi(t)dt.$$

If $K(X) \subseteq M(X)$ and $L(X) \subseteq M(X)$, then each of the pairs (K, M) and (L, M) has a coincident point in X. Moreover if the pairs (K, M) and (L, M) are weakly compatible, then the mappings K, L and M have a unique common fixed point in X.

By setting $L = K$ and $N = M$ in Theorem 2.5, we get the following corollary involving a pair of mappings.

Corollary 2.7. *Let (X, d) be a complex valued metric space and $K, M : X \to X$ selfmappings satisfying the conditions:*

1. *the pair (K, M) has (CLR_K) property;*
2. *for each $x, y \in X$ and $0 \leq \lambda < 1$, one has*

$$\int_0^{d(Kx,Ky)} \varphi(t)dt \precsim \lambda \int_0^{d(Kx,Mx)+d(Ky,My)} \varphi(t)dt.$$

If $K(X) \subseteq M(X)$, then the pair (K, M) have a coincident point in X. Moreover if the pair (K, M) are weakly compatible, then the mappings K and M have a unique common fixed point in X.

If we replace (CLR) property by (E.A) property in Theorem 2.5, we get the following corollary:

Corollary 2.8. *Let (X, d) be a complex valued metric space and $K, L, M, N : X \to X$ four selfmappings satisfying condition (2) of* Theorem 2.5 *and one of the pairs (K, M) and (L, N) has $(E.A)$ property such that $N(X)$ (or $M(X)$) is a closed subspace of X. If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, then each pair (K, M) and (L, N) have a coincidence point in X. Moreover if the pairs (K, M) and (L, N) are weakly compatible, then the mappings K, L, M and N have a unique common fixed point in X.*

Proof. Since the (E.A) property together with the closedness property of a suitable subspace imply closed range property, the proof follows on the lines of the proof of Theorem 2.5. Hence, it is omitted. □

To illustrate Theorem 2.5, we construct the following example:

Example 2.9. Let $X = (-1,1) \cup (1,5)$ be a metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x,y) = e^{im}|x-y|$, where $x,y \in X$ and $0 \le m \le \frac{\pi}{6}$. Define selfmaps K, L, M and N on X by

$$Kx = \begin{cases} 3 & \text{if } x \in (-1,1)\cup(1,3] \\ \frac{x-1}{2} & \text{if } x \in (3,5) \end{cases}; \qquad Lx = \begin{cases} 3 & \text{if } x \in (-1,1)\cup(1,3] \\ \frac{x+1}{2} & \text{if } x \in (3,5) \end{cases};$$

$$Mx = \begin{cases} 2x-3 & \text{if } x \in (1,3] \\ 4 & \text{if } x \in (-1,1)\cup(3,5) \end{cases} \text{ and } Nx = \begin{cases} x-1 & \text{if } x \in (1,3) \\ 3 & \text{if } x = 3 \\ 5 & \text{if } x \in (-1,1)\cup(3,5) \end{cases}.$$

Also define $\varphi : \mathbb{R}^2 \to \mathbb{C}$ by $\varphi(t) = 3z^2$, where $t = (a,b)$ and $z = a+ib$. Then

$$K(X) = (1,2)\cup\{3\}, \quad L(X) = (2,3], \quad M(X) = (-1,3]\cup\{4\}, \quad N(X) = (0,2)\cup\{3,5\}.$$

Firstly, we verify condition (1) of Theorem 2.5. For this, let $\{x_n\} = \{3 - \frac{1}{n^2+1}\}_{n\ge1}$ be a sequence in X. Then

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} K\left(3 - \frac{1}{n^2+1}\right) = \lim_{n\to\infty} 3 = 3 \quad \text{and}$$
$$\lim_{n\to\infty} Mx_n = \lim_{n\to\infty} M\left(3 - \frac{1}{n^2+1}\right) = \lim_{n\to\infty}\left(2(3 - \frac{1}{n^2+1}) - 3\right) = 3,$$

i.e., there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = 3 = Kx$ for all $x \in (1,3]$. Thus

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = 3 = Kx \quad \text{for some} \quad x \in X.$$

Hence (K,M) has (CLR_K) property.

