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Omega open sets in generalized topological spaces

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Abstract

We extend the notion of omega open set in ordinary topological spaces to generalized topological spaces. We obtain several characterizations of omega open sets in generalized topological spaces and prove that they form a generalized topology. Using omega open sets we introduce characterizations of Lindelöf, compact, and countably compact concepts generalized topological spaces. Also, we generalize the concepts of continuity in generalized topological spaces via omega open sets.

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1. Introduction and preliminaries

Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a condensation point of A [18] if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω-closed sets and ω-open sets as follows: A is called ω-closed [19] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. The family of all ω-open subsets of X forms a topology on X, denoted by τ_ω. Many topological concepts and results related to ω-closed and ω-open sets appeared in [1, 2, 5, 6, 7, 8, 10, 11, 20, 29, 31] and in the references therein. In 2002, Császár [12] defined generalized topological spaces as follows: the pair (X, μ) is a generalized topological space if X is a nonempty set and μ is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. For a generalized topological space (X, μ), the elements of μ are called μ-open sets, the complements of μ-open sets are called μ-closed sets, the union of all elements of μ will be denoted by M_μ, and (X, μ) is said to be strong if $M_\mu = X$. Recently many topological concepts have been modified to give new concepts in the structure of generalized

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topological spaces, see [3, 4, 9, 13, 14, 15, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28, 30] and others. In this paper, we introduce the notion of ω-open sets in generalized topological spaces, and we use them to introduce new classes of mappings in generalized topological spaces. We present several characterizations, properties, and examples related to the new concepts. In Section 2, we introduce and study ω-open sets in generalized topological spaces. In Section 3, we introduce and study the concept of ω-(μ_1, μ_2)-continuous function.

Definition 1.1 ([15])**.** Let (X, μ) be a generalized topological space and $\mathcal{B}$ a collection of subsets of X such that $\emptyset \in \mathcal{B}$. Then $\mathcal{B}$ is called a base for μ if $\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\} = \mu$. We also say that μ is generated by $\mathcal{B}$.

Definition 1.2. Let (X, μ) be a generalized topological space.

a. [30] A collection $\mathcal{F}$ of subsets of X is said to be a cover of M_μ if M_μ is a subset of the union of the elements of $\mathcal{F}$.

b. [30] A subcover of a cover $\mathcal{F}$ is a subcollection $\mathcal{G}$ of $\mathcal{F}$ which itself is a cover.

c. [30] A cover $\mathcal{F}$ of M_μ is said to be a μ-open cover if the elements of $\mathcal{F}$ are μ-open subsets of (X, μ).

d. [30] (X, μ) is said to be μ-compact if each μ-open cover of M_μ has a finite μ-open subcover.

e. (X, μ) is said to be countably compact if each countable μ-open cover of M_μ has a finite μ-open subcover.

f. (X, μ) is said to be Lindelöf if each μ-open cover of M_μ has a countable μ-open subcover.

Definition 1.3 ([3])**.** Suppose (X, μ) is a generalized topological space and A a nonempty subset of X. The subspace generalized topology of A on X is generalized topological $\mu_A = \{A \cap U\ : U \in \mu\}$ on A. The pair (A, μ_A) is called a subspace generalized topological space of (X, μ).

A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called a function on generalized topological spaces if (X, μ_1) and (Y, μ_2) are generalized topological spaces. From now on, each function is a function on generalized topological spaces unless otherwise stated.

Definition 1.4 ([12])**.** A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called (μ_1, μ_2)-continuous at a point $x \in X$, if for every μ_2-open set V containing $f(x)$ there is a μ_1-open set U containing x such that $f(U) \subseteq V$. If f is (μ_1, μ_2)-continuous at each point of X, then f is said to be (μ_1, μ_2)-continuous.

Definition 1.5 ([16])**.** A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called (μ_1, μ_2)-closed if $f(C)$ is μ_2-closed in (Y, μ_2) for each μ_1-closed set C.

2. ω-Open sets in generalized topological spaces

In this section, we introduce and study ω-open sets in generalized topological spaces. We obtain several characterizations of omega open sets in generalized topological spaces and prove that they form a generalized topology. Using omega open sets we introduce characterizations of Lindelöf, compact, and countably compact concepts in generalized topological spaces.

Definition 2.1. Let (X, μ) be a generalized topological space and B a subset of X.

a. A point $x \in X$ is a condensation point of B if for all $A \in \mu$ such that $x \in A$, $A \cap B$ is uncountable.

b. The set of all condensation points of B is denoted by $Cond(B)$.

c. B is ω-μ-closed if $Cond(B) \subseteq B$.

d. B is ω-μ-open if $X - B$ is ω-μ-closed.

e. The family of all ω-μ-open sets of (X,μ) will be denoted by μ_ω.

