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Dislocated quasi-b-metric spaces and fixed point theorems for cyclic weakly contractions

Cholatis Suanoom[a], Chakkrid Klin-eam[a,b,*], Suthep Suantai[c]

[a] Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.
[b] Research Center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok, 65000, Thailand.
[c] Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand.

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Abstract

In this paper, we introduce the notions of type dqb-cyclic-weak Banach contraction, dqb-cyclic-ϕ-contraction and derive the existence of fixed point theorems on dislocated quasi-b-metric spaces. Our main theorem extends and unifies existing results in the recent literature.

Keywords: Fixed points, dqb-cyclic-ϕ-contraction, dislocated quasi-b-metric spaces, dqb-converges sequence theorems, dqb-Cauchy sequence theorems.
2010 MSC: 47H05, 47H10, 47J25.

1. Introduction and Preliminaries

Banach contraction principle was introduced in 1922 by Banach [3]. In 2001, Rhoades [7] introduced weakly contractive as follows:
(i) A mapping $T : X \to X$ is said to be a *weakly contractive* if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1 - k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

*Corresponding author
Email addresses: Cholatis.Suanoom@gmail.com (Cholatis Suanoom), chakkridk@nu.ac.th (Chakkrid Klin-eam), scmti005@chiangmai.ac.th (Suthep Suantai)

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Now, we recall the definition of cyclic map. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$, then T is called a *cyclic map* iff $T(A) \subseteq B$ and $T(B) \subseteq A$. In 2003, Kirk et al. [4] introduced cyclic contraction as follows:
(ii) A cyclic map $T : A \cup B \to A \cup B$ is said to be a *cyclic contraction* if there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x \in A$ and $y \in B$.

In 2013, K. Zoto [9] introduced d-cyclic-ϕ-contraction follows:
(iii) A cyclic map $T : A \cup B \to A \cup B$ is said to be a *d-cyclic-ϕ-contraction* if $\phi \in \Phi$ such that

$$d(Tx, Ty) \leq \phi(d(x, y))$$

for all $x \in A$, $y \in B$, where Φ the family of non-decreasing functions: $\phi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where n is the n-th iterate of ϕ.

Lemma 1.1. *Suppose that the function $\phi : [0, \infty) \to [0, \infty)$ is non-decreasing, then for each $t > 0$, $\lim_{n\to\infty} \phi^n(t) = 0$ implies $\phi(t) < t$.*

If (X, d) is complete metric spaces, at least one of (i), (ii) and (iii) holds, then T has a unique fixed point (see[7]-[9]). Recently, Klin-eam and Suanoom [5] introduced dislocated quasi b-metric spaces, which is a new generalization of quasi b-metric space (see[8]), b-metric-like space (see[1]), b-metric space (see[2]), metric space, etc. as follows:

Definition 1.2 ([5])**.** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

$(d1)$ $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;

$(d2)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is then called a *dislocated quasi b-metric space (or simply dqb-metric)*. The number s is called to be the coefficient of (X, d).

Remark 1.3. When, in addition, the conditions $d(x, y) = d(y, x)$ and $d(x, x) = 0$ are true, then d is a b-metric.

Definition 1.4. Let $\{x_n\}$ be a sequence in a dqb-metric space (X, d).

(1) A sequence $\{x_n\}$ *dislocated quasi-b-converges (for short, dqb-converges)* to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

In this case x is called a dqb-limit of $\{x_n\}$ and we write $(x_n \to x)$.

(2) A sequence $\{x_n\}$ is called *dislocated quasi-b-Cauchy (for short, dqb-Cauchy)*, if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n,m\to\infty} d(x_m, x_n).$$

(3) A dqb-metric space (X, d) is complete if every dqb-Cauchy sequence is dqb-convergent in X.

Moreover, they introduced the notion of dqb-cyclic-Banach and dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such space.

In this paper, we study the properties of dislocated quasi-b-metric spaces and introduce dqb-cyclic-weak Banach contraction, dqb-cyclic-ϕ-contraction and derive the existence of fixed point theorems in dislocated quasi-b-metric spaces. Our main theorem extends and unifies existing results in the recent literature.

