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Viscosity splitting methods for variational inclusion and fixed point problems in Hilbert spaces

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Abstract

In this paper, a viscosity splitting method is investigated for treating variational inclusion and fixed point problems. Strong convergence theorems of common solutions are established in the framework of Hilbert spaces. Applications are also provided to support the main results.

Keywords: Convex feasibility problem, fixed point, iterative process, monotone operator, splitting algorithm.
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1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle x, y\rangle$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$ for $x, y \in H$. Let C be a nonempty convex and closed subset of H.

Let $T : C \to C$ be a mapping. In this paper, we use $Fix(T)$ to stand for the set of fixed points of T. Recall that T is said to be an α-contractive mapping iff there exists a constant α with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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If C is also bounded, then the set of fixed points of S is not empty; see [5] and the references therein. In the real world, many important problems have reformulations which require finding fixed points of nonexpansive mapping. Mann iteration is powerful to study fixed points of nonexpansive mappings. However it is only weakly convergent. Recently, many authors studied the problem of modifying Mann iteration so that strong convergence is guaranteed without any compactness assumption; see [7, 8, 12, 13, 14, 16, 24, 25, 26, 27] and the references therein.

T is said to be a λ-strict pseudocontraction iff there exists a constant λ with $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\| + \lambda\|x - Tx - y + Ty\|^2, \quad \forall x, y \in C.$$

The class of λ-strict pseudocontractions was introduced by Browder and Petryshyn [6] in 1967. It is clear that the class of λ-strict pseudocontractions strictly include the class of nonexpansive mappings as a special cases. It is also known that every λ-strict pseudocontraction is Lipschitz continuous; see [6] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be λ-strongly monotone iff there exists a positive constant $\lambda > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \lambda\|x - y\|^2, \quad \forall x, y \in C.$$

A is said to be inverse λ-strongly monotone iff there exists a positive constant $\lambda > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \lambda\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

From the above, we see that A is inverse λ-strongly monotone iff A^{-1} is strongly monotone. A is said to be L-Lipschitz continuous iff there exists a positive constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

It is obvious that A is inverse λ-strongly monotone, then A is also monotone and $\frac{1}{\lambda}$-Lipschitz continuous.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of variational inequality (1.1) is denoted by $VI(C, A)$. Projection methods have been recently investigated for solving variational inequality (1.1). Let $Proj_C$ be the metric projection from H onto C and I the identity on H. It is known that x is a solution to (1.1) iff x is a fixed point of mapping $Proj_C(I - rA)$. If A is inverse λ-strongly monotone, then $Proj_C(I - rA)$ is nonexpansive. If C is bounded, closed and convex, then the existence of solutions of variational inequality (1.1) is guaranteed by the nonexpansivity of mapping $Proj_C(I - rA)$.

Recall that an operator $B : H \rightrightarrows H$ is said to be monotone iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \geq 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \rightrightarrows H$ is maximal iff the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in G(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r^B = (I + rB)^{-1}$, where $D(B)$ denote the domain of B. It is known that $J_r^B : H \to D(B)$ is firmly nonexpansive, and $B^{-1}(0) = Fix(J_r^B)$. The property of the resolvent ensures that the Picard iterative algorithm $x_{n+1} = J_r^B x_n$ converge weakly to a fixed point of J_r^B, which is necessarily a zero point of B. Rockafellar introduced this iteration method and call it the proximal point algorithm (PPA); for more detail, see [20, 23] and the references therein. The PPA and its dual version in the context of convex programming, the method of multipliers of Hesteness and

Powell, have been extensively studied and are known to yield as special cases decomposition methods such as the method of partial inverses [22], the Douglas-Rachford splitting method, and the alternating direction method of multipliers [10]. In the case of $B = B_1 + B_2$, where B_1 and B_2 are maximal monotone on H, the forward-backward splitting method $x_{n+1} = (I + r_nB_1)^{-1}(I - r_nB_2)x_n$, $n = 0, 1, \cdots$, where $r_n > 0$, was proposed by Lions and Mercier [15], and in a dual form for convex programming, by Han and Lou [11]. In the case where $B_1 = N_C$, this method reduces to a projection method proposed by Sibony [21] for monotone variational inequalities (1.1). Recently, many authors have studied the splitting algorithm; see [2, 3, 4, 9, 17, 18, 19] and the references therein.