Theorem 2.2. *A subset G of a generalized topological space (X,μ) is ω-μ-open if and only if for every $x \in G$ there exists a $U \in \mu$ such that $x \in U$ and $U - G$ is countable.*

Proof. G is ω-μ-open if and only if $X - G$ is ω-μ-closed if and only if $Cond(X - G) \subseteq X - G$ if and only if for each $x \in G$, $x \notin Cond(X - G)$ if and only if for each $x \in G$, there exists a $U \in \mu$ such that $x \in U$ and $U \cap (X - G) = U - G$ is countable. □

Corollary 2.3. *A subset G of a generalized topological space (X,μ) is ω-μ-open if and only if for every $x \in G$ there exists a $U \in \mu$ and a countable set $C \subseteq M_\mu$ such that $x \in U - C \subseteq G$.*

Proof. $\Longrightarrow$) Suppose G is ω-μ-open and let $x \in G$. By Theorem 2.2, there exists a $U \in \mu$ such that $x \in U$ and $U - G$ is countable. Set $C = U - G$. Then C is countable, $C \subseteq M_\mu$ and $x \in U - C = U - (U - G) \subseteq G$.

$\Longleftarrow$) Let $x \in G$. Then by assumption there exists a $U \in \mu$ and a countable set $C \subseteq M_\mu$ such that $x \in U - C \subseteq G$. Since $U - G \subseteq C$, then $U - G$ is countable, which ends the proof. □

Corollary 2.4. *Let (X,μ) be a generalized topological space. Then $\mu \subseteq \mu_\omega$.*

Proof. Let $G \in \mu$ and $x \in G$. Set $U = G$, $C = \emptyset$. Then $U \in \mu$, $C \subseteq M_\mu$ such that $x \in U - C \subseteq G$. Therefore, by Corollary 2.3, it follows that $G \in \mu_\omega$. □

Theorem 2.5. *For any generalized topological space (X,μ), μ_ω is a generalized topology on X.*

Proof. By Corollary 2.4, $\emptyset \in \mu_\omega$. Let $\{G_\alpha : \alpha \in J\}$ be a collection of ω-μ-open subsets of (X,μ) and $x \in \bigcup_{\alpha\in J} G_\alpha$. There exists an $\alpha_\circ \in J$ such that $x \in G_{\alpha_\circ}$. Since $G_{\alpha_\circ}$is ω-μ-open set, then by Corollary 2.4, there exist $U \in \mu$ and a countable set $C \subseteq M_\mu$ such that $x \in U - C \subseteq G_{\alpha_\circ} \subseteq \bigcup_{\alpha\in J} G_\alpha$. By Corollary 2.4, it follows that $\bigcup_{\alpha\in J} G_\alpha$ is ω-μ-open. □

The following example shows that $\mu \neq \mu_\omega$ in general.

Example 2.6. Consider $X = \mathbb{R}$ and $\mu = \{\emptyset, [-3,-1], [-2,0] \cup \mathbb{N}, [-3,0] \cup \mathbb{N}\}$. Then (X,μ) is a generalized topological space. Let $A = [-2,0]$. It is easy to check that $Cond(\mathbb{R} - A) = ((\mathbb{R} - A) - \mathbb{N}) \subseteq \mathbb{R} - A$. Then $A \in \mu_\omega - \mu$.

Theorem 2.7. *Let (X,μ) be a generalized topological space. Then $M_\mu = M_{\mu_\omega}$.*

Proof. Since $\mu \subseteq \mu_\omega$, then $M_\mu \subseteq M_{\mu_\omega}$. On the other hand, let $x \in M_{\mu_\omega}$. Since $M_{\mu_\omega} \in \mu_\omega$ by Corollary 2.3, there exists a $U \in \mu$ and a countable set $C \subseteq M_\mu$ such that $x \in U - C \subseteq M_{\mu_\omega}$. Since $U \subseteq M_\mu$, it follows that $x \in M_\mu$. □

For a nonempty set X, we denote the cocountable topology on X by $(\tau_{coc})_X$.

Theorem 2.8. *Let (X,μ) be a generalized topological space. Then $(\tau_{coc})_U \subseteq \mu_\omega$ for all $U \in \mu - \{\emptyset\}$.*

Proof. Let $U \in \mu - \{\emptyset\}$, $V \in (\tau_{coc})_U$ and $x \in V$. Since $V \subseteq U$, we have $x \in U$. Also, as $U - V$ is countable, then by Theorem 2.2, it follows that $V \in \mu_\omega$. □

Theorem 2.9. *Let (X,μ) be a generalized topological space. Then $\mu = \mu_\omega$ if and only if $(\tau_{coc})_U \subseteq \mu$ for all $U \in \mu - \{\emptyset\}$.*

Proof. $\Longrightarrow$) Suppose $\mu = \mu_\omega$ and $U \in \mu - \{\emptyset\}$. Then by Theorem 2.8, $(\tau_{coc})_U \subseteq \mu_\omega = \mu$.