2. Main results

Every dislocated quasi-b-metric space (X, d) can be considered as a topological space on which the topology is introduced by taking, for any $x \in X$, the collection $\{B_r(x)|r > 0\}$ as a base of the neighborhood filter of the point x. Here the ball $B_r(x)$ is defined by the equality $B_r(x) = \{y \in X|\max\{d(x, y), d(y, x)\} < r\}$.

Definition 2.1 ([6])**.** Let X be topological space. Then X is said to be *Hausdorff topological space* if for any distinct points $x, y \in X$, there exists two open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$.

Proposition 2.2. *Every dqb-metric space is Hausdorff topological space.*

Proof. Let x and y be two distinct points in X. Then $d(x, y) > 0$ and $d(y, x) > 0$. Choose $\delta = \frac{d(x,y)}{2s}$. Then, there exists

$$B_\delta(x) = \{z \in X|\max\{d(x, z), d(z, x)\} < \delta\}$$

and

$$B_\delta(y) = \{z \in X|\max\{d(y, z), d(z, y)\} < \delta\}$$

such that $x \in B_\delta(x)$ and $y \in B_\delta(y)$.

To show that $B_\delta(x) \cap B_\delta(y) = \emptyset$, suppose that $B_\delta(x) \cap B_\delta(y) \neq \emptyset$. Then, there exists $z \in B_\delta(x) \cap B_\delta(y)$. We have

$$\begin{aligned} d(x, y) &\leq sd(x, z) + sd(z, y) \\ &\leq s\max\{d(x, z), d(z, x)\} + s\max\{d(y, z), d(z, y)\} \\ &< s\delta + s\delta = d(x, y). \end{aligned}$$

So, $d(x, y) < d(x, y)$ which is a contradiction. Therefore $B_\delta(x) \cap B_\delta(y) = \emptyset$. □

Proposition 2.3. *Every dqb-convergent sequence in a dqb-metric space (X, d) is dqb-Cauchy sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-convergent. Then there exists $x \in X$ such that $x_n \to x$, that is

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

Consider,

$$d(x_n, x_m) \leq sd(x_n, x) + sd(x, x_m).$$

Taking limit as $n, m \to \infty$ we obtain

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Similarly,

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Therefore $\{x_n\}$ is dqb-Cauchy. □

Definition 2.4. A subset S of a dqb-metric space (X, d) is bounded if there exists $\bar{x}$, $M \in (0, \infty)$ such that $d(x, \bar{x}) \leq M$ for all $x \in S$.

Proposition 2.5. *Every dqb-convergent sequence in a dqb-metric space (X, d) is bounded sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-convergent. Then there exists $x \in X$ such that $x_n \to x$, that is

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

Let $\epsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ and $d(x, x_n) < \epsilon$ for all $n \geq n_0$. Choose

$$K = \max\{d(x_1, x), d(x_2, x), ..., d(x_{n_0-1}, x), 1\}.$$

Thus, $d(x_n, x) \leq K$ for all $n \in \mathbb{N}$ and so $\{x_n\}$ is bounded sequence. □

Proposition 2.6. *Every dqb-Cauchy sequence in a dqb-metric space* (X, d) *is bounded sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-Cauchy. Then

$$\lim_{n\to\infty} d(x_n, x_m) = 0 = \lim_{n\to\infty} d(x_m, x_n).$$

Let $\epsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ and $d(x_m, x_n) < 1$ for all $n, m \geq n_0$. Let p be any point in the space and let

$$k = \max_{i\leq m} d(x_i, p).$$

The maximum exists, since $\{x_i : i \leq m\}$ is a finite set. If $n \leq m$, then $d(x_n, p) \leq k$. If $n > m$, then $d(x_n, p) \leq d(x_n, x_m) + d(x_m, p) \leq 1 + k$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded sequence. □

The next two propositions for subsequence follow immediately from definitions of dqb-convergent sequence and dqb-Cauchy sequence respectively.

Proposition 2.7. *Every subsequence of dqb-convergent sequence in a dqb-metric space* (X, d) *is dqb-convergent sequence.*

Proposition 2.8. *Every subsequence of dqb-Cauchy sequence in a dqb-metric space* (X, d) *is dqb-Cauchy sequence.*

Proposition 2.9. *Let* $\{x_n\}$ *be sequence in a dqb-metric space* (X, d)*. Then* $x_n \to x$ *if and only if* $d(x_n, x) \to 0$ *and* $d(x, x_n) \to 0$.

