Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2846–2856
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On the approximation of a convex body by its radial mean bodies

Lvzhou Zheng

School of Mathematics and Statistics, Hubei Normal University, 435002 Huangshi, P. R. China.

Communicated by R. Saadati

Abstract

In this paper, we consider the approximation problem on the volume of a convex body K in $\mathbb{R}^n$ by those of its radial mean bodies R_pK. Specifically, we establish the identity

$$\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\frac{|R_p(K)|}{|K|})=\frac{n(n+1)}{2},$$

when K is an ellipsoid in $\mathbb{R}^n$.

Keywords: Convex body, radial mean body, difference body, restricted chord projection function.
2010 MSC: 53A40, 41A25.

1. Introduction

In convex geometry, corresponding to each convex body $K \subset \mathbb{R}^n$, there are two important geometric objects called difference body DK and polar projection body Π^*K. The difference body was studied by Minkowski, and has found many applications in mathematical physics and PDEs. See, for example, the books of Bandle [2] and Kawohl [11]. Projection bodies also originated in the work of Minkowski, and are widely used in the local theory of Banach spaces, stochastic geometry, mathematical economics, and other areas [4, 9]. The polar projection body, the polar body of the projection body, appears explicitly in the more recent literature; its behavior under linear transformations often renders it more natural than the projection body itself.

Both the difference body and the polar projection body appear in known affine inequalities. The first is an ingredient in the famous Rogers - Shephard inequality [6, 21], that is,

$$V(DK)\leq\binom{2n}{n}V(K). \tag{1.1}$$

Email address: oasiszlz@sina.com (Lvzhou Zheng)

Received 2015-11-06

The second appears in the celebrated Zhang projection inequality [8, 27, 28]

$$n^{-n}\binom{2n}{n} \leq V(K)^{n-1}V(\Pi^*K), \tag{1.2}$$

where the equality holds in (1.1) as well as in (1.2), if and only if K is a simplex.

Given a convex body $K \in \mathbb{R}^n$ and $p > -1$, Gardner and Zhang [8] originally introduced an important geometric body, called *radial* p-th *mean body* R_pK of K, whose radial function is defined by

$$\rho_{R_pK}(u) = (\frac{1}{V(K)}\int_K \rho_K(x,u)^p dx)^{\frac{1}{p}}, \quad \forall u \in S^{n-1}. \tag{1.3}$$

It is remarkable that the bodies R_pK form a *spectrum* linking the difference body DK of K and the polar projection body Π^*K of K, which correspond to $p = \infty$ and $p = -1$, respectively. More importantly, for $-1 < p < q$, the following strong and sharp affine inequality

$$V(DK) \leq c_{n,q}^n V(R_qK) \leq c_{n,p}^n V(R_pK) \leq n^n V(K)^n V(\Pi^*K), \tag{1.4}$$

which was established in [8] , implies the above mentioned Rogers-Shephard inequality and Zhang projection inequality as special cases. In (1.4), each equality holds if and only if K is a *simplex*, and $c_{n,p} = (nB(p+1,n))^{-\frac{1}{p}}$ is a constant.

Specifically, when $p = n$ and $q \to \infty$, the middle inequality in (1.4) becomes the Rogers-Shephard inequality, and when $p \to -1$ and $q = n$, it becomes the Zhang projection inequality. Therefore, in some sense, radial mean bodies R_pK exhibit a strong unity in convex geometry. In [26] , the authors established the identity related chord power integrals of convex body K and dual quermassintegrals of R_pK.

It is proved in [8] that for $p \geq 0$, the radial p-th mean body R_pK is an origin-symmetric convex body. Now, a problem is naturally asked,

Problem. *If K is a convex body in $\mathbb{R}^n$, how about the rate of approximation on the volume $V(K)$ of convex body K by the volume $V(R_pK)$ of its radial p-th mean body R_pK ?*

It is noted that the approximation problem of a convex body by its associated bodies, such as floating bodies, convolution bodies, and centroid bodies, projection bodies etc, have been intensively investigated. We refer to e.g. [5, 8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 23, 24, 25] for further details, extensions and applications. As an aside, we observe that throughout the whole paper [8], all affine inequalities attain extremum if and only if the convex body is a simplex. Therefore, it will naturally lead us to study the radial mean bodies R_pK when K is an origin-symmetric convex body.

From now on, we shall use $|\cdot|$ to represent the $n-$dimensional volume $V(\cdot)$ of a convex body in $\mathbb{R}^n$. In this paper, for the affine invariant ratio $\frac{|R_pK|}{|K|}$, we will prove the following theorems.