In this paper, a viscosity splitting method is investigated for treating a inclusion problem with two monotone operators and a fixed point problem of λ-strict pseudocontractions. Strong convergence theorems of common solutions are established in the framework of Hilbert spaces. Applications are also provided to support the main results.

The following lemmas are essential to prove our main results.

Lemma 1.1 ([3]). *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $A : C \to H$ be a mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $F(J_r(I - rA)) = (A + B)^{-1}(0)$.*

Lemma 1.2 ([25]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \leq (1 - t_n)a_n + t_nb_n + c_n$, $\forall n \geq 0$, where $\{t_n\}$ is a number sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \leq 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.3 ([1]). *Let H be a Hilbert space, and A an maximal monotone operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 1.4 ([6]). *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudocontraction. Define a mapping S by $S = \beta I + (1 - \beta)T$. If beta $\in [\lambda, 1)$, then S is nonexpansive and $Fix(T) = Fix(S)$.*

Lemma 1.5 ([6]). *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudocontraction. Then T is Lipschitz continuous and $I - T$ is demiclosed at zero.*

2. Main results

Theorem 2.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an inverse κ-strongly monotone mapping and let B be a maximal monotone operator on H. Let $f : C \to C$ be a fixed α-contraction and let $T : C \to C$ be a λ-strict pseudocontraction. Assume that $(A + B)^{-1}(0) \cap Fix(T)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real number sequences in $(0, 1)$ and let $\{r_n\}$ be a real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C generated in the following process: $x_0 \in C$, $x_{n+1} = \beta_ny_n + (1-\beta_n)Ty_n$, $\forall n \geq 0$, where $\{y_n\}$ is a sequence in C such that $\|y_n - (I + r_nB)^{-1}\big(\alpha_nf(x_n) + (1-\alpha_n)x_n - r_nA\big(\alpha_nf(x_n) + (1 - \alpha_n)x_n\big)\big)\| \leq e_n$. Assume that the control sequences satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $0 < a \leq r_n \leq a' < 2\kappa$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, and $\lambda \leq \beta_n \leq a'' < 1$, where a, a' and a'' are three real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T) \cap (A + B)^{-1}(0)$, where $\bar{x} = Proj_{Fix(T)\cap(A+B)^{-1}(0)}f(\bar{x})$, that is, $\bar{x}$ solves the following variational inequality $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$, $\forall x \in Fix(T) \cap (A + B)^{-1}(0)$.*

Proof. Since A is inverse κ-strongly monotone, one has

$$\begin{aligned}\|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + {r_n}^2\|Ax - Ay\|^2\\ &\leq \|x - y\|^2 - r_n(2\kappa - r_n)\|Ax - Ay\|^2.\end{aligned}$$

From the restriction imposed on $\{r_n\}$, one has $I - r_nA$ is nonexpansive. Setting $T_n = \beta_nI + (1 - \beta_n)T$, where I is the identity, one sees from Lemma 1.4 that T_n is nonexpansive with $Fix(T_n) = Fix(T)$. Fixing $p \in Fix(T) \cap (A+B)^{-1}(0)$, one has from Lemma 1.1 that $p = T_n p = (I + r_n B)^{-1}(p - r_n Ap)$. Setting $z_n = \alpha_n f(x_n) + (1-\alpha_n)x_n$, one has

$$\begin{aligned}\|z_n - p\| &\le \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \big(1 - \alpha_n(1-\alpha)\big)\|x_n - p\| + \alpha_n\|f(p) - p\|.\end{aligned}$$

