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The mixed L_p-dual affine surface area for multiple star bodies

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Abstract

Associated with the notion of the mixed L_p-affine surface area for multiple convex bodies for all real p ($p \neq -n$) which was introduced by Ye, et al. [D. Ye, B. Zhu, J. Zhou, arXiv, **2013** (2013), 38 pages], we define the concept of the mixed L_p-dual affine surface area for multiple star bodies for all real p ($p \neq -n$) and establish its monotonicity inequalities and cyclic inequalities. Besides, the Brunn-Minkowski type inequalities of the mixed L_p-dual affine surface area for multiple star bodies with two addition are also presented.

Keywords: L_p-affine surface area, L_p-dual affine surface area, multiple star bodies, Hölder inequality.
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1. Introduction

During the past three decades, the investigations of the classical affine surface area have received great attention from many articles (see [7, 8, 9, 10, 11, 12, 13, 14]). Based on the classical affine surface area, Lutwak (see [14]) introduced the notion of L_p-affine surface area and established its some inequalities. Wang and He (see [19, 20]) introduced the notion of L_p-dual affine surface area. Regarding studies of the L_p-affine surface area and L_p-dual affine surface area also see [16, 17, 21, 22, 23, 24, 25, 26].

We say that K is a convex body if K is a compact and convex subset in n-dimensional Euclidean space $\mathbb{R}^n$ with non-empty interior. The set of all convex bodies in $\mathbb{R}^n$ is written as $\mathcal{K}$, and its subset $\mathcal{K}_o$ denote the set of convex bodies containing the origin in their interiors. Similarly, $\mathcal{K}_c$ denote the set of convex bodies

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with centroid at the origin. Besides $\mathcal{S}_o$ denotes the set of star bodies (with respect to the origin) and $\mathcal{S}_c$ denotes the set of star bodies whose centroid lie at the origin in $\mathbb{R}^n$. Let $\mathcal{F}_o$ denotes the subset of $\mathcal{K}_o$ that has a positive continuous curvate function. Let S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and $V(K)$ denotes the n-dimensional volume of the body K.

The notion of classical affine surface area was proposed by Leichtweiβ (see [7]). For $K \in \mathcal{K}$, the affine surface area, $\Omega(K)$ of K is defined by

$$n^{-\frac{1}{n}}\Omega(K)^{\frac{n+1}{n}} = \inf_{L\in\mathcal{S}_o}\{nV_1(K,L^*)V(L)^{\frac{1}{n}}\}.$$

Here L^* denotes the polar body of L.

According to the L_p-mixed volume, Lutwak introduced the notion of L_p-affine surface area in [14]. For $K \in \mathcal{K}_o$, $p \geq 1$, the L_p-affine surface area, $\Omega_p(K)$ of K is defined by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf_{L\in\mathcal{S}_o}\{nV_p(K,L^*)V(L)^{\frac{p}{n}}\}.$$

Obviously, if $p = 1$, $\Omega_1(K)$ is the classical affine surface area $\Omega(K)$.

Based on above the notion of L_p-affine surface area, Wang and He (see [19]) presented the notion of L_p-dual affine surface area associated with the L_p-dual mixed volume. For $K \in \mathcal{S}_o$ and $1 \leq p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$ of K is defined by

$$n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf_{L\in\mathcal{K}_c}\{n\widetilde{V}_{-p}(K,L^*)V(L)^{-\frac{p}{n}}\}.$$

According to the definition of L_p-dual affine surface area, Wang and He (see [19]) proved the following result:

Theorem 1.1. *If $K, L \in \mathcal{K}_c$ and $1 \leq p < n$, then*

$$\widetilde{\Omega}_{-p}(K\widetilde{+}_{n+p}L)^{\frac{n-p}{n}} \geq \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} + \widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}},$$

with equality if and only if K and L dilates. Here $K\widetilde{+}_{n+p}L$ denotes the L_{n+p}-radial combination of K and L.

In fact, Wang and Wang in [18] extend the definition of L_p-dual affine surface area which was introduced by Wang and He (see [19]) from $L \in \mathcal{K}_c$ to $L \in \mathcal{S}_c$, as follows:

For $K \in \mathcal{S}_o$ and $1 \leq p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$ of K is defined by

$$n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf_{L\in\mathcal{S}_c}\{n\widetilde{V}_{-p}(K,L^*)V(L)^{-\frac{p}{n}}\}. \tag{1.1}$$