Proof. Suppose that $x_n \to x$. Then

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Thus $d(x_n, x) \to 0$ and $d(x, x_n) \to 0$.

Conversely, Suppose that $d(x_n, x) \to 0$ and $d(x, x_n) \to 0$. Then

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

By definition of dqb-convergent sequence, we get $x_n \to x$. □

Proposition 2.10. *Let* $\{x_n\}$ *be sequence in a dqb-metric space* (X, d)*. If* $x_n \to x$ *and* $x_n \to y$*, then* $x = y$.

Proof. Suppose that $x_n \to x$ and $x_n \to y$. Then

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = \lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(y, x_n) = 0.$$

Consider,

$$0 \leq d(x, y) \leq sd(x, x_n) + sd(x_n, y)$$

and

$$0 \leq d(y, x) \leq sd(y, x_n) + sd(x_n, x).$$

Taking limit as $n, m \to \infty$, we obtain

$$d(x, y) = d(y, x) = 0.$$

Therefore $x = y$. □

Now, we begin with introducing the property of a continuous function.

Definition 2.11. Suppose that (X, d_X) and (Y, d_Y) are dislocated quasi-b-metric spaces, $E \subset X$, $f : E \to Y$ and $p \in E$. Then f is continuous at p iff for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\max\{d_Y(fx, fp), d_Y(fp, fx)\} < \epsilon$$

for all $x \in E$, when $\max\{d_X(x, p), d_X(p, x)\} < \delta$.

Theorem 2.12. *Let (X, d_X) and (X, d_Y) be dislocated quasi-b-metric spaces, $E \subset X$, $f : E \to Y$ and $p \in E$. Then f is continuous at p if and only if for every dislocated quasi-b-converges sequence $\{x_n\}$ in X, $\lim_{n\to\infty} fx_n = fx$.*

Proof. Suppose that f is continuous at p and $\{x_n\}$ converges to p. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $\max\{d_Y(fx, fp), d_Y(fp, fx)\} < \epsilon$, when $\max\{d_X(x, p), d_X(p, x)\} < \delta$ for all $x \in E$.

Since $\{x_n\}$ converges to p, there exists $N \in \mathbb{N}$ such that $\max\{d_X(x_n, p), d_Y(p, x_n)\} < \delta$ for all $n \geq N$. Since f is continuous at p, we have $\max\{d_Y(fx_n, fp), d_Y(fp, fx_n)\} < \epsilon$, for all $n \geq N$.

Hence $\lim_n fx_n = fx$.

Conversely, let $x \in X$ and assume in the contrary that

$$\exists\epsilon > 0\ \forall\delta > 0 :\ \max\{d_X(x, p), d_X(p, x)\} < \delta, \max\{d_Y(fx, fp), d_Y(fp, fx)\} \geq \epsilon.$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence $\{x_k\}$ such that $\max\{d_X(x_k, p), d_X(p, x_k)\} < \frac{1}{k}$ and $\max\{d_Y(fx_k, fp), d_Y(fp, fx_k)\} \geq \epsilon'$. Thus

$$\lim_{k\to\infty} x_k = p.$$

By assumption, we have

$$\lim_{k\to\infty} fx_k = fp.$$

Hence, there exists a k_0 such that for all $k > k_0$

$$\max\{d_Y(fx_k, fp), d_Y(fp, fx_k)\} < \epsilon,$$

which is a contradiction. □

Definition 2.13. Suppose that (X, d_X) and (Y, d_Y) are dislocated quasi-b-metric spaces, $E \subset X$, $f : E \to Y$ and $p \in E$. Then f is continuous on E iff f is continuous at p for all $p \in E$.

Next, we begin with prove fixed point theorems.