Hence, one has

$$\begin{aligned}\|x_{n+1} - p\| &\le \|y_n - p\| \\ &\le \|y_n - (I + r_n B)^{-1}\big(z_n - r_n A z_n\big)\| + \|\big(z_n - r_n A z_n\big) - \big(p - r_n Ap\big)\| \\ &\le \|z_n - p\| + e_n \\ &\le \big(1 - \alpha_n(1-\alpha)\big)\|x_n - p\| + \alpha_n\|f(p) - p\| + e_n \\ &\le \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha}\} + e_n.\end{aligned}$$

By mathematical induction, one finds that sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Notice that

$$\|z_n - z_{n-1}\| \le |\alpha_n - \alpha_{n-1}|\|x_{n-1} - f(x_{n-1})\| + \big(1 - \alpha_n(1-\alpha)\big)\|x_{n-1} - x_n\|. \tag{2.1}$$

Putting $w_n = z_n - r_n A z_n$, we find from (2.1) that

$$\begin{aligned}\|w_n - w_{n-1}\| &\le \|z_n - z_{n-1}\| + \|r_n - r_{n-1}\|\|Az_{n-1}\| \\ &\le |\alpha_n - \alpha_{n-1}|\|x_{n-1} - f(x_{n-1})\| + \big(1 - \alpha_n(1-\alpha)\big)\|x_{n-1} - x_n\| \\ &\quad + |r_n - r_{n-1}|\|Az_{n-1}\|.\end{aligned} \tag{2.2}$$

Set $J_{r_n}^B = (I + r_n B)^{-1}$. Using Lemma 1.3, one has

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|T_{n-1}y_{n-1} - T_n y_n\| \\ &\le \|y_{n-1} - y_n\| + |\beta_n - \beta_{n-1}|\|y_n - Ty_n\| \\ &\le \|J_{r_n}^B w_n - J_{r_{n-1}}^B w_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_n - Ty_n\| + e_{n-1} + e_n \\ &\le \|(1 - \frac{r_{n-1}}{r_n})(J_{r_n}^B w_n - w_{n-1}) + \frac{r_{n-1}}{r_n}(w_n - w_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}|\|y_n - Ty_n\| + e_{n-1} + e_n \\ &\le \|(1 - \frac{r_{n-1}}{r_n})(J_{r_n}^B w_n - w_n) + (w_n - w_{n-1})\| \\ &\quad + |\beta_n - \beta_{n-1}|\|y_n - Ty_n\| + e_{n-1} + e_n \\ &\le \frac{|r_n - r_{n-1}|}{r_n}\|w_n - J_{r_n}^B w_n\| + \|w_{n-1} - w_n\| \\ &\quad + |\beta_n - \beta_{n-1}|\|y_n - Ty_n\| + e_{n-1} + e_n.\end{aligned} \tag{2.3}$$

Combining (2.2) with (2.3), one has

$$\begin{aligned}\|x_n - x_{n+1}\| &\le \frac{|r_n - r_{n-1}|}{r_n}\|w_n - J_{r_n}^B w_n\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - f(x_{n-1})\| \\ &\quad + \big(1 - \alpha_n(1-\alpha)\big)\|x_{n-1} - x_n\| + |r_n - r_{n-1}|\|Az_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|\|y_n - Ty_n\| + e_{n-1} + e_n.\end{aligned}$$

Using the restrictions imposed on $\{r_n\}$, $\{e_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ and Lemma 1.1, we find $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Since $\alpha_n \to 0$ as $n \to \infty$, we find

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{2.4}$$

Since $\|\cdot\|^2$ is convex, we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2.\end{aligned}$$

This in turn implies

$$\begin{aligned}\|x_{n+1} - p\|^2 \le\ & \|y_n - J^B_{r_n}(z_n - r_n A z_n)\|^2 + \|J^B_{r_n}(z_n - r_n A z_n) - p\|^2 \\ & + 2\|J^B_{r_n}(z_n - r_n A z_n) - p\|\|y_n - J^B_{r_n}(z_n - r_n A z_n)\| \\ \le\ & \|(z_n - r_n A z_n) - (p - r_n A p)\|^2 + 2e_n \|J^B_{r_n}(z_n - r_n A z_n) - p\| + e_n^2 \\ \le\ & \|z_n - p\|^2 - r_n(2\kappa - r_n)\|A z_n - A p\|^2 + 2e_n\|z_n - p\| + e_n^2 \\ \le\ & \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - r_n(2\kappa - r_n)\|A z_n - A p\|^2 + 2e_n\|z_n - p\| + e_n^2.\end{aligned}$$