Recently, L_p-affine surface area was successfully extended to any real p ($p \neq -n$) by Ye (see [24, 25]). Moreover, Ye, Zhu and Zhou [26] studied the mixed L_p-affine surface area for multiple star bodies for all real p ($p \neq -n$). Let $\mathbf{K} = (K_1, \cdots, K_n)$ be a sequence with each $K_i \subset \mathbb{R}^n$ ($i = 1, \cdots, n$) and $\mathbf{K} \in \mathcal{F}_o^n$ means each $K_i \in \mathcal{F}_o$, $L \in \mathcal{S}_o$. They defined the mixed L_p-affine surface areas for multiple convex bodies $\Omega_p(\mathbf{K})$ as follows:

for $p > 0$,

$$\Omega_p(\mathbf{K}) = \inf_{L\in\mathcal{S}_o}\{nV_p(\mathbf{K};\underbrace{L^*,\cdots,L^*}_{n})^{\frac{n}{n+p}}V(L)^{\frac{p}{n+p}}\};$$

for $-n \neq p < 0$,

$$\Omega_p(\mathbf{K}) = \sup_{L\in\mathcal{S}_o}\{nV_p(\mathbf{K};\underbrace{L^*,\cdots,L^*}_{n})^{\frac{n}{n+p}}V(L)^{\frac{p}{n+p}}\}.$$

In this paper, combining with (1.1) and the notion of the mixed L_p-affine surface area for multiple star bodies, we first introduce the notion of the mixed L_p-dual affine surface area for multiple star bodies for all real p ($p \neq -n$). Here, we write that $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$ be a sequence with each $K_i \in \mathcal{S}_o$.

Definition 1.2. Let $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$, the mixed L_p-dual affine surface area for multiple star bodies, $\widetilde{\Omega}_{-p}(\mathbf{K})$ of $\mathbf{K}$ is defined by:

for $p > 0$,

$$\widetilde{\Omega}_{-p}(\mathbf{K}) = \inf_{L\in\mathcal{S}_c}\{n\widetilde{V}_{-p}(\mathbf{K}; \underbrace{L^*, \cdots, L^*}_{n})^{\frac{n}{n-p}} V(L)^{-\frac{p}{n-p}}\}; \tag{1.2}$$

for $-n \neq p < 0$,

$$\widetilde{\Omega}_{-p}(\mathbf{K}) = \sup_{L\in\mathcal{S}_c}\{n\widetilde{V}_{-p}(\mathbf{K}; \underbrace{L^*, \cdots, L^*}_{n})^{\frac{n}{n-p}} V(L)^{-\frac{p}{n-p}}\}. \tag{1.3}$$

Further, we establish monotonicity inequalities and cyclic inequalities of the mixed L_p-dual affine surface area for multiple star bodies. Our results can be stated as follows:

Theorem 1.3. *Let $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$. If $0 < p < q < n$, then*

$$\left(\frac{\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p}}{n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{\Omega}_{-q}(\mathbf{K})^{n-q}}{n^{n-q}\widetilde{V}(\mathbf{K})^{n+q}}\right)^{\frac{1}{q}}; \tag{1.4}$$

if $-n < q < p < 0$, then

$$\left(\frac{\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p}}{n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}}\right)^{\frac{1}{p}} \geq \left(\frac{\widetilde{\Omega}_{-q}(\mathbf{K})^{n-q}}{n^{n-q}\widetilde{V}(\mathbf{K})^{n+q}}\right)^{\frac{1}{q}}. \tag{1.5}$$

Here $\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p}/n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}$ denotes L_p-dual affine area ratio of the sequence $\mathbf{K}$.

Theorem 1.4. *Let $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$. If $0 < r < q < p < n$, then*

$$\widetilde{\Omega}_{-p}(\mathbf{K})^{(n-p)(q-r)} \geq \widetilde{\Omega}_{-q}(\mathbf{K})^{(n-q)(p-r)}\widetilde{\Omega}_{-r}(\mathbf{K})^{(n-r)(q-p)}; \tag{1.6}$$

if $-n \neq r < p < q < 0$, then

$$\widetilde{\Omega}_{-p}(\mathbf{K})^{(n-p)(q-r)} \leq \widetilde{\Omega}_{-q}(\mathbf{K})^{(n-q)(p-r)}\widetilde{\Omega}_{-r}(\mathbf{K})^{(n-r)(q-p)}. \tag{1.7}$$

Besides, associated with the combination $\lambda \circ \mathbf{K}\tilde{+}_q\mu \circ \mathbf{L} = (\lambda \circ K_1\tilde{+}_q\mu \circ L_1, \cdots, \lambda \circ K_n\tilde{+}_q\mu \circ L_n)$, where $\lambda \circ K\tilde{+}_q\mu \circ L$ is the L_q-radial combination of star bodies K and L, and corresponding to Theorem 1.1, *we give a Brunn-Minkowski type inequality of the mixed L_p-dual affine surface area for multiple star bodies.*

Theorem 1.5. *Let $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$, $\mathbf{L} = (L_1, \ldots, L_n) \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero). If $0 < p < n$, $q > n + p$, then*

$$\widetilde{\Omega}_{-p}(\lambda \circ \mathbf{K}\tilde{+}_q\mu \circ \mathbf{L})^{\frac{q(n-p)}{n(n+p)}} \geq \lambda\widetilde{\Omega}_{-p}(\mathbf{K})^{\frac{q(n-p)}{n(n+p)}} + \mu\widetilde{\Omega}_{-p}(\mathbf{L})^{\frac{q(n-p)}{n(n+p)}}; \tag{1.8}$$

with equality if and only if K_i and L_i are dilates.