To check condition (2) of Theorem 2.5, we distinguish the following three cases.

Case 1. Let $x,y \in (1,3)$; then $Kx = Ky = 3, Mx = 2x-3$ and $Ny = y-1$.

Now, for all $\lambda \in [0,1)$ we have

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 \precsim \lambda(|6-2x|+|4-y|)^3 e^{3im} = \lambda\int_0^{d(Kx,Mx)+d(Ly,Ny)} \varphi(t)dt.$$

Case 2. Let $x,y \in (-1,1)\cup\{3\}$; then $Kx = Ly = 3, Mx = 4, Ny = 5$ and

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 = \lambda\int_0^{d(Kx,Mx)+d(Ly,Ny)} \varphi(t)dt, \quad \forall \ \lambda \in [0,1).$$

Case 3. Let $x,y \in (3,5)$; then $Kx = \frac{x-1}{2}, Ly = \frac{y+1}{2}, Mx = 4$ and $Ny = 5$.

Now

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = \int_0^{e^{im}|\frac{x-1}{2} - \frac{y+1}{2}|} 3z^2dt = z^3\Big|_0^{e^{im}|\frac{x-y-2}{2}|} = \frac{1}{8}|x-y-2|^3e^{3im} \prec 8e^{3im},$$

and

$$\int_0^{d(Kx,Mx)+d(Ly,Ny)} \varphi(t)dt = z^3\Big|_0^{e^{im}(|\frac{x-1}{2}-4|+|\frac{y+1}{2}-5|)} = \frac{1}{8}(|x-9|+|y-9|)^3e^{3im} \succ 64e^{3im}.$$

Thus

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \prec 8e^{3im} \precsim \lambda 64 e^{3im} \prec \lambda \int_0^{d(Kx,Mx)+d(Ly,Ny)} \varphi(t)dt,$$

$$\Rightarrow \int_0^{d(Kx,Ly)} \varphi(t)dt \prec \lambda \int_0^{d(Kx,Mx)+d(Ly,Ny)} \varphi(t)dt, \ \forall \ \lambda \in \left[\frac{1}{8}, 1\right).$$

Therefore, in view of foregoing three cases, the integral contractive condition (2) is satisfied.

Also $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$ and the pairs (K, M) and (L, N) are weakly compatible. Thus all the conditions of Theorem 2.5 are satisfied and 3 is a unique common fixed point of K, L, M and N.

To illustrate Corollary 2.8, we construct the following example.

Example 2.10. Let $X = (1, 3] \cup [4, 6]$ be a metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = e^{im}|x - y|$, where $x, y \in X$ and $0 \leq m \leq \frac{\pi}{6}$. Define selfmaps K, L, M and N on X by

$$Kx = \begin{cases} \frac{3}{2} & \text{if } x \in (1, \frac{3}{2}] \cup [4, 6] \\ 2 & \text{if } x \in (\frac{3}{2}, 3] \end{cases}; \qquad Lx = \begin{cases} \frac{3}{2} & \text{if } x \in (1, \frac{3}{2}] \cup [4, 6] \\ 3 & \text{if } x \in (\frac{3}{2}, 3] \end{cases};$$

$$Mx = \begin{cases} 3 & \text{if } x \in \left(1, \frac{3}{2}\right) \\ \frac{3}{2} & \text{if } x \in \{\frac{3}{2}\} \cup [4, 6] \\ 2x & \text{if } x \in \left(\frac{3}{2}, 3\right] \end{cases} \text{ and } \quad Nx = \begin{cases} 3 - x & \text{if } x \in \left(1, \frac{3}{2}\right) \\ \frac{3}{2} & \text{if } x \in \{\frac{3}{2}\} \cup [4, 6] \\ 4 & \text{if } x \in \left(\frac{3}{2}, 3\right] \end{cases}.$$

Also define $\varphi : \mathbb{R}^2 \to \mathbb{C}$ by $\varphi(t) = 3z^2$, where $t = (a, b)$ and $z = a + ib$. Then by routine calculation one can verify all the conditions of Corollary 2.8, so that $\frac{3}{2}$ is a unique common fixed point of K, L, M and N.