Theorem 1.1. *Suppose K is an ellipsoid in $\mathbb{R}^n$ and R_pK is the radial p-th mean body of K. Then*

$$\lim_{p\to\infty} \frac{p}{\log p}(1 - 2^{-n}\frac{|R_pK|}{|K|}) = \frac{n(n+1)}{2}. \tag{1.5}$$

Theorem 1.2. *Suppose K is a simplex in $\mathbb{R}^n$. Then*

$$\lim_{p\to\infty} \frac{p}{\log p}(1 - \binom{2n}{n}^{-1}\frac{|R_pK|}{|K|}) = n^2. \tag{1.6}$$

.

This paper is organized as follows. In Section 2, we develop some notation and list, some basic facts regarding convex bodies. Good general references for the theory of convex bodies are provided by the books of Gardner [7] and Schneider [22]. In Section 3, we give some bounds for the approximation of volume in the case of a general convex body. The proofs of Theorems 1.1 and 1.2 will be arranged in the Section 4.

2. Notations and Preliminaries

The setting for this paper is $n-$dimensional Euclidean space, $\mathbb{R}^n$. A *convex body* K in $\mathbb{R}^n$ is a compact convex set that has a non-empty interior. As usual, S^{n-1} denotes the unit sphere, B_n the unit ball and o the origin in $\mathbb{R}^n$. The volume of B_n is denoted by ω_n. If $u \in S^{n-1}$, we denote by $u^\perp$ the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to u and by l_u the line through o parallel to u. We write V_k for the $k-$dimensional Lebesgue measure in $\mathbb{R}^n$.

Let K be a convex body in $\mathbb{R}^n$. The *radial function* $\rho_K(x,\cdot)$ of K with respect to $x \in \mathbb{R}^n$, is defined by $\rho_K(x,u) = \max\{c : x + cu \in K\}$, $\forall u \in S^{n-1}$. If x is the origin, we usually denote $\rho_K(o,u)$ by $\rho_K(u)$.

For $u \in S^{n-1}$ and $y \in u^\perp$, let $X_uK(y) = V_1(K \cap (l_u + y))$, the function is called the $X-ray$ of K in the direction u. See [7] for details.

Let

$$E_K(r,u) = \{y \in u^\perp : X_uK(y) \geq r\}$$

and

$$a_K(r,u) = V_{n-1}(E_K(r,u))$$

for $r \geq 0$, and $u \in S^{n-1}$. In [27] the function $a_K(r,u)$ is called the *restricted chord projection function* of K. Note that if $u \in S^{n-1}$, then $E_K(0,u) = K|u^\perp$ and $a_K(0,u) = V_{n-1}(K|u^\perp)$, and when $r > \rho_{DK}(u)$, we have $E_K(r,u) = \emptyset$ and $a_K(r,u) = 0$.

The *difference body* of the convex body K, denoted by DK, is the centrally symmetric convex body (centered at the origin) defined by

$$DK = K + (-K) = \{x - y : x, y \in K\}.$$

It is not difficult to verify that

$$\rho_{DK}(u) = \max_{x\in K} \rho_K(x,u) = \max_{y \in u^\perp} V_1(K \cap (l_u + y)), \quad u \in S^{n-1}.$$

An often used fact in both convex and Banach space geometry is that associated with each convex body K in $\mathbb{R}^n$ is a unique ellipsoid JK of maximal volume contained in K. The ellipsoid is called the *John ellipsoid* of K and the center of this ellipsoid is called the *John point* of K. The John ellipsoid is extremely useful; see, for example, [1, 17]and [20] for applications.

Two important results concerning the John ellipsoid are John's inclusion and Ball's volume-ratio inequality. $John's\ inclusion$ states that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\mathrm{JK} \subseteq K \subseteq \sqrt{n}\mathrm{JK}. \tag{2.1}$$

Among a slew of applications, John's inclusion gives the best upper bound, $\sqrt{n}$, for the Banach-Mazur distance of an $n-$dimensional normed space to $n-$dimensional Euclidean space. Ball's $volume - ratio$ *inequality* is the following: if K is an origin-symmetric convex body in $\mathbb{R}^n$, then

$$\frac{|K|}{|\mathrm{JK}|} \leq \frac{2^n}{\omega_n}, \tag{2.2}$$

with equality if and only if K is a parallelotope. The fact that there is equality in (2.2) only for parallelotopes was established by Barthe [3].

Lemma 2.1 (Markov's Inequality)**.** *Suppose $(X, \sum, \mu)$ is a measure space, f is a measurable extended real-valued function and $\varepsilon > 0$. Then*

$$\mu(\{x \in X : |f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X |f| d\mu. \tag{2.3}$$

The invariant property of radial p-th mean body under non-singular linear transformation shows that they are natural objects in affine geometry.