Finally, combining with the combination $\lambda \star \mathbf{K} +_{-q} \mu \star \mathbf{L} = (\lambda \star K_1 +_{-q} \mu \star L_1, \cdots, \lambda \star K_n +_{-q} \mu \star L_n)$, where $\lambda \star K +_{-q} \mu \star L$ denote the L_q-harmonic radial combination of star bodies K and L, and corresponding to Theorem 1.1, we get another Brunn-Minkowski type inequality of the mixed L_p-dual affine surface area for multiple star bodies.

Theorem 1.6. *For $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$ and $\mathbf{L} = (L_1, \ldots, L_n) \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero). If $p > n > 0$, $q \geq 1$, then*

$$\widetilde{\Omega}_{-p}(\lambda \star \mathbf{K} +_{-q} \mu \star \mathbf{L})^{-\frac{q(n-p)}{n(n+p)}} \geq \lambda\widetilde{\Omega}_{-p}(\mathbf{K})^{-\frac{q(n-p)}{n(n+p)}} + \mu\widetilde{\Omega}_{-p}(\mathbf{L})^{-\frac{q(n-p)}{n(n+p)}}; \tag{1.9}$$

equality holds if and only if K_i and L_i are dilates.

The proofs of Theorems 1.3–1.6 will be completed in Section 3 of this paper.

2. Notations and Background Materials

2.1. Radial functions and polar set

If K is a compact star-shaped (with respect to the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [4, 15])

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is positive and continuous, K will be called a star body (respect to the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is a nonempty subset in $\mathbb{R}^n$, the polar set, E^*, of E is defined by (see [4, 15])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}.$$

2.2. L_p-dual mixed volume

Lutwak ([14]) introduced the L_p-dual mixed volume. For $K, L \in \mathcal{S}_o$ and $p \geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$ of K and L is defined by

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u). \tag{2.1}$$

Obviously, $\widetilde{V}_{-p}(K, K) = V(K)$.

Now we extend the L_p-dual mixed volume (2.1) to multiple star bodies as follows: For $\mathbf{K} = (K_1, \ldots, K_n) \in \mathcal{S}_o^n$, $\mathbf{L} = (L_1, \ldots, L_n) \in \mathcal{S}_o^n$, $p \in \mathbb{R}$ ($p \neq -n$ and $p \neq 0$), the L_p-dual mixed volume, $\widetilde{V}_{-p}(\mathbf{K}; \mathbf{L})$ of $\mathbf{K}$ and $\mathbf{L}$ is defined by

$$\widetilde{V}_{-p}(\mathbf{K}; \mathbf{L}) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} [\rho(K_i, u)^{n+p} \rho(L_i, u)^{-p}]^{\frac{1}{n}} dS(u). \tag{2.2}$$

From (2.2), when all K_i coincide with K and all L_i coincide with L, one can easily get $\widetilde{V}_{-p}(\mathbf{K}; \mathbf{L}) = \widetilde{V}_{-p}(K, L)$.

When $L_1 = L_2 = \cdots = L_n = L$, we rewrite (2.2) as follows:

$$\widetilde{V}_{-p}(\mathbf{K}; \underbrace{L, L, \cdots, L}_{n}) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} [\rho(K_i, u)^{n+p}]^{\frac{1}{n}} \rho(L, u)^{-p} dS(u). \tag{2.3}$$

We use $\widetilde{V}(\mathbf{L})$ to denote the dual mixed volume of $\mathbf{L} = (L_1, \ldots, L_n) \in \mathcal{S}_o^n$. That is,

$$\widetilde{V}(\mathbf{L}) = \widetilde{V}(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} \rho_{L_i}(u) dS(u).$$

When $L_1 = \ldots = L_n = L$, one has $\widetilde{V}(\mathbf{L}) = V(L)$. It is easy to get the following inequality for the dual mixed volume:

$$\widetilde{V}(\mathbf{L})^n = \widetilde{V}(L_1, \ldots, L_n)^n \leq V(L_1) \cdots V(L_n),$$

with equality if and only if L_i $(1 \leq i \leq n)$ are dilates of each other.