Our next theorem is proved under maximum integral contractive condition.

Theorem 2.11. *Let (X, d) be a complex valued metric space and $K, L, M, N : X \to X$ four selfmappings satisfying the conditions:*

(1) *either the pair (K, M) has (CLR_K) property or the pair (L, N) has (CLR_L) property;*
(2) *for each $x, y \in X$ and $0 \leq \lambda < 1$, one has*

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda \ \max\left(\int_0^{d(Ly,Mx)[1+d(Kx,Mx)d(Kx,Ny)]} \varphi(t)dt, \int_0^{d(Kx,Ny)[1+d(Ly,Mx)d(Ly,Ny)]} \varphi(t)dt \right).$$

If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, then each of the pairs (K, M) and (L, N) have a coincidence point in X. Moreover if the pairs (K, M) and (L, N) are weakly compatible, then the mappings K, L, M and N have a unique common fixed point in X.

Proof. Let the pair (K, M) have (CLR_K) property; then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = Kx \ \text{ for some } \ x \in X. \tag{2.10}$$

Since $K(X) \subseteq N(X)$, there exists a $u \in X$ such that $Kx = Nu$.

We claim that $Lu = Nu$. If not, then on using condition (2) of Theorem 2.11 with $x = x_n$ and $y = u$, it would follow that

$$\int_0^{d(Kx_n,Lu)} \varphi(t)dt \precsim \lambda \ \max\left(\int_0^{d(Mx_n,Lu)[1+d(Mx_n,Kx_n)d(Nu,Kx_n)]} \varphi(t)dt, \right.$$
$$\left. \int_0^{d(Nu,Kx_n)[1+d(Mx_n,Lu)d(Nu,Lu)]} \varphi(t)dt \right).$$

Taking limit as $n \to \infty$ and using (2.10), we would have

$$\int_0^{d(Kx,Lu)} \varphi(t)dt \precsim \lambda \ \max\left(\int_0^{d(Kx,Lu)} \varphi(t)dt, 0\right) = \lambda \int_0^{d(Kx,Lu)} \varphi(t)dt,$$

implies that $\left|\int_0^{d(Kx,Lu)} \varphi(t)dt\right| \leq \lambda \ \left|\int_0^{d(Kx,Lu)} \varphi(t)dt\right|$, which is possible only if $\left|\int_0^{d(Kx,Lu)} \varphi(t)dt\right| = 0$, thus $Lu = Kx$ and $Lu = Nu = Kx$. But $L(X) \subseteq M(X)$, so there exists $v \in X$ such that $Lu = Mv$. Hence

$$Lu = Nu = Mv = Kx. \tag{2.11}$$

Next, we show that $Kv = Mv$. Let, on contrary, $Kv \neq Mv$; then on putting $x = v$ and $y = u$ in condition (2) of Theorem 2.11, we would obtain

$$\int_0^{d(Kv,Lu)} \varphi(t)dt \precsim \lambda \ \max\left(\int_0^{d(Mv,Lu)[1+d(Mv,Kv)d(Nu,Kv)]} \varphi(t)dt, \int_0^{d(Nu,Kv)[1+d(Mv,Lu)d(Nu,Lu)]} \varphi(t)dt\right).$$

Using equation (2.11), we would have

$$\int_0^{d(Kv,Kx)} \varphi(t)dt \precsim \lambda \ \max\left(0, \int_0^{d(Kx,Kv)} \varphi(t)dt\right), = \lambda \int_0^{d(Kx,Kv)} \varphi(t)dt$$
$$\Rightarrow \ \left|\int_0^{d(Kv,Kx)} \varphi(t)dt\right| \leq \lambda \ \left|\int_0^{d(Kx,Kv)} \varphi(t)dt\right|,$$

which is contradiction. Thus $Kv = Kx$ and hence $Kv = Mv = Kx$. Therefore from equation (2.11), it follows that

$$Kv = Lu = Nu = Mv = Kx = z(say). \tag{2.12}$$

Now, using the weak compatibility of the pairs (K, M), (L, N) and equation (2.12) it follows that

$$Kv = Mv \quad \Rightarrow KMv = MKv \quad \Rightarrow \quad Kz = Mz \tag{2.13}$$

and

$$Nu = Lu \quad \Rightarrow LNu = NLu \quad \Rightarrow \quad Lz = Nz. \tag{2.14}$$

That is z is a coincident point of each pair (K, M) and (L, N) in X.