Lemma 2.2 ([26])**.** *Let K be a convex body in $\mathbb{R}^n$ and $\mathrm{GL(n)}$ the nonsingular linear transformation group. Then for $\varphi \in \mathrm{GL(n)}$ and $p > -1$, we have $R_p(\varphi K) = \varphi(R_p K)$.*

Lemma 2.3 ([8])**.** *Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. Then for $p > -1$, we have*

$$\int_K \rho_K(x,u)^p dx = \int_0^{\rho_{DK}(u)} a_K(r,u) r^p dr. \tag{2.4}$$

We will also use the following lemma.

Lemma 2.4 ([18])**.** *Let $p > 0$. Then*

$$B\left(p+1,n\right)^{\frac{n}{p}} = 1 - \frac{n^2}{p}\log p + \frac{n}{p}\log(\Gamma(n)) + \frac{n^4}{2p^2}(\log p)^2 - \frac{n^3}{p^2}\log(\Gamma(n))\log p \pm o(p^{-2}). \tag{2.5}$$

3. General Bounds

Lemma 3.1. *Let K be a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$. Then*

$$(1 - \frac{r}{\rho_{DK}(u)})^{n-1} V_{n-1}(K|u^\perp) \le a_K(r,u) \le V_{n-1}(K|u^\perp), \tag{3.1}$$

where the equality holds in the left hand if and only if $a_K^{\frac{1}{n-1}}(r,u)$ is linear in r, the equality holds in the right hand if and only if $r = 0$.

Proof. The right inequality is obvious. Since $a_K(r,u)^{\frac{1}{n-1}}$ is concave in r, we have

$$\begin{aligned} a_K(r,u)^{\frac{1}{n-1}} &= a_K(\frac{r}{\rho_{DK}(u)}\rho_{DK}(u) + (1-\frac{r}{\rho_{DK}(u)})0, u)^{\frac{1}{n-1}} \\ &\ge \frac{r}{\rho_{DK}(u)} a_K(\rho_{DK}(u),u)^{\frac{1}{n-1}} + \left(1 - \frac{r}{\rho_{DK}(u)}\right) a_K(0,u)^{\frac{1}{n-1}} \\ &\ge \left(1 - \frac{r}{\rho_{DK}(u)}\right) a_K(0,u)^{\frac{1}{n-1}} \\ &= \left(1 - \frac{r}{\rho_{DK}(u)}\right) V_{n-1}(K|u^\perp)^{\frac{1}{n-1}}. \end{aligned}$$

The equality condition can be derived from the arguments easily. This completes the proof. □

Lemma 3.2. *Let K be a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$.*

(1) For $-1 < p < 0$, we have

$$\left(\frac{n}{p+1}\right)^{\frac{1}{p}} \le \frac{\rho_{R_pK}(u)}{\rho_{DK}(u)} \le B(p+1,n)^{\frac{1}{p}}.$$

(2) For $p > 0$, we have

$$B(p+1,n)^{\frac{1}{p}} \le \frac{\rho_{R_pK}(u)}{\rho_{DK}(u)} \le \left(\frac{n}{p+1}\right)^{\frac{1}{p}}.$$

Proof. According to the formula $|K| = \int_0^{\rho_{DK}(u)} a_K(r,u)dr$ and (3.1), we have

$$|K| = \int_0^{\rho_{DK}(u)} a_K(r,u)dr \le \rho_{DK}(u) a_K(0,u) = \rho_{DK}(u) V_{n-1}(K|u^\perp),$$

and

$$
\begin{aligned}
|K| &= \int_0^{\rho_{DK}(u)} a_K(r,u)dr \\
&\geq \int_0^{\rho_{DK}(u)} \left(1-\frac{r}{\rho_{DK}(u)}\right)^{n-1} V_{n-1}(K|u^{\perp})dr \\
&= \frac{1}{n}\rho_{DK}(u)V_{n-1}(K|u^{\perp}).
\end{aligned}
$$

It yields that

$$
\frac{1}{n}\rho_{DK}(u)V_{n-1}(K|u^{\perp}) \leq |K| \leq \rho_{DK}(u)V_{n-1}(K|u^{\perp}). \tag{3.2}
$$

Combined with (3.1) and (3.2), we have

$$
\begin{aligned}
\rho^p_{R_pK}(u) &= \frac{1}{|K|}\int_K \rho_K(x,u)^p dx \\
&= \frac{1}{|K|}\int_0^{\rho_{DK}(u)} r^p a_K(r,u)dr \\
&\leq \frac{V_{n-1}(K|u^{\perp})}{|K|}\frac{1}{p+1}\rho^{p+1}_{DK}(u) \\
&\leq \frac{n}{p+1}\rho^p_{DK}(u).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\rho^p_{R_pK}(u) &= \frac{1}{|K|}\int_0^{\rho_{DK}(u)} r^p a_K(r,u)dr \\
&\geq \frac{1}{|K|}\int_0^{\rho_{DK}(u)} r^p \left(1-\frac{r}{\rho_{DK}(u)}\right)^{n-1} V_{n-1}(K|u^{\perp})dr \\
&= \frac{V_{n-1}(K|u^{\perp})}{|K|}\rho^{p+1}_{DK}(u)\int_0^1 s^p(1-s)^{n-1}ds \\
&\geq \rho^p_{DK}(u)B(p+1,n).
\end{aligned}
$$