2.3. Two L_q-combinations

1. L_q-radial combination. For $K, L \in \mathcal{S}_o$, $q > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_q-radial combination, $\lambda \circ K \widetilde{+}_q \mu \circ L \in \mathcal{S}_o$ of K and L is defined by (see [4])

$$\rho(\lambda \circ K \widetilde{+}_q \mu \circ L, \cdot)^q = \lambda \rho(K, \cdot)^q + \mu \rho(L, \cdot)^q, \tag{2.4}$$

where the operation "$\widetilde{+}_q$" is called L_q-radial addition and $\lambda \circ K$ denotes the L_q-radial scalar multiplication. From (2.4), we easily get $\lambda \circ K = \lambda^{\frac{1}{q}} K$. For $q = 1$, L_q-radial combination (2.4) is the classical radial combination (see [4]).

2. L_q-harmonic radial combination. For $K, L \in \mathcal{S}_o$, $q \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_q-harmonic radial combination, $\lambda \star K +_{-q} \mu \star L \in \mathcal{S}_o$ of K and L is defined by (see [3, 1, 2, 14])

$$\rho(\lambda \star K +_{-q} \mu \star K, \cdot)^{-q} = \lambda \rho(K, \cdot)^{-q} + \mu \rho(L, \cdot)^{-q}, \tag{2.5}$$

where the operation " $+_{-q}$ " is called L_q-harmonic radial addition and $\lambda \star K$ denotes the L_q-harmonic radial scalar multiplication. From (2.5), we can obtain $\lambda \star K = \lambda^{-\frac{1}{q}} K$. For $q = 1$, L_q-harmonic radial combination (2.5) is the classical harmonic radial combination (see [14]).

3. Results and Proofs

In this section, we complete the proofs of Theorems 1.3–1.6.

Proof of Theorem 1.3. For $\mathbf{K} = (K_1, \cdots, K_n) \in \mathcal{S}_o^n$, $L \in \mathcal{S}_o$, using (2.3), we obtain

$$\begin{aligned}\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*) &= \frac{1}{n} \int_{S^{n-1}} \rho(L^*, u)^{-p} \prod_{i=1}^{n} \left[\rho(K_i, u)^{n+p}\right]^{\frac{1}{n}} dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \left[\rho(L^*, u)^{-q} \prod_{i=1}^{n} \rho(K_i, u)^{\frac{n+q}{n}}\right]^{\frac{p}{q}} \prod_{i=1}^{n} \rho(K_i, u)^{\frac{q-p}{q}} dS(u).\end{aligned}$$

If $0 < p < q$, i.e., $q/p > 1$, together with the Hölder inequality, then

$$\begin{aligned}\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*) &\leq \left\{\frac{1}{n} \int_{S^{n-1}} \left[\rho(L^*, u)^{-p} \prod_{i=1}^{n} \rho(K_i, u)^{\frac{np+pq}{nq}}\right]^{\frac{q}{p}} dS(u)\right\}^{\frac{p}{q}} \\ &\quad \left\{\frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} (\rho(K_i, u)^{\frac{q-p}{q}})^{\frac{q}{q-p}} dS(u)\right\}^{\frac{q-p}{q}} \\ &\leq \widetilde{V}_{-q}(\mathbf{K}; L^*, \ldots, L^*)^{\frac{p}{q}} \widetilde{V}(\mathbf{K})^{\frac{q-p}{q}}.\end{aligned} \tag{3.1}$$

Thus

$$\left(\frac{\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{V}_{-q}(\mathbf{K}; L^*, \ldots, L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{1}{q}}. \tag{3.2}$$

If $q < p < 0$, i.e. $q/p > 1$, then (3.1) is also hold. Since $p < 0$, we give

$$\left(\frac{\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{1}{p}} \geq \left(\frac{\widetilde{V}_{-q}(\mathbf{K}; L^*, \ldots, L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{1}{q}}. \tag{3.3}$$

(i) For $0 < p < q < n$, applying (1.2) and (3.2), we have

$$\begin{aligned}\left(\frac{\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p}}{n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}}\right)^{\frac{1}{p}} &= \inf_{L \in \mathcal{S}_c} \left\{\left[\frac{n^{n-p}\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*)^n V(L)^{-p}}{n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}}\right]^{\frac{1}{p}}\right\} \\ &= \inf_{L \in \mathcal{S}_c} \left\{\left(\frac{\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{n}{p}} V(L)^{-1} \widetilde{V}(\mathbf{K})^{-1}\right\} \\ &\leq \inf_{L \in \mathcal{S}_c} \left\{\left(\frac{\widetilde{V}_{-q}(\mathbf{K}; L^*, \ldots, L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{n}{q}} V(L)^{-1} \widetilde{V}(\mathbf{K})^{-1}\right\} \\ &= \left(\frac{\widetilde{\Omega}_{-q}(\mathbf{K})^{n-q}}{n^{n-q}\widetilde{V}(\mathbf{K})^{n+q}}\right)^{\frac{1}{q}}.\end{aligned}$$

So (1.4) is obtained.