It follows that

$$r_n(2\kappa - r_n)\|A z_n - A p\|^2 \le \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - \|x_{n+1} - p\|^2 + (2\|z_n - p\| + e_n)e_n.$$

Therefore, one finds

$$\lim_{n\to\infty} \|Ap - Az_n\| = 0. \tag{2.5}$$

Since $J^B_{r_n}$ is firmly nonexpansive, one has

$$\begin{aligned}\|J^B_{r_n}(z_n - r_n A z_n) - p\|^2 &\le \langle (z_n - r_n A z_n) - (p - r_n A p), J^B_{r_n}(z_n - r_n A z_n) - p\rangle \\ &\le \frac{1}{2}\Big(\|(z_n - r_n A z_n) - (p - r_n A p)\|^2 + \|J^B_{r_n}(z_n - r_n A z_n) - p\|^2 \\ &\quad - \|z_n - J^B_{r_n}(z_n - r_n A z_n) - r_n(A z_n - A p)\|^2\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\|J^B_{r_n}(z_n - r_n A z_n) - p\|^2 &\le \|z_n - p\|^2 - \|z_n - J^B_{r_n}(z_n - r_n A z_n)\|^2 \\ &\quad - r_n\|A z_n - A p\|^2 + 2r_n\|z_n - J^B_{r_n}(z_n - r_n A z_n)\|\|A z_n - A p\|.\end{aligned}$$

Hence, one has

$$\begin{aligned}\|x_{n+1} - p\|^2 \le\ & \|y_n - J^B_{r_n}(z_n - r_n A z_n)\|^2 + \|J^B_{r_n}(z_n - r_n A z_n) - p\|^2 \\ & + 2\|J^B_{r_n}(z_n - r_n A z_n) - p\|\|y_n - J^B_{r_n}(z_n - r_n A z_n)\| \\ \le\ & \|J^B_{r_n}(z_n - r_n A z_n) - p\|^2 + e_n(2\|J^B_{r_n}(z_n - r_n A z_n) - p\| + e_n) \\ \le\ & \|z_n - p\|^2 - \|z_n - J^B_{r_n}(z_n - r_n A z_n)\|^2 \\ & + 2r_n\|z_n - J^B_{r_n}(z_n - r_n A z_n)\|\|A z_n - A p\| \\ & + e_n(2\|J^B_{r_n}(z_n - r_n A z_n) - p\| + e_n) \\ \le\ & \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - \|z_n - J^B_{r_n}(z_n - r_n A z_n)\|^2 \\ & + 2r_n\|z_n - J^B_{r_n}(z_n - r_n A z_n)\|\|A z_n - A p\| \\ & + e_n(2\|J^B_{r_n}(z_n - r_n A z_n) - p\| + e_n),\end{aligned}$$

which further implies from (2.5)

$$\lim_{n\to\infty} \|z_n - J^B_{r_n}(z_n - r_n A z_n)\| = 0. \tag{2.6}$$

Since T_n is nonexpansive, one has

$$\begin{aligned}\|\beta_n x_n + (1-\beta_n)Tx_n - x_n\| &\le \|\beta_n x_n + (1-\beta_n)Tx_n - \beta_n y_n - (1-\beta_n)Ty_n\| \\ &\quad + \|x_n - \beta_n y_n - (1-\beta_n)Ty_n\| \\ &\le \|x_n - y_n\| + \|x_n - x_{n+1}\| \\ &\le \|x_n - z_n\| + \|z_n - J^B_{r_n}(z_n - r_nAz_n)\| + \|x_n - x_{n+1}\| + e_n.\end{aligned}$$

In view of (2.4) and (2.6), one has $\lim_{n\to\infty}\|\beta_n x_n + (1-\beta_n)Tx_n - x_n\| = 0$. Note that