Next, we have to show that z is a common fixed point of K, L, M and N in X. For this, by putting $x = z$ and $y = u$ in condition (2) of Theorem 2.11, we have

$$\int_0^{d(Kz,Lu)} \varphi(t)dt \precsim \lambda \ \max\left(\int_0^{d(Lu,Mz)[1+d(Kz,Mz)d(Kz,Nu)]} \varphi(t)dt, \int_0^{d(Kz,Nu)[1+d(Lu,Mz)d(Lu,Nu)]} \varphi(t)dt\right).$$

Using equations (2.12) and (2.13), we get

$$\int_0^{d(Kz,z)} \varphi(t)dt \precsim \lambda \max\left(\int_0^{d(z,Kz)} \varphi(t)dt, \int_0^{d(Kz,z)} \varphi(t)dt\right) = \int_0^{d(Kz,z)} \varphi(t)dt,$$

which is possible if $\int_0^{d(Kz,z)} \varphi(t)dt = 0$, thus $Kz = z$ and hence from equation (2.13), we get

$$Kz = Mz = z. \tag{2.15}$$

Similarly, if we put $x = v$ and $y = z$ in condition (2) of Theorem 2.11, we get

$$Lz = Nz = z. \tag{2.16}$$

Using equations (2.15) and (2.16), we get

$$Kz = Mz = Lz = Nz = z. \tag{2.17}$$

That is z is a common fixed point of K, L, M and N in X.

To prove the uniqueness of the common fixed point, let $z^* \neq z$ be another fixed point of K, L, M and N, i.e., $Kz^* = Lz^* = Mz^* = Nz^* = z^*$. Then using condition (2) of Theorem 2.11, we would have

$$\begin{aligned}
\int_0^{d(z,z^*)} \varphi(t)dt = \int_0^{d(Kz,Lz^*)} \varphi(t)dt \\
&\precsim \lambda \max\left(\int_0^{d(Lz^*,Mz)[1+d(Kz,Mz)d(Kz,Nz^*)]} \varphi(t)dt, \int_0^{d(Kz,Nz^*)[1+d(Lz^*,Mz)d(Lz^*,Nz^*)]} \varphi(t)dt \right) \\
&\precsim \lambda \max\left(\int_0^{d(z,z^*)} \varphi(t)dt, \int_0^{d(z,z^*)} \varphi(t)dt \right) \\
&= \int_0^{d(z,z^*)} \varphi(t)dt,
\end{aligned}$$

which is not possible. Thus $z = z^*$ and z remains a unique common fixed point of K, L, M and N in X. □

By setting $N = M$ in Theorem 2.11, we get the following corollary involving three mappings.

Corollary 2.12. *Let (X, d) be a complex valued metric space and $K, L, M : X \to X$ three selfmappings satisfying the conditions:*

1. *either the pair (K, M) has (CLR_K) property or the pair (L, M) has (CLR_L) property;*
2. *for each $x, y \in X$ and $0 \leq \lambda < 1$, one has*

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda \max\left(\int_0^{d(Mx,Ly)[1+d(Mx,Kx)d(My,Kx)]} \varphi(t)dt, \int_0^{d(My,Kx)[1+d(Mx,Ly)d(My,Ly)]} \varphi(t)dt \right).$$

If $K(X) \subseteq M(X)$ and $L(X) \subseteq M(X)$, then each of the pairs (K, M) and (L, M) have a coincidence point in X. Moreover if the pairs (K, M) and (L, M) are weakly compatible, then the mappings K, L and M have a unique common fixed point in X.

By setting $L = K$ and $N = M$ in Theorem 2.11, we get the following corollary involving a pair of mappings.