(ii) For $-n < q < p < 0$, by (1.3) and (3.3), we obtain

$$\left(\frac{\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p}}{n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}}\right)^{\frac{1}{p}} = \sup_{L\in\mathcal{S}_c}\left\{\left[\frac{n^{n-p}\widetilde{V}_{-p}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-p}}{n^{n-p}\widetilde{V}(\mathbf{K})^{n+p}}\right]^{\frac{1}{p}}\right\}$$
$$= \sup_{L\in\mathcal{S}_c}\left\{\left(\frac{\widetilde{V}_{-p}(\mathbf{K};L^*,\ldots,L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{n}{p}} V(L)\widetilde{V}(\mathbf{K})\right\}$$
$$\geq \sup_{L\in\mathcal{S}_c}\left\{\left(\frac{\widetilde{V}_{-q}(\mathbf{K};L^*,\ldots,L^*)}{\widetilde{V}(\mathbf{K})}\right)^{\frac{n}{q}} V(L)\widetilde{V}(\mathbf{K})\right\}$$
$$= \left(\frac{\widetilde{\Omega}_{-q}(\mathbf{K})^{n-q}}{n^{n-q}\widetilde{V}(\mathbf{K})^{n+q}}\right)^{\frac{1}{q}}.$$

This gives (1.5). □

Proof of Theorem 1.4. For $\mathbf{K} = (K_1, \cdots, K_n) \in \mathcal{S}_o^n$ and $L \in \mathcal{S}_o$, from (2.3), we obtain

$$\widetilde{V}_{-p}(\mathbf{K};L^*,\ldots,L^*) = \int_{S^{n-1}} \rho(L^*,u)^{-p}\left[\prod_{i=1}^{n}\left(\rho(K_i,u)^{n+p}\right)^{\frac{1}{n}}\right]dS(u)$$
$$= \int_{S^{n-1}}\left[\rho(L^*,u)^{-q}\left(\prod_{i=1}^{n}\rho(K_i,u)^{\frac{n+q}{n}}\right)\right]^{\frac{p-r}{q-r}}\left[\rho(L^*,u)^{-r}\left(\prod_{i=1}^{n}\rho(K_i,u)^{\frac{n+r}{n}}\right)\right]^{\frac{q-p}{q-r}}dS(u).$$

If $0 < r < q < p < n$, i.e., $0 < \frac{q-r}{p-r} < 1$, then by the Hölder inequality, we get

$$\begin{aligned}\widetilde{V}_{-p}(\mathbf{K};L^*,\ldots,L^*) &\geq \left[\int_{S^{n-1}}\rho(L^*,u)^{-q}\left(\prod_{i=1}^{n}\rho(K_i,u)^{\frac{n+q}{n}}\right)dS(u)\right]^{\frac{p-r}{q-r}}\\ &\quad\left[\int_{S^{n-1}}\rho(L^*,u)^{-r}\left(\prod_{i=1}^{n}\rho(K_i,u)^{\frac{n+r}{n}}\right)dS(u)\right]^{\frac{q-p}{q-r}}\\ &= \widetilde{V}_{-q}(\mathbf{K};L^*,\ldots,L^*)^{\frac{p-r}{q-r}}\widetilde{V}_{-r}(\mathbf{K};L^*,\ldots,L^*)^{\frac{q-p}{q-r}}.\end{aligned} \tag{3.4}$$

Thus

$$\widetilde{V}_{-p}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-p} \geq \left[\widetilde{V}_{-q}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-q}\right]^{\frac{p-r}{q-r}}\left[\widetilde{V}_{-r}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-r}\right]^{\frac{q-p}{q-r}}.$$

This combining with (1.2), and notice $n > p$, yields

$$\begin{aligned}\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p} &= \inf_{L\in\mathcal{S}_c}\left\{n^{n-p}\widetilde{V}_{-p}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-p}\right\}\\ &\geq \inf_{L\in\mathcal{S}_c}\left\{n^{n-q}\widetilde{V}_{-q}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-q}\right\}^{\frac{p-r}{q-r}}\\ &\quad\inf_{L\in\mathcal{S}_c}\left\{n^{n-r}\widetilde{V}_{-r}(\mathbf{K};L^*,\ldots,L^*)^n V(L)^{-r}\right\}^{\frac{q-p}{q-r}}\\ &= \widetilde{\Omega}_{-q}(\mathbf{K})^{\frac{(n-q)(p-r)}{q-r}}\widetilde{\Omega}_{-r}(\mathbf{K})^{\frac{(n-r)(q-p)}{q-r}}.\end{aligned}$$

So (1.6) is obtained.