This completes the proof. □

From Lemma 3.2, we can get immediately that,

Corollary 3.3. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$.*

(1) For $-1 < p < 0$, we have

$$
2(\frac{n}{p+1})^{\frac{1}{p}} \leq \frac{\rho_{R_pK}(u)}{\rho_{DK}(u)} \leq 2B(p+1,n)^{\frac{1}{p}}.
$$

(2) For $p > 0$, we have

$$
2B(p+1,n)^{\frac{1}{p}} \leq \frac{\rho_{R_pK}(u)}{\rho_{DK}(u)} \leq 2(\frac{n}{p+1})^{\frac{1}{p}}.
$$

By using Markov's inequality, we can obtain a new upper bound for $\frac{\rho_{R_pK}(u)}{\rho_{DK}(u)}$.

Lemma 3.4. *Let K be a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$. If $p > 0$, we have*

$$
\frac{\rho_{R_pK}(u)}{\rho_{DK}(u)} \leq \left(\frac{1}{p}\right)^{\frac{1}{p}}. \tag{3.3}
$$

Proof. According to Markov's inequality (2.3), we have

$$\begin{aligned} a_K(r,u) &= V_{n-1}(\{y \in K|u^\perp : X_uK(y) \geq r\}) \\ &\leq \frac{1}{r}\int_{K|u^\perp} X_uK(y)dy = \frac{|K|}{r}. \end{aligned}$$

With Lemma 2.3, it yields

$$\rho^p_{R_pK}(u) = \frac{1}{|K|}\int_K \rho_K(x,u)^p dx \leq \frac{1}{|K|}\int_0^{\rho_{DK}(u)} \frac{|K|}{r} r^p dr = \frac{1}{p}\rho^p_{DK}(u).$$

This completes the proof. □

Theorem 3.5. *Let K be a convex body in $\mathbb{R}^n$ and $p > 0$. Then*

$$n|DK| \leq \lim_{p\to\infty} \frac{p}{\log p}(|DK| - |R_pK|) \leq n^2|DK|. \tag{3.4}$$

Proof. We have that

$$\begin{aligned} |DK| - |R_pK| &= \frac{1}{n}\int_{S^{n-1}} \left(\rho^n_{DK}(u) - \rho^n_{R_pK}(u)\right) du \\ &= \frac{1}{n}\int_{S^{n-1}} \rho^n_{DK}(u)\left(1 - \frac{\rho^n_{R_pK}(u)}{\rho^n_{DK}(u)}\right) du. \end{aligned}$$

From Lemma 3.4 , it follows

$$\frac{\rho^n_{R_pK}(u)}{\rho^n_{DK}(u)} \leq \left(\frac{1}{p}\right)^{\frac{n}{p}} = 1 - \frac{n\log p}{p} \pm o(\frac{\log p}{p}).$$

From Lemma 3.2 and Lemma 2.4, it implies

$$\begin{aligned} \frac{\rho^n_{R_pK}(u)}{\rho^n_{DK}(u)} &\geq B(p+1,n)^{\frac{n}{p}} \\ &= 1 - \frac{n^2\log p}{p} + \frac{n}{p}\log(\Gamma(n)) + \frac{n^4}{2p^2}(\log p)^2 \\ &\quad - \frac{n^3}{p^2}\log(\Gamma(n))\log p \pm o(p^{-2}). \end{aligned}$$

By using Lebesgue convergence Theorem, it gives

$$\begin{aligned} \lim_{p\to\infty}\frac{p}{\log p}(|DK| - |R_pK|) &= \frac{1}{n}\int_{S^{n-1}} \rho^n_{DK}(u)\lim_{p\to\infty}\frac{p}{\log p}\left(1 - \frac{\rho^n_{R_pK}(u)}{\rho^n_{DK}(u)}\right) du \\ &\geq \frac{1}{n}\int_{S^{n-1}} \rho^n_{DK}(u)\lim_{p\to\infty}\frac{p}{\log p}\left(\frac{n\log p}{p} \pm o(\frac{\log p}{p})\right) du \\ &= n|DK|, \end{aligned}$$