Corollary 2.13. *Let (X, d) be a complex valued metric space and $K, M : X \to X$ two selfmappings satisfying the conditions:*

1. *the pair (K, M) has (CKR_K) property;*
2. *for each $x, y \in X$ and $0 \leq \lambda < 1$, one has*

$$\int_0^{d(Kx,Ky)} \varphi(t)dt \precsim \lambda \max\left(\int_0^{d(Ky,Mx)[1+d(Kx,Mx)d(Kx,My)]} \varphi(t)dt, \int_0^{d(Kx,My)[1+d(Ky,Mx)d(Ky,My)]} \varphi(t)dt \right).$$

If $K(X) \subseteq M(X)$, then each pair (K, M) have a coincidence point in X. Moreover if the pair (K, M) is weakly compatible, then the mappings K and M have a unique common fixed point in X.

If we replace (CLR) property by (E.A) property in Theorem 2.11, we get the following corollary.

Corollary 2.14. *Let (X, d) be a complex valued metric space and $K, L, M, N : X \to X$ four selfmappings satisfying condition (2) of* Theorem 2.11 *and one of the pairs (K, M) and (L, N) has $(E.A)$ property such that $N(X)$ (or $M(X)$) is a closed subspace of X. If $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$, then each of the pairs (K, M) and (L, N) have a coincidence point in X. Moreover if the pairs (K, M) and (L, N) are weakly compatible, then the mappings K, L, M and N have a unique common fixed point in X.*

Proof. As indicated earlier, the proof easily follows on the lines of the proof of Theorem 2.11, so we omit it. □

To illustrate Theorem 2.11, we provide the following example.

Example 2.15. Let $X = \{\frac{-1}{3}\} \cup (0, \frac{3}{2})$ be a metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = i|x - y|$, where $x, y \in X$. Define selfmaps K, L, M and N on X by

$$Kx = \begin{cases} \frac{1}{6} & \text{if } x \in \{\frac{-1}{3}\} \cup (0, \frac{1}{2}) \cup [1, \frac{3}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases}; \qquad Lx = \begin{cases} \frac{1}{7} & \text{if } x \in \{\frac{-1}{3}\} \cup (0, \frac{1}{2}) \cup [1, \frac{3}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases};$$
$$Mx = \begin{cases} \frac{-1}{3} & \text{if } x \in \{\frac{-1}{3}\} \cup (0, \frac{1}{2}) \cup [1, \frac{3}{2}) \\ 1 - x & \text{if } x \in [\frac{1}{2}, 1) \end{cases}; \quad Nx = \begin{cases} \frac{1}{6} & \text{if } x \in \{\frac{-1}{3}\} \cup (0, \frac{1}{2}) \cup [1, \frac{3}{2}) \\ 2x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases}.$$

Also define $\varphi : \mathbb{R}^2 \to \mathbb{C}$ by $\varphi(t) = 5z^4$, where $t = (a, b)$ and $z = a + ib$. Then

$$K(X) = \left\{\frac{1}{6}, \frac{1}{2}\right\}, \quad L(X) = \left\{\frac{1}{7}, \frac{1}{2}\right\}, \quad M(X) = \left(0, \frac{1}{2}\right] \cup \left\{\frac{-1}{3}\right\}, \quad N(X) = \left\{\frac{1}{6}\right\} \cup \left[\frac{1}{2}, \frac{3}{2}\right).$$

Firstly, we verify condition (2) of Theorem 2.11. For this let $\{x_n\} = \{\frac{1}{2} + \frac{1}{n+2}\}_{n \geq 1}$ be a sequence in X. Then

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} K\left(\frac{1}{2} + \frac{1}{n+2}\right) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$
$$\text{and} \quad \lim_{n\to\infty} Mx_n = \lim_{n\to\infty} M\left(\frac{1}{2} + \frac{1}{n+2}\right) = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2},$$

that is, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Mx_n = \frac{1}{2} = Kx$ for some $x \in X$.

Hence (K, M) has (CLR_K) property.

To check condition (2) of Theorem 2.11, we distinguish the following cases.