If $r < p < q < 0$, i.e., $\frac{q-r}{p-r} > 1$, then inequality (3.4) is reversed, that is

$$\widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*) \leq \widetilde{V}_{-q}(\mathbf{K}; L^*, \ldots, L^*)^{\frac{p-r}{q-r}} \widetilde{V}_{-r}(\mathbf{K}; L^*, \ldots, L^*)^{\frac{q-p}{q-r}}.$$

This combining with (1.3), we have that

$$\begin{aligned}
\widetilde{\Omega}_{-p}(\mathbf{K})^{n-p} &= \sup_{L\in\mathcal{S}_c} \left\{ n^{n-p} \widetilde{V}_{-p}(\mathbf{K}; L^*, \ldots, L^*)^n V(L)^{-p} \right\} \\
&\leq \sup_{L\in\mathcal{S}_c} \left\{ n^{n-q} \widetilde{V}_{-q}(\mathbf{K}; L^*, \ldots, L^*)^n V(L)^{-q} \right\}^{\frac{p-r}{q-r}} \\
&\quad \sup_{L\in\mathcal{S}_c} \left\{ n^{n-r} \widetilde{V}_{-r}(\mathbf{K}; L^*, \ldots, L^*)^n V(L)^{-r} \right\}^{\frac{q-p}{q-r}} \\
&= \widetilde{\Omega}_{-q}(\mathbf{K})^{\frac{(n-q)(p-r)}{q-r}} \widetilde{\Omega}_{-r}(\mathbf{K})^{\frac{(n-r)(q-p)}{q-r}}.
\end{aligned}$$

This yields (1.7). □

In the following we will prove Theorem 1.5 and 1.6. The Minkowski's produce type inequality obtained by Kuang [6] is needed.

Lemma 3.1 (Minkowski's product type inequality)**.** *Let a_k, $b_k \geq 0$, then*

$$\left\{ \prod_{k=1}^{n} (a_k + b_k) \right\}^{\frac{1}{n}} \geq \left(\prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k \right)^{\frac{1}{n}},$$

with equality if and only if a_k and b_k are proportional.

Lemma 3.2. *For $\mathbf{K} = (K_1, \cdots, K_n) \in \mathcal{S}_o^n$, $\mathbf{L} = (L_1, \cdots, L_n) \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero). If $p > 0$ and $q > n + p$, then for any $\mathbf{Q} = (Q_1, \cdots, Q_n) \in \mathcal{S}_o^n$,*

$$\widetilde{V}_{-p}(\lambda \circ \mathbf{K} \widetilde{+}_q \mu \circ \mathbf{L}; \mathbf{Q})^{\frac{q}{n+p}} \geq \lambda \widetilde{V}_{-p}(\mathbf{K}; \mathbf{L})^{\frac{q}{n+p}} + \mu \widetilde{V}_{-p}(\mathbf{L}; \mathbf{Q})^{\frac{q}{n+p}}, \tag{3.5}$$

with equality if and only if K_i and L_i are dilates.

Proof. Since $p > 0$, $q > n + p$, thus $0 < \frac{n+p}{q} < 1$. Hence, from (2.2), (2.4), Lemma 3.1 and the Minkowski's integral inequality (see [5]), we get

$$\begin{aligned}
\widetilde{V}_{-p}(\lambda \circ \mathbf{K} \widetilde{+}_q \mu \circ \mathbf{L}; \mathbf{Q})^{\frac{q}{n+p}} &= \left\{ \frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} \rho(\lambda \circ K_i \widetilde{+}_q \mu \circ L_i, u)^{\frac{n+p}{n}} \rho(Q_i, u)^{-\frac{p}{n}} dS(u) \right\}^{\frac{q}{n+p}} \\
&= \left\{ \frac{1}{n} \int_{S^{n-1}} \left[\prod_{i=1}^{n} \rho(\lambda \circ K_i \widetilde{+}_q \mu \circ L_i, u)^{\frac{q}{n}} \rho(Q_i, u)^{-\frac{pq}{n(n+p)}} \right]^{\frac{n+p}{q}} dS(u) \right\}^{\frac{q}{n+p}} \\
&= \left\{ \frac{1}{n} \int_{S^{n-1}} \left[\prod_{i=1}^{n} \left(\lambda \rho(K_i, u)^q + \mu \rho(L_i, u)^q \right)^{\frac{1}{n}} \rho(Q_i, u)^{-\frac{pq}{n(n+p)}} \right]^{\frac{n+p}{q}} dS(u) \right\}^{\frac{q}{n+p}} \\
&\geq \left\{ \frac{1}{n} \int_{S^{n-1}} \left[\lambda \prod_{i=1}^{n} \rho(K_i, u)^{\frac{q}{n}} \rho(Q_i, u)^{-\frac{pq}{n(n+p)}} \right. \right. \\
&\qquad \left. \left. + \mu \prod_{i=1}^{n} \rho(L_i, u)^{\frac{q}{n}} \rho(Q_i, u)^{-\frac{pq}{n(n+p)}} \right]^{\frac{n+p}{q}} dS(u) \right\}^{\frac{q}{n+p}}
\end{aligned}$$