and

$$\begin{aligned} \lim_{p\to\infty}\frac{p}{\log p}(|DK| - |R_pK|) &= \frac{1}{n}\int_{S^{n-1}} \rho^n_{DK}(u)\lim_{p\to\infty}\frac{p}{\log p}\left(1 - \frac{\rho^n_{R_pK}(u)}{\rho^n_{DK}(u)}\right) du \\ &\leq \frac{1}{n}\int_{S^{n-1}} \rho^n_{DK}(u)\lim_{p\to\infty}\frac{p}{\log p}\left(\frac{n^2\log p}{p} \pm o(\frac{\log p}{p})\right) du \\ &= n^2|DK|, \end{aligned}$$

which yields the required inequalities. This completes the proof. □

If convex body K is origin-symmetric, then $DK = K + (-K) = 2K$. Hence, we have:

Corollary 3.6. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and $p > 0$. Then*

$$n \leq \lim_{p\to\infty} \frac{p}{\log p}(1 - 2^{-n}\frac{|R_pK|}{|K|}) \leq n^2.$$

4. Proof of Main Results

Firstly, we prove Theorem 1.1, which is involved in a large number of estimations.

Proof of Theorem 1.1. We first consider the radial p-th mean body of the unit ball B_n. Obviously, we have

$$a_{B_n}(r, u) = V_{n-1}(\sqrt{1 - (\frac{r}{2})^2}B_{n-1}), \quad \rho_{DB_n}(u) = 2,$$

for all $u \in S^{n-1}$. From Lemma 2.3, it gives

$$\begin{aligned}
\rho^p_{R_pB_n}(u) &= \frac{1}{V(B_n)}\int_{B_n} \rho_{B_n}(x, u)^p dx \\
&= \frac{1}{V(B_n)}\int_0^{\rho_{DB_n}(u)} a_{B_n}(r, u)r^p dr \\
&= \frac{\omega_{n-1}}{\omega_n}\int_0^2 (1 - \frac{r^2}{4})^{\frac{n-1}{2}} r^p dr \\
&= \frac{2^p\omega_{n-1}}{\omega_n}\int_0^1 (1 - r)^{\frac{n-1}{2}} r^{\frac{p-1}{2}} dr \\
&= \frac{2^p\omega_{n-1}}{\omega_n} B(\frac{n+1}{2}, \frac{p+1}{2})
\end{aligned}$$

for all $p > 0$. Let

$$d_{n,p} = \{\frac{2^p\omega_{n-1}}{\omega_n} B(\frac{n+1}{2}, \frac{p+1}{2})\}^{\frac{1}{p}},$$

we have

$$\rho_{R_pB_n}(u) = d_{n,p},$$

which is a constant depending only on the numbers p and n. Hence the radial p-th mean body of unit ball B_n is still a ball centered at the origin with radius $d_{n,p}$, i.e., $R_pB_n = d_{n,p}B_n$.

When K is a centered ellipsoid, there exists $\varphi \in \mathrm{GL}(n)$ such that $K = \varphi B_n$. From Lemma 2.2 and the above argument, we have

$$R_pK = R_p(\varphi B_n) = \varphi(R_pB_n) = \varphi(d_{n,p}B_n) = d_{n,p}\varphi(B_n) = d_{n,p}K,$$

which is still a centered ellipsoid.

Now, we have

$$\begin{aligned}
\lim_{p\to\infty} \frac{p}{\log p}(1 - 2^{-n}\frac{|R_pK|}{|K|}) &= \lim_{p\to\infty} \frac{p}{\log p}(1 - 2^{-n}\frac{|R_pB_n|}{|B_n|}) \\
&= \lim_{p\to\infty} \frac{p}{\log p}\left[1 - \left(\frac{\omega_{n-1}}{\omega_n} B(\frac{n+1}{2}, \frac{p+1}{2})\right)^{\frac{n}{p}}\right].
\end{aligned}$$

With the fact that,

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}\left[1 + \frac{1}{12x} + \frac{1}{288x^2} \pm o(x^{-2})\right], x \to \infty,$$

we have

$$\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\frac{|R_pK|}{|K|})=\lim_{p\to\infty}\frac{p}{\log p}\left(1-\left(\frac{\omega_{n-1}}{\omega_n}B(\frac{n+1}{2},\frac{p+1}{2})\right)^{\frac{n}{p}}\right).$$