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - \beta_n x_n - (1-\beta_n)Tx_n\| + \|\beta_n x_n + (1-\beta_n)Tx_n - x_n\| \\ &\le \beta_n\|x_n - Tx_n\| + \|\beta_n x_n + (1-\beta_n)Tx_n - x_n\|.\end{aligned}$$

From the restriction imposed on sequence $\{\beta_n\}$, one finds that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, z_n - \bar{x}\rangle \le 0, \tag{2.7}$$

where $\bar{x}$ is the unique fixed point of the mapping $Proj_{(A+B)^{-1}(0)\cap Fix(T)}f$. To show this inequality, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, z_n - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, z_{n_i} - \bar{x}\rangle \le 0.$$

Since $\{z_{n_i}\}$ is bounded, we find that there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we assume that $z_{n_i} \rightharpoonup \hat{x}$. Putting $\mu_n = J^B_{r_n}(z_n - r_nAz_n)$, we find that $\mu_{n_i} \rightharpoonup \hat{x}$.

Next, we show $\hat{x} \in (A+B)^{-1}(0)$. Notice that $z_n - r_nAz_n \in \mu_n + r_nB\mu_n$; that is,

$$\frac{z_n - r_nAz_n - \mu_n}{r_n} \in B\mu_n.$$

Let $\mu \in B\nu$. Since B is maximal monotone, we find

$$\left\langle \frac{z_n - \mu_n}{r_n} - Az_n - \mu, \mu_n - \nu \right\rangle \ge 0.$$

It follows that $\langle -A\hat{x} - \mu, \hat{x} - \nu\rangle \ge 0$. This in turn implies that $-A\hat{x} \in B\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}(0)$.

Now, we are in a position to show that $\hat{x}$ is also in $Fix(T)$. Since $x_{n_i} \rightharpoonup \hat{x}$, we find from Lemma 1.5 that $\hat{x} \in Fix(T)$ immediately. This proves that (2.7) holds.

Finally, we show that $\{x_n\}$ converges strongly to $\bar{x}$, where $\bar{x}$ is the unique fixed point of mapping $Proj_{(A+B)^{-1}(0)\cap Fix(T)}f$.

Note that

$$\|z_n - \bar{x}\|^2 \le \alpha_n\langle f(\bar{x}) - \bar{x}, z_n - \bar{x}\rangle + \big(1 - \alpha_n(1-\alpha)\big)\|z_n - \bar{x}\|\|x_n - \bar{x}\|.$$

It follows that $\|z_n - \bar{x}\|^2 \le 2\alpha_n\langle f(\bar{x}) - \bar{x}, z_n - \bar{x}\rangle + \big(1 - \alpha_n(1-\alpha)\big)\|x_n - \bar{x}\|^2$. Hence, one has

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le \|y_n - \bar{x}\|^2 \\ &\le \|J^B_{r_n}(z_n - r_nAz_n) - \bar{x}\|^2 + e_n(2\|J^B_{r_n}(z_n - r_nAz_n) - \bar{x}\| + e_n) \\ &\le \|z_n - \bar{x}\|^2 + e_n(2\|J^B_{r_n}(z_n - r_nAz_n) - \bar{x}\| + e_n) \\ &\le 2\alpha_n\langle f(\bar{x}) - \bar{x}, z_n - \bar{x}\rangle + \big(1 - \alpha_n(1-\alpha)\big)\|x_n - \bar{x}\|^2 \\ &\quad + e_n(2\|J^B_{r_n}(z_n - r_nAz_n) - \bar{x}\| + e_n).\end{aligned}$$

An application of Lemma 1.2 to the above inequality yields that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$. This completes the proof. □

From Theorem 2.1, the following results are not hard to derive.