$$\geq \lambda\left[\frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\rho(K_i,u)^{\frac{n+p}{n}}\rho(Q_i,u)^{-\frac{p}{n}}dS(u)\right]^{\frac{q}{n+p}}$$
$$+\mu\left[\frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\rho(L_i,u)^{\frac{n+p}{n}}\rho(Q_i,u)^{-\frac{p}{n}}dS(u)\right]^{\frac{q}{n+p}}$$
$$=\lambda\widetilde{V}_{-p}(\mathbf{K};\mathbf{Q})^{\frac{q}{n+p}}+\mu\widetilde{V}_{-p}(\mathbf{L};\mathbf{Q})^{\frac{q}{n+p}}.$$

According to the equality conditions of Lemma 3.1 and Minkowski's integral inequality, we see that equality holds in (3.5) if and only if K_i and L_i are dilates. □

Proof of Theorem 1.5. Since $0<p<n$, $q>1+\frac{p}{n}$, thus by (1.2) and (3.5), we have

$$\left\{\widetilde{\Omega}_{-p}(\lambda\circ\mathbf{K}\widetilde{+}_q\mu\circ\mathbf{L})^{\frac{n-p}{n}}\right\}^{\frac{q}{n+p}}=\left\{\inf_{Q\in\mathcal{S}_c}\{n^{\frac{n-p}{n}}\widetilde{V}_{-p}(\lambda\circ\mathbf{K}\widetilde{+}_q\mu\circ\mathbf{L};Q^*,\cdots,Q^*)V(Q)^{-\frac{p}{n}}\}\right\}^{\frac{q}{n+p}}$$
$$=\inf_{Q\in\mathcal{S}_c}\left\{\left[n^{\frac{n-p}{n}}\widetilde{V}_{-p}(\lambda\circ\mathbf{K}\widetilde{+}_q\mu\circ\mathbf{L};Q^*,\cdots,Q^*)\right]^{\frac{q}{n+p}}\left[V(Q)^{-\frac{p}{n}}\right]^{\frac{q}{n+p}}\right\}$$
$$\geq\inf_{Q\in\mathcal{S}_c}\left\{\lambda\left[n^{\frac{n-p}{n}}\widetilde{V}_{-p}(\mathbf{K};Q^*,\cdots,Q^*)V(Q)^{-\frac{p}{n}}\right]^{\frac{q}{n+p}}\right\}$$
$$+\inf_{Q\in\mathcal{S}_c}\left\{\mu\left[n^{\frac{n-p}{n}}\widetilde{V}_{-p}(\mathbf{L};Q^*,\cdots,Q^*)V(Q)^{-\frac{p}{n}}\right]^{\frac{q}{n+p}}\right\}$$
$$=\lambda\left[\widetilde{\Omega}_{-p}(\mathbf{K})^{\frac{n-p}{n}}\right]^{\frac{q}{n+p}}+\mu\left[\widetilde{\Omega}_{-p}(\mathbf{L})^{\frac{n-p}{n}}\right]^{\frac{q}{n+p}}.$$

According to the equality condition of (3.5), we see that equality holds in (1.8) if and only if K_i and L_i are dilates. □

Using the proof method of Lemma 3.2 and combining with L_q-harmonic radial combination (2.5), we easily obtain the following result for the L_p-dual mixed volume.

Lemma 3.3. *If $\mathbf{K}=(K_1,\cdots,K_n)\in\mathcal{S}_o^n$, $\mathbf{L}=(L_1,\cdots,L_n)\in\mathcal{S}_o^n$, $\lambda,\mu\geq 0$ (not both zero). If $p>0$, $q\geq 1$, then for any $\mathbf{Q}=(Q_1,\cdots,Q_n)\in\mathcal{S}_o^n$,*

$$\widetilde{V}_{-p}(\lambda\star\mathbf{K}+_{-q}\mu\star\mathbf{L};\mathbf{Q})^{-\frac{q}{n+p}}\geq\lambda\widetilde{V}_{-p}(\mathbf{K};\mathbf{Q})^{-\frac{q}{n+p}}+\mu\widetilde{V}_{-p}(\mathbf{L};\mathbf{Q})^{-\frac{q}{n+p}},\tag{3.6}$$

with equality if and only if K_i and L_i are dilates.