Definition 2.14. Let A and B be nonempty closed subsets of a dislocated quasi-b-metric spaces (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a *dqb-cyclic-weak contraction* or *dqb-cyclic-weakly contraction* if for all $x \in A$, $y \in B$,

$$sd(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \tag{2.1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

Lemma 2.15. *Let (X, d_X) and (Y, d_Y) be dislocated quasi-b-metric spaces and A and B be nonempty closed subsets of a dislocated quasi-b-metric spaces (X, d). Consider a cyclic map $T : A \cup B \to A \cup B$. If T is dqb-cyclic-weak contraction, then T is continuous.*

Proof. Let $\epsilon > 0$, all $x \in A \cup B$ and fixed $p \in A \cup B$. Suppose that $\max\{d_X(x, p), d_C(p, x)\} < \delta$. Choose $\epsilon = \frac{\delta}{s}$. Since T is dqb-cyclic-weak contraction, we have

$$\begin{aligned} sd(Tx, Tp) &\leq d(x, p) - \psi(d(x, p)) \\ &\leq d(x, p) < \delta \end{aligned}$$

and

$$\begin{aligned} sd(Tp, Tx) &\leq d(p, x) - \psi(d(p, x)) \\ &\leq d(p, x) < \delta. \end{aligned}$$

So, $d(Tx, Tp) < \epsilon$ and $d(Tp, Tx) < \epsilon$. Thus T is continuous at p and hence T is continuous on $A \cup B$. □

Now, we present a fixed point theorem related to dqb-cyclic-weak contraction.

Theorem 2.16. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-weak contraction. Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$ be fixed. Using contractive condition in assumptions, we have

$$\begin{aligned} d(T^2x, Tx) &\leq sd(T^2x, Tx) \\ &= sd(T(Tx), Tx) \\ &\leq d(Tx, x) - \psi(d(Tx, x)), \\ &\leq d(Tx, x) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(Tx, T^2x) &\leq sd(Tx, T^2x) \\ &= sd(Tx, T(Tx)) \\ &\leq d(x, Tx) - \psi((x, Tx)), \\ &\leq d(x, Tx). \end{aligned} \tag{2.3}$$

So

$$d(T^3x, T^2x) \leq d(T^2x, Tx) - \psi(d(T^2x, Tx)) \tag{2.4}$$

and

$$d(T^2x, T^3x) \leq d(Tx, T^2x) - \psi(d(Tx, T^2x)). \tag{2.5}$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+2}x, T^{n+1}x) \leq d(T^{n+1}x, T^nx) - \psi(d(T^{n+1}x, T^nx)) \tag{2.6}$$

and

$$d(T^{n+1}x, T^{n+2}x) \leq d(T^nx, T^{n+1}x) - \psi(d(T^nx, T^{n+1}x)). \tag{2.7}$$

Set $\varsigma_n = d(T^{n+1}x, T^nx)$ and $\tau_n = d(T^nx, T^{n+1}x)$. By inequalities (2.6) and (2.7), we get

$$\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma_n) \leq \varsigma_n \tag{2.8}$$

and

$$\tau_{n+1} \leq \tau_n - \psi(\tau_n) \leq \tau_n. \tag{2.9}$$

Thus $\{\varsigma_n\}$ and $\{\tau_n\}$ are decreasing sequences of non-negative real numbers, and hence possess a $\lim_{n\to\infty} \varsigma_n = \varsigma \geq 0$ and $\lim_{n\to\infty} \tau_n = \tau \geq 0$. Suppose that $\varsigma > 0$. Since ψ is nondecreasing, $\psi(\varsigma_n) \geq \psi(\varsigma) > 0$. By inequality (2.8), we have $\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma)$. Thus $\varsigma_{N+m} \leq \varsigma_m - N\psi(\varsigma)$, a contradiction for N large enough. Therefore $\varsigma = 0$.

Similarly, $\tau = 0$.

Next, we prove that $\{T^nx\}$ is a Cauchy sequence. Suppose that $\{T^nx\}$ is not Cauchy, then there exist $\epsilon > 0$ and subsequence $\{T^{m_k}x\}$ and $\{T^{n_k}x\}$ with $m_k > n_k \geq n$ such that $d(T^{m_k}x, T^{n_k}x) \geq \epsilon$ and $d(T^{m_k-1}x, T^{n_k}x) < \epsilon$. Now, we consider

$$\begin{aligned} sd(T^{m_k}x, T^{n_k}x) &\leq d(T^{m_k-1}x, T^{n_k-1}x) - \psi(d(T^{m_k-1}x, T^{n_k-1}x)) \\ &\leq d(T^{m_k-1}x, T^{n_k-1}x), \end{aligned} \tag{2.10}$$

which implies that

$$s\epsilon \leq d(T^{m_k-1}x, T^{n_k-1}x). \tag{2.11}$$

Take limit inferior in (2.11) as $k \to \infty$, we get

$$\epsilon s \leq \liminf d(T^{m_k-1}x, T^{n_k-1}x). \tag{2.12}$$

We have

$$\begin{aligned} d(T^{m_k-1}x, T^{n_k-1}x) &\leq sd(T^{m_k-1}x, T^{n_k}x) + sd(T^{n_k}x, T^{n_k-1}x) \\ &< s\epsilon + sd(T^{n_k}x, T^{n_k-1}x). \end{aligned} \tag{2.13}$$