Since

$$\left(\frac{\omega_{n-1}}{\omega_n}B(\frac{n+1}{2},\frac{p+1}{2})\right)^{\frac{n}{p}}=\left(\frac{\Gamma(1+\frac{n}{2})\sqrt{2\pi}(\frac{p+1}{2})^{\frac{p}{2}}e^{-\frac{p+1}{2}}\left[1+\frac{1}{12(\frac{p+1}{2})}+\frac{1}{288(\frac{p+1}{2})^2}\pm o(p^{-2})\right]}{\sqrt{\pi}\sqrt{2\pi}(\frac{n+p+2}{2})^{\frac{n+p+1}{2}}e^{-\frac{n+p+2}{2}}\left[1+\frac{1}{12(\frac{n+p+2}{2})}+\frac{1}{288(\frac{n+p+2}{2})^2}\pm o(p^{-2})\right]}\right)^{\frac{n}{p}}$$
$$=\left(\frac{\Gamma(1+\frac{n}{2})e^{\frac{n+1}{2}}}{\sqrt{\pi}}\right)^{\frac{n}{p}}\left(\frac{\frac{p+1}{2}}{\frac{n+p+2}{2}}\right)^{\frac{n}{2}}\left(\frac{1}{\frac{n+p+2}{2}}\right)^{\frac{n(n+1)}{2p}}\left(1\pm o(p^{-2})\right),$$

and each term can be computed as

$$\left(\frac{\Gamma(1+\frac{n}{2})e^{\frac{n+1}{2}}}{\sqrt{\pi}}\right)^{\frac{n}{p}}=e^{\frac{n}{p}\log(\frac{\Gamma(1+\frac{n}{2})}{\sqrt{\pi}})}=1+\frac{n}{p}(\log\frac{\Gamma(1+\frac{n}{2})}{\sqrt{\pi}}+\frac{n+1}{2})$$
$$+\frac{n^2}{2p^2}\left(\log\frac{\Gamma(1+\frac{n}{2})}{\sqrt{\pi}}+\frac{n+1}{2}\right)^2\pm o(p^{-2}),$$

$$\left(\frac{\frac{p+1}{2}}{\frac{n+p+2}{2}}\right)^{\frac{n}{2}}=\left(1+\frac{n+1}{p+1}\right)^{-\frac{n}{2}}=1-\frac{n(n+1)}{2(p+1)}+\frac{n(n+2)(n+1)^2}{8(p+1)^2}\pm o(p^{-2}),$$

$$\left(\frac{1}{\frac{n+p+2}{2}}\right)^{\frac{n(n+1)}{2p}}=e^{-\frac{n(n+1)}{2p}\log(\frac{n+p+2}{2})}$$
$$=1-\frac{n(n+1)}{2p}\log\frac{n+p+2}{2}+\frac{n^2(n+1)^2}{8p^2}(\log\frac{n+p+2}{2})^2$$
$$-\frac{n(n+1)(n+2)}{2p^2}\pm o(p^{-2}).$$

So we have

$$\left(\frac{\omega_{n-1}}{\omega_n}B(\frac{n+1}{2},\frac{p+1}{2})\right)^{\frac{n}{p}}=1+\frac{n}{p}\log\frac{\Gamma(1+\frac{n}{2})}{\sqrt{\pi}}-\frac{n(n+1)}{2p}\log\frac{n+p+2}{2}$$
$$+\frac{n^2(n+1)^2}{8p^2}\left(\log(\frac{n+p+2}{2})\right)^2+\frac{n(n+2)(n+1)^2}{8p^2}$$
$$-\frac{n^2(n+1)}{2p^2}\log\frac{n+p+2}{2}\left(\log\frac{\Gamma(1+\frac{n}{2})}{\sqrt{\pi}}+\frac{n+1}{2}\right)$$
$$+\frac{n}{2p^2}\left[n\left(\log\frac{\Gamma(1+\frac{n}{2})}{\sqrt{\pi}}+\frac{n+1}{2}\right)^2-(n+1)(n+3)\right]\pm o(p^{-2}).$$

Consequently, it yields

$$\lim_{p\to\infty}\frac{p}{\log p}\left[1-\left(\frac{\omega_{n-1}}{\omega_n}B(\frac{n+1}{2},\frac{p+1}{2})\right)^{\frac{n}{p}}\right]=\frac{n(n+1)}{2}.$$

This completes the proof. □

Proof of Theorem 1.2.

By using the left inequality in (1.4) and its equality condition, it has

$$DK = c_{n,p} R_p K,$$

here

$$c_{n,p} = (nB(p+1,n))^{-\frac{1}{p}}.$$

Moreover,

$$\begin{aligned}
\lim_{p\to\infty} \frac{p}{\log p}(|DK| - |R_pK|) &= \lim_{p\to\infty} \frac{p}{\log p}|DK|(1 - (\frac{1}{c_{n,p}})^n) \\
&= \lim_{p\to\infty} \frac{p}{\log p}|DK|(1 - (nB(p+1,n))^{\frac{n}{p}}) \\
&= \lim_{p\to\infty} \frac{p}{\log p}|DK|(1 - (n)^{\frac{n}{p}}(1 - \frac{n^2 \log p}{p})) \\
&= \lim_{p\to\infty} \frac{p}{\log p}|DK|\frac{n^2 \log p}{p} \\
&= n^2|DK|.
\end{aligned}$$

When K is a simplex, the equality holds in Rogers-Shephard inequality (1.2). That is, $|DK| = \begin{pmatrix} 2n \\ n \end{pmatrix} |K|$. So the desired identity is obtained.