Proof. Since $p>0$, $q\geq 1$, thus $-\frac{n+p}{q}<0$. Hence, by (2.2), (2.5), Lemma 3.1 and the Minkowski's integral inequality (see [5]), we get

$$\widetilde{V}_{-p}(\lambda\star\mathbf{K}\widetilde{+}_{-q}\mu\star\mathbf{L};\mathbf{Q})^{-\frac{q}{n+p}}$$
$$=\left\{\frac{1}{n}\int_{S^{n-1}}\prod_{i=1}^{n}\rho(\lambda\star K_i\widetilde{+}_{-q}\mu\star L_i,u)^{\frac{n+p}{n}}\rho(Q_i,u)^{-\frac{p}{n}}dS(u)\right\}^{-\frac{q}{n+p}}$$
$$=\left\{\frac{1}{n}\int_{S^{n-1}}\left[\prod_{i=1}^{n}\rho(\lambda\star K_i\widetilde{+}_{-q}\mu\star L_i,u)^{-\frac{q}{n}}\rho(Q_i,u)^{\frac{pq}{n(n+p)}}\right]^{-\frac{n+p}{q}}dS(u)\right\}^{-\frac{q}{n+p}}$$
$$=\left\{\frac{1}{n}\int_{S^{n-1}}\left[\prod_{i=1}^{n}\left(\lambda\rho(K_i,u)^{-q}+\mu\rho(L_i,u)^{-q}\right)^{\frac{1}{n}}\rho(Q_i,u)^{\frac{pq}{n(n+p)}}\right]^{-\frac{n+p}{q}}dS(u)\right\}^{-\frac{q}{n+p}}$$

$$\geq \left\{ \frac{1}{n} \int_{S^{n-1}} \left[\lambda \prod_{i=1}^{n} \rho(K_i,u)^{-\frac{q}{n}} \rho(Q_i,u)^{\frac{pq}{n(n+p)}} + \mu \prod_{i=1}^{n} \rho(L_i,u)^{-\frac{q}{n}} \rho(Q_i,u)^{\frac{pq}{n(n+p)}} \right]^{-\frac{n+p}{q}} dS(u) \right\}^{-\frac{q}{n+p}}$$

$$\geq \lambda \left[\frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} \rho(K_i,u)^{\frac{n+p}{n}} \rho(Q_i,u)^{-\frac{p}{n}} dS(u) \right]^{-\frac{q}{n+p}}$$

$$+\mu \left[\frac{1}{n} \int_{S^{n-1}} \prod_{i=1}^{n} \rho(L_i,u)^{\frac{n+p}{n}} \rho(Q_i,u)^{-\frac{p}{n}} dS(u) \right]^{-\frac{q}{n+p}}$$

$$= \lambda \widetilde{V}_{-p}(\mathbf{K};\mathbf{Q})^{-\frac{q}{n+p}} + \mu \widetilde{V}_{-p}(\mathbf{L};\mathbf{Q})^{-\frac{q}{n+p}}.$$

According to the equality conditions of Lemma 3.1 and the Minkowski's integral inequality, we see that equality holds in (3.6) if and only if K_i and L_i are dilates. □

Proof of Theorem 1.6. If $p > n > 0$, $q \geq 1$, then from (1.2) and (3.6), and notice that $n - p < 0$ and $-\frac{n+p}{q} < 0$ we have

$$\left[\widetilde{\Omega}_{-p}(\lambda \star \mathbf{K} +_{-q} \mu \star \mathbf{L})^{\frac{n-p}{n}} \right]^{-\frac{q}{n+p}} \geq \inf_{Q \in \mathcal{S}_c} \left\{ \lambda \left[n^{\frac{n-p}{n}} \widetilde{V}_{-p}(\mathbf{K}; Q^*, \cdots, Q^*) V(Q)^{-\frac{p}{n}} \right]^{-\frac{q}{n+p}} \right\}$$

$$+ \inf_{Q \in \mathcal{S}_c} \left\{ \mu \left[n^{\frac{n-p}{n}} \widetilde{V}_{-p}(\mathbf{L}; Q^*, \cdots, Q^*) V(Q)^{-\frac{p}{n}} \right]^{-\frac{q}{n+p}} \right\}$$

$$= \lambda \left[\widetilde{\Omega}_{-p}(\mathbf{K})^{\frac{n-p}{n}} \right]^{-\frac{q}{n+p}} + \mu \left[\widetilde{\Omega}_{-p}(\mathbf{L})^{\frac{n-p}{n}} \right]^{-\frac{q}{n+p}}.$$

This gives (1.9).

According to the equality condition of (3.6), we see that equality holds in (1.9) if and only if K_i and L_i are dilates. □

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