Take limit superior in (2.13) as $k \to \infty$, we get

$$\limsup d(T^{m_k-1}x, T^{n_k-1}x) \leq s\epsilon. \tag{2.14}$$

By (2.12) and (2.14), we get

$$\lim d(T^{m_k-1}x, T^{n_k-1}x) = s\epsilon. \tag{2.15}$$

Letting $k \to \infty$ in (2.10), by property of ψ and (2.15), we get

$$s\epsilon \leq s\epsilon - \psi(s\epsilon) < s\epsilon, \tag{2.16}$$

which is a contradiction. Hence $\{T^n x\}$ is a dqb-Cauchy sequence. Since (X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$. We note that, $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to same limit z. Since A and B are closed, we have $z \in A \cap B$ and hence $A \cap B \neq \emptyset$. The continuity of T implies that the limit is a fixed point. Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$. Then, we have

$$d(z, z^*) = d(Tz, Tz^*) \leq sd(Tz, Tz^*) \leq d(z, z^*) - \psi(d(z, z^*)) \leq d(z, z^*). \tag{2.17}$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq sd(Tz^*, Tz) \leq d(z^*, z) - \psi(d(z, z^*)) \leq d(z^*, z). \tag{2.18}$$

By forms (2.17) and (2.18), we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$. Therefore z is a unique fixed point of T. This completes the proof. □

Example 2.17. Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ be defined by $Tx = \frac{-x}{3}$ and $\psi(t) = \frac{t}{50}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Defined the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X (see[[5]]).

Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{3} \leq \frac{1}{3}$. Thus, $Tx \in B$. On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{3} \leq \frac{-x}{3} \leq 0$. Thus, $Tx \in A$.

Hence, the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned} 2d(Tx, Ty) &= 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \\ &= 2(|\frac{-x}{3} - \frac{-y}{3}|^2 + \frac{1}{10}|\frac{-x}{3}| + \frac{1}{11}|\frac{-y}{3}|) \end{aligned}$$

$$
\begin{aligned}
&= \frac{49}{50}(\frac{100}{441}|x-y|^2 + \frac{50}{1470}|x| + \frac{100}{539}|y|) \\
&\leq \frac{49}{50}\big(|x-y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|\big) \\
&= |x-y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y| - \psi\big(|x-y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|\big) \\
&= d(x,y) - \psi(d(x,y)).
\end{aligned}
$$

Thus, T satisfies dqb-cyclic-weak contraction of Theorem 2.16 and 0 is the unique fixed point of T.

Definition 2.18. Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces. (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a *dqb-cyclic-ϕ-contraction* and if there exists $k \in [0, 1)$ and $s \geq 1$ such that

$$
sd(Tx, Ty) \leq \phi(d(x, y)) \tag{2.19}
$$

for all $x \in A$, $y \in B$, where Φ the family of non-decreasing functions: $\phi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where n is the n-th iterate of ϕ.

Theorem 2.19. *Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a* dqb-cyclic-ϕ-contraction*. Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$ be fixed, then using contractive condition of theorem, we have

$$
\begin{aligned}
sd(T^2x, Tx) &= sd(T(Tx), Tx) \\
&\leq \phi(d(Tx, x))
\end{aligned}
$$

and

$$
\begin{aligned}
sd(Tx, T^2x) &= sd(Tx, T(Tx)) \\
&\leq \phi(d(x, Tx)).
\end{aligned}
$$