This completes the proof. □

Finally, we discuss the case when K is a general origin-symmetric convex body in $\mathbb{R}^n$. The following lemma can be derived through the definition of ρ_{R_pK} directly.

Lemma 4.1. *Let K_1 and K_2 be convex bodies in $\mathbb{R}^n$. If $K_1 \subseteq K_2$ and $p > 0$, then*

$$\rho_{R_pK_1}(u) \leq (\frac{|K_2|}{|K_1|})^{\frac{1}{p}} \rho_{R_pK_2}(u), \quad \text{for all } u \in S^{n-1}.$$

By using the important John's inclusion (2.1) and Theorem 1.1, we have

$$\begin{aligned}
\rho_{R_pK}(u) &\leq \left(\frac{|\sqrt{n}\mathrm{JK}|}{|K|}\right)^{\frac{1}{p}} \cdot \rho_{R_p(\sqrt{n}\mathrm{JK})}(u) \\
&\leq n^{\frac{n}{2p}} \cdot n^{\frac{1}{2}} \cdot \left(\frac{|\mathrm{JK}|}{|K|}\right)^{\frac{1}{p}} \cdot \rho_{R_p(\mathrm{JK})}(u) \\
&\leq n^{\frac{n}{2p}} \cdot n^{\frac{1}{2}} \cdot \rho_{R_p(\mathrm{JK})}(u), \quad u \in S^{n-1},
\end{aligned}$$

which implies

$$R_pK \subseteq n^{\frac{n}{2p}} \cdot n^{\frac{1}{2}} R_p(\mathrm{JK}).$$

So it gives

$$\frac{|R_pK|}{|K|} \leq \frac{|n^{\frac{n}{2p}} \cdot n^{\frac{1}{2}} R_p(\mathrm{JK})|}{|\mathrm{JK}|} = n^{\frac{n^2}{2p}} \cdot n^{\frac{n}{2}} \frac{|R_p(\mathrm{JK})|}{|\mathrm{JK}|}.$$

Hence, it has

$$1-2^{-n}\cdot n^{\frac{-n}{2}}\cdot n^{\frac{-n^2}{2p}}\frac{|R_pK|}{|K|}\geq 1-2^{-n}\frac{|R_p(\mathrm{JK})|}{|\mathrm{JK}|}.$$

Consequently, from Lemma 4.1 it yields

$$\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\cdot n^{\frac{-n}{2}}\cdot n^{\frac{-n^2}{2p}}\frac{|R_pK|}{|K|})\geq \lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\frac{|R_p(\mathrm{JK})|}{|\mathrm{JK}|})=\frac{n(n+1)}{2}. \tag{4.1}$$

Similarly, by using John's inclusion (2.1), followed by Lemma 4.1 and Ball's volume-ratio inequality (2.2), it gives

$$\begin{aligned}\rho_{R_p(\mathrm{JK})}(u)&\leq\left(\frac{|K|}{|\mathrm{JK}|}\right)^{\frac{1}{p}}\cdot\rho_{R_pK}(u)\\&\leq\left(\frac{2^n}{\omega_n}\right)^{\frac{1}{p}}\cdot\rho_{R_pK}(u),\quad u\in S^{n-1},\end{aligned}$$

which implies

$$R_p(\mathrm{JK})\subseteq\left(\frac{2^n}{\omega_n}\right)^{\frac{1}{p}}R_pK.$$

So it gives

$$\frac{|R_pK|}{|K|}\geq\frac{|(\frac{2^n}{\omega_n})^{-\frac{1}{p}}R_p(\mathrm{JK})|}{|\sqrt{n}\mathrm{JK}|}=2^{-\frac{n^2}{p}}\cdot n^{-\frac{n}{2}}\cdot\omega_n^{\frac{n}{p}}\cdot\frac{|R_p(\mathrm{JK})|}{|\mathrm{JK}|}.$$

Hence, it has

$$1-2^{-n}\cdot 2^{\frac{n^2}{p}}\cdot n^{\frac{n}{2}}\cdot\omega_n^{-\frac{n}{p}}\cdot\frac{|R_pK|}{|K|}\leq 1-2^{-n}\frac{|R_p(\mathrm{JK})|}{|\mathrm{JK}|}.$$