Case 1. Let $x, y \in \{\frac{-1}{3}\} \cup (0, \frac{1}{2}) \cup [1, \frac{3}{2})$; then $Kx = \frac{1}{6}, Ly = \frac{1}{7}, Mx = \frac{-1}{3}$ and $Ny = \frac{1}{6}$. Now

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = z^5\Big|_0^{\frac{i}{42}} = \left(\frac{i}{42}\right)^5 = 7.651622719 \times 10^{-9} i.$$

Also

$$\int_0^{d(Ly,Mx)[1+d(Kx,Mx)d(Kx,Ny)]} \varphi(t)dt = z^5\Big|_0^{\frac{10i}{21}[1-\frac{1}{2}.0]} = \left(\frac{10i}{21}\right)^5 = 0.0244851927i.$$

Hence for all $\lambda \in \left[\frac{1}{3200000}, 1\right)$ we have

$$\begin{aligned} \int_0^{d(Kx,Ly)} \varphi(t)dt =& 7.651622719 \times 10^{-9} i \\ \precsim& \lambda 0.0244851927i \\ =& \lambda \max\left(\int_0^{d(Ly,Mx)[1+d(Kx,Mx)d(Kx,Ny)]} \varphi(t)dt, \int_0^{d(Kx,Ny)[1+d(Ly,Mx)d(Ly,Ny)]} \varphi(t)dt\right). \end{aligned}$$

Case 2. Let $x, y \in [\frac{1}{2}, 1)$; then $Kx = Ly = \frac{1}{2}, Mx = 1 - x$ and $Ny = 2y - \frac{1}{2}$. Now, for all $\lambda \in [0, 1)$ we have

$$\int_0^{d(Kx,Ly)} \varphi(t)dt = 0 \precsim \lambda \max\left(\int_0^{d(Ly,Mx)[1+d(Kx,Mx)d(Kx,Ny)]} \varphi(t)dt, \int_0^{d(Kx,Ny)[1+d(Ly,Mx)d(Ly,Ny)]} \varphi(t)dt\right).$$

Therefore from the above two cases, it follows that for all $\lambda \in \left[\frac{1}{3200000}, 1\right)$ we have

$$\int_0^{d(Kx,Ly)} \varphi(t)dt \precsim \lambda \max\left(\int_0^{d(Ly,Mx)[1+d(Kx,Mx)d(Kx,Ny)]} \varphi(t)dt. \int_0^{d(Kx,Ny)[1+d(Ly,Mx)d(Ly,Ny)]} \varphi(t)dt\right).$$

Also $K(X) \subseteq N(X)$ and $L(X) \subseteq M(X)$ and the pairs (K, M) and (L, N) are weakly compatible. Thus all the conditions of Theorem 2.11 are satisfied, and $\frac{1}{2}$ is a unique common fixed point of K, L, M and N.

To illustrate Corollary 2.14, we furnish the following example.

Example 2.16. Let $X = \{\frac{-1}{3}\} \cup [0, 1)$ be a metric space with metric $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = i|x - y|$, where $x, y \in X$. Define selfmaps K, L, M and N on X by

$$Kx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{8} & \text{if } x \in \{\frac{-1}{3}\} \cup [\frac{1}{2}, 1) \end{cases}; \qquad Lx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{9} & \text{if } x \in \{\frac{-1}{3}\} \cup [\frac{1}{2}, 1) \end{cases};$$

$$Mx = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}) \\ \frac{-1}{3} & \text{if } x \in \{\frac{-1}{3}\} \cup [\frac{1}{2}, 1) \end{cases} \quad \text{and} \quad Nx = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in \{\frac{-1}{3}\} \cup [\frac{1}{2}, 1) \end{cases}.$$

Also define $\varphi : \mathbb{R}^2 \to \mathbb{C}$ by $\varphi(t) = 5z^4$, where $t = (a, b)$ and $z = a + ib$. Then by routine calculations, one can verify all the conditions of Corollary 2.14 and 0 is a unique common fixed point of K, L, M and N.

Remark 2.17. The derived results generalize the results contained in [18].

Acknowledgements

The authors are grateful to the editor and the anonymous referee for his/her helpful comments.

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