Inductively, we have for all $n \in \mathbb{N}$, we get

$$
s^n d(T^{n+1}x, T^n x) \leq \phi^n(d(Tx, x))
$$

and

$$
s^n d(T^n x, T^{n+1}x) \leq \phi^n(d(x, Tx)).
$$

Let $\epsilon > 0$ be fixed and $n(\epsilon) \in \mathbb{N}$, such that

$$
\Sigma_{n \geq n(\epsilon)} \phi^n(d(Tx, x)) < \epsilon
$$

and

$$
\Sigma_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.
$$

Let $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, using the triangular inequality, we have:

$$
\begin{aligned}
d(T^m x, T^n x) &\leq s^{m-n} d(T^m x, T^{m-1}x) + s^{m-n-1} d(T^{m-1}x, T^{m-2}x) + ... + sd(T^{n+1}x, T^n x) \\
&\leq s^{m-1} d(T^m x, T^{m-1}x) + s^{m-2} d(T^{m-1}x, T^{m-2}x) + ... + s^n d(T^{n+1}x, T^n x) \\
&\leq \phi^{m-1}(d(Tx, x)) + \phi^{m-2}(d(Tx, x)) + \phi^{m-3}(d(Tx, x)) + ... + \phi^n(d(Tx, x)) \\
&= \Sigma_{k=n}^{m-1} \phi^k(d(x, Tx)) \\
&\leq \Sigma_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.
\end{aligned}
$$

Similarly,

$$d(T^n x, T^m x) < \epsilon.$$

Thus $\{T^n x\}$ is a Cauchy sequence. Since(X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$. We note that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to same limit z. Since A and B are closed, we have $z \in A \cap B$ and then $A \cap B \neq \emptyset$. Now, we will show that $Tz = z$. By using (2.19), consider

$$\begin{aligned} d(z, Tz) &\leq sd(z, T^{2n}x) + sd(T^{2n}x, Tz) \\ &\leq sd(z, T^{2n}x) + d(T^{2n-1}x, z). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(z, Tz) = 0.$$

Similarly considering form (2.19), we get

$$\begin{aligned} d(Tz, z) &\leq sd(Tz, T^{2n}x) + sd(T^{2n}x, z) \\ &\leq d(z, T^{2n-1}x) + sd(T^{2n}x, z). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(Tz, z) = 0.$$

Hence $d(z, Tz) = d(Tz, z) = 0$. This implies that $Tz = z$ that is z is a fixed point of T.

Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$. Then, we have

$$d(z^*, z) \leq sd(Tz^*, T^n x) + sd(T^n x, Tz) \leq \phi(d(Tz^*, T^n x)) + \phi(d(T^n x, Tz)) \tag{2.20}$$

and on the other hand,

$$d(z, z^*) \leq sd(Tz, T^n x) + sd(T^n x, Tz^*) \leq \phi(d(Tz, T^n x)) + \phi(d(T^n x, Tz^*)). \tag{2.21}$$

Letting $n \to \infty$ we obtain that $d(z, z^*) = d(z^*, z) = 0$, which implies that $z^* = z$. Therefore z is a unique fixed point of T. This completes the proof. □

Example 2.20. Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ be defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Defined the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X, where $s = 2$. Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{5} \leq \frac{1}{5}$. Thus, $Tx \in B$. On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{5} \leq \frac{-x}{5} \leq 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned} sd(Tx, Ty) &= 2d(Tx, Ty) \\ &= 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \\ &= 2(|\frac{-x}{5} - \frac{-y}{5}|^2 + \frac{1}{10}|\frac{-x}{5}| + \frac{1}{11}|\frac{-y}{5}|) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}(\frac{3}{25}|x-y|^2 + \frac{3}{50}|x| + \frac{3}{55}|y|) \\
&\leq \frac{2}{3}(|x-y|^2 + \frac{5}{50}|x| + \frac{5}{55}|y|) \\
&= \frac{2}{3}(|x-y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\
&= \phi(d(x,y)),
\end{aligned}
$$

where the function $\phi \in \Phi$ is $\phi(t) = \frac{2t}{3}$. Clearly, 0 is the unique fixed point of T.

The following corollary can be taken as a particular case of Theorem 2.19 if we take $\phi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. That is the dqb-cyclic-Banach contraction, in the setting of dislocated quasi-b-metric spaces.

Corollary 2.21. *Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-Banach contraction; that is, if there exists $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \tag{2.22}$$

for all $x \in A$, $y \in B$ and $s \geq 1$ and $sk \leq 1$. Then, T has a unique fixed point in $A \cap B$.

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