Corollary 2.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an inverse κ-strongly monotone mapping and let B be a maximal monotone operator on H. Let $f : C \to C$ be a fixed α-contraction and let $T : C \to C$ be a nonexpansive mapping. Assume that $(A+B)^{-1}(0) \cap Fix(T)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real number sequences in $(0,1)$ and let $\{r_n\}$ be a real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C generated in the following process: $x_0 \in C$, $x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n$, $\forall n \geq 0$, where $\{y_n\}$ is a sequence in C such that $\|y_n - (I + r_nB)^{-1}\big(\alpha_n f(x_n) + (1-\alpha_n)x_n - r_nA\big(\alpha_n f(x_n) + (1-\alpha_n)x_n\big)\big)\| \leq e_n$. Assume that the control sequences satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $0 < a \leq r_n \leq a' < 2\kappa$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, and $0 \leq \beta_n \leq a'' < 1$, where a, a' and a'' are three real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T) \cap (A+B)^{-1}(0)$, where $\bar{x} = Proj_{Fix(T)\cap(A+B)^{-1}(0)} f(\bar{x})$, that is, $\bar{x}$ solves the following variational inequality $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$, $\forall x \in Fix(T) \cap (A+B)^{-1}(0)$.*

Corollary 2.3. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an inverse κ-strongly monotone mapping and let B be a maximal monotone operator on H. Let $f : C \to C$ be a fixed α-contraction. Assume that $(A+B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real number sequences in $(0,1)$ and let $\{r_n\}$ be a real number sequence in $(0, 2\kappa)$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence in C such that $\|x_{n+1} - (I + r_nB)^{-1}\big(\alpha_n f(x_n) + (1-\alpha_n)x_n - r_nA\big(\alpha_n f(x_n) + (1-\alpha_n)x_n\big)\big)\| \leq e_n$. Assume that the control sequences satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $0 < a \leq r_n \leq a' < 2\kappa$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$, where a and a' are three real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)} f(\bar{x})$, that is, $\bar{x}$ solves the following variational inequality $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$, $\forall x \in (A+B)^{-1}(0)$.*

Let C be a nonempty closed and convex subset of a Hilbert space H. Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Since i_C is a proper lower and semicontinuous convex function on H, the subdifferential ∂i_C of i_C is maximal monotone. So, we can define the resolvent $J_r^{\partial i_C}$ of ∂i_C for $r > 0$, i.e., $J_r^{\partial i_C} := (I + r\partial i_C)^{-1}$. Letting $x = J_r^{\partial i_C} y$, we find that

$$\begin{aligned} y \in x + r\partial i_C x &\iff y \in x + rN_C x \\ &\iff \langle y - x, v - x\rangle \leq 0, \forall v \in C \\ &\iff x = Proj_C y, \end{aligned}$$

where $Proj_C$ is the metric projection from H onto C and $N_C x := \{e \in H : \langle e, v - x\rangle, \forall v \in C\}$.

From Theorem 2.1, we have the following results on variational inequality (1.1).

Corollary 2.4. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let $A : C \to H$ be an inverse κ-strongly monotone mapping. Let $f : C \to C$ be a fixed α-contraction and let $T : C \to C$ be a λ-strict pseudocontraction. Assume that $VI(C, A) \cap Fix(T)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real number sequences in $(0,1)$ and let $\{r_n\}$ be a real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C generated in the following process: $x_0 \in C$, $x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n$, $\forall n \geq 0$, where $\{y_n\}$ is a sequence in C such that $\|y_n - Proj_C\big(\alpha_n f(x_n) + (1-\alpha_n)x_n - r_nA\big(\alpha_n f(x_n) + (1-\alpha_n)x_n\big)\big)\| \leq e_n$. Assume that the control sequences satisfy the following restrictions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$, $0 < a \leq r_n \leq a' < 2\kappa$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$, and $\lambda \leq \beta_n \leq a'' < 1$, where a, a' and a'' are three real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T) \cap VI(C, A)$, where $\bar{x} = Proj_{Fix(T)\cap VI(C,A)} f(\bar{x})$, that is, $\bar{x}$ solves the following variational inequality $\langle f(\bar{x}) - \bar{x}, \bar{x} - x\rangle \geq 0$, $\forall x \in Fix(T) \cap VI(C, A)$.*

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