Consequently, from Lemma 4.1 it yields

$$\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\cdot 2^{\frac{n^2}{p}}\cdot n^{\frac{n}{2}}\cdot\omega_n^{-\frac{n}{p}}\frac{|R_pK|}{|K|})\leq\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\frac{|R_p(\mathrm{JK})|}{|\mathrm{JK}|})=\frac{n(n+1)}{2}. \tag{4.2}$$

Combined with (4.1) and (4.2), it yields

$$\begin{aligned}\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\cdot 2^{\frac{n^2}{p}}\cdot n^{\frac{n}{2}}\cdot\omega_n^{-\frac{n}{p}}\frac{|R_pK|}{|K|})&\leq\frac{n(n+1)}{2}\\&\leq\lim_{p\to\infty}\frac{p}{\log p}(1-2^{-n}\cdot n^{\frac{-n}{2}}\cdot n^{\frac{-n^2}{2p}}\frac{|R_pK|}{|K|}).\end{aligned}$$

Acknowledgements

The research is partially supported by Hubei Provincial Department of Education No. 2013361.

References

[1] K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc., **44** (1991), 351–359. 2

[2] C. Bandle, *Isoperimetric inequalities and applications*, Pitman, London, (1980). 1

[3] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math., **134** (1998), 335–361. 2

[4] J. Bourgain, J. Lindenstrauss, *Projection bodies*, Geometric Aspects of Functional Analysis, Lecture Notes in Math., **1317** (1988), 250–270. 1

[5] S. Campi, P. Gronchi, *The L^p Busemann-Petty centroid inequality*, Adv. Math., **167** (2002), 128–141. 1

[6] G. Chakerian, *Inequalities for the difference body of a convex body*, Proc. Amer. Math. Soc., **18** (1967), 879–884. 1

[7] R. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, Cambridge, (2006). 1, 2

[8] R. Gardner, G. Zhang, *Affine inequalities and radial mean bodies*, Amer. J. Math., **120** (1998), 505–528. 1, 1, 1, 2.3

[9] P. Goodey, W. Weil, *Zonoids and generalizations*, Handbook of convex geometry ed. by P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, (1993), 1297–1326. 1

[10] E. Grinberg, G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc., **78** (1999), 77–115. 1

[11] B. Kawohl, *Rearrangements and convexity of level sets in PDES*, Lecture Notes in Mathematics, **1150** Springer, Berlin, (1985). 1

[12] M. Ludwig, C. Schütt, E. Werner, *Approximation of the Euclidean ball by polytopes*, Studia Math., **173** (2006), 1–18. 1

[13] E. Lutwak, D. Yang, G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom., **47** (1997), 1–16. 1

[14] E. Lutwak, D. Yang, G. Zhang, *A new ellipsoid associated with convex bodies*, J. Duke Math., **104** (2000), 375–390. 1

[15] E. Lutwak, D. Yang, G. Zhang, *L^p affine isoperimetric inequalities*, J. Differential Geom., **56** (2000), 111–132. 1

[16] E. Lutwak, D. Yang, G. Zhang, *The Cramer-Rao inequality for star bodies*, J. Duke Math., **112** (2002), 59–81. 1

[17] E. Lutwak, D. Yang, G. Zhang, *L_p John ellipsoids*, Proc. London Math. Soc., **90** (2005), 497–520. 1, 2

[18] G. Paouris, E. Werner, *Relative entropy of cone measures and Lp-centroid bodies*, Proc. London Math. Soc., **104** (2012), 253–286. 1, 2.4

[19] G. Paouris, M. E. Werner, *On the approximation of a polytope by its dual Lp-centroid bodies*, Indiana Univ. Math. J., **62** (2013), 235–248. 1

[20] G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge university Press, Cambridge, (1989). 2

[21] C. Rogers, G. Shephard, *The difference body of a convex body*, Arch. Math., **8** (1957), 220–233. 1

[22] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Cambridge university Press, Cambridge, (1993). 1

[23] C. Schütt, E. Werner, *The convex floating body*, Math. Scand., **66** (1990), 275–290. 1

[24] C. Schütt, E. Werner, *Surface bodies and $p-$affine surface area*, Adv. Math., **187** (2004), 98–145. 1

[25] G. Xiong, *Extremum problems for the cone volume functional of convex polytopes*, Adv. Math., **225** (2010), 3214–3228. 1

[26] G. Xiong, W. Cheung, *Chord power integrals and radial mean bodies*, J. Math. Anal. Appl., **342** (2008), 629–637. 1, 2.2

[27] G. Zhang, *Restricted chord projection and affine inequalities*, Geom. Dedicata, **39** (1991), 213–222. 1, 2

[28] G. Zhang, *Geometric inequalities and inclusion measures of convex bodies*, Mathematika, **41** (1994), 95–116. 1