$\Longleftarrow$) Suppose $(\tau_{coc})_U \subseteq \mu$ for all $U \in \mu - \{\emptyset\}$. It is enough to show that $\mu_\omega \subseteq \mu$. Let $A \in \mu_\omega - \{\emptyset\}$. By Corollary 2.3, for each $x \in A$ there exists a $U_x \in \mu$ and a countable set $C_x \subseteq M_\mu$ such that $x \in U_x - C_x \subseteq A$. Thus, $U_x - C_x \in (\tau_{coc})_{U_x} \subseteq \mu$ for all $x \in A$, and so $U_x - C_x \in \mu$. It follows that $A = \bigcup \{U_x - C_x : x \in A\} \in \mu$. □

Definition 2.10. A generalized topological space (X, μ) is called locally countable if M_μ is nonempty and for every point $x \in M_\mu$, there exists a $U \in \mu$ such that $x \in U$ and U is countable.

Theorem 2.11. *If (X, μ) is a locally countable generalized topological space, then μ_ω is the discrete topology on M_μ.*

Proof. We show that every singleton subset of M_μ is ω-μ-open. For $x \in M_\mu$, since (X, μ) is locally countable, there exists a $U \in \mu$ such that $x \in U$ and U is countable. By Theorem 2.8, $(\tau_{coc})_U \subseteq \mu_\omega$. Hence $U - (U - \{x\}) = \{x\} \in \mu_\omega$. □

Corollary 2.12. *If (X, μ) is generalized topological space such that M_μ is a countable nonempty set, then μ_ω is the discrete topology on M_μ.*

Proof. Since M_μ is countable, it follows directly that (X, μ) is locally countable. By Theorem 2.11, it follows that μ_ω is the discrete topology on M_μ. □

Corollary 2.13. *If (X, μ) is a generalized topological space such that X is a countable nonempty set and M_μ is nonempty, then μ_ω is the discrete topology on M_μ.*

Theorem 2.14. *Let (X, μ) be a generalized topological space. Then (X, μ_ω) is countably compact if and only if M_μ is finite.*

Proof. $\Longrightarrow$) Suppose (X, μ_ω) is countably compact and suppose on the contrary that M_μ is infinite. Choose a denumerable subset $\{a_n : n \in \mathbb{N}\}$ with $a_i \neq a_j$ when $i \neq j$ of M_μ. For each $n \in \mathbb{N}$, set $A_n = M_\mu - \{a_k : k \geq n\}$. Then $\{A_n : n \in \mathbb{N}\}$ is a μ_ω-open cover of $M_{\mu_\omega} = M_\mu$ and so it has a finite subcover, say $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$ where $n_1 < n_2 < \cdots < n_k$. Thus $\bigcup_{i=1}^{k} A_{n_i} = A_{n_k} = M_{\mu_\omega} = M_\mu$, a contradiction.

$\Longleftarrow$) Suppose M_μ is finite. If $M_\mu = \emptyset$, we are done. If $M_\mu \neq \emptyset$, then by Corollary 2.12, μ_ω is the discrete topology on M_μ where M_μ is finite. Hence (X, μ_ω) is countably compact. □

Corollary 2.15. *Let (X, μ) be a generalized topological space. Then (X, μ_ω) is compact if and only if M_μ is finite.*

The following lemma will be used in the next main result; its proof is obvious and left to the reader.

Lemma 2.16. *Let (X, μ) be a generalized topological space and let $\mathcal{B}$ be a base of μ. Then (X, μ) is Lindelöf if and only if every μ-open cover of M_μ consisting of elements of $\mathcal{B}$ has a countable subcover.*

Theorem 2.17. *A generalized topological space (X, μ) is Lindelöf if and only if (X, μ_ω) is Lindelöf.*

Proof. $\Longrightarrow$) Suppose (X, μ) is Lindelöf. Set $\mathcal{B} = \{U - C : U \in \mu \text{ and } C \text{ is countable}\}$. By Corollary 2.3, $\mathcal{B}$ is a base of μ_ω. We are going to apply Lemma 2.16. Let $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{A} = M_{\mu_\omega}$, say

$$\mathcal{A} = \{U_\alpha - C_\alpha : \text{ where } U_\alpha \in \mu \text{ and } C_\alpha \text{ is a countable subset of } M_\mu : \alpha \in \Delta\}$$

for some index set Δ. By Theorem 2.7, $M_\mu = M_{\mu_\omega}$. Since $\bigcup\{U_\alpha : \alpha \in \Delta\} = M_\mu$ and (X, μ) is Lindelöf, there exists a $\Delta_1 \subseteq \Delta$ such that Δ_1 is countable and $\bigcup\{U_\alpha : \alpha \in \Delta_1\} = M_\mu$. Put $C = \bigcup\{C_\alpha : \alpha \in \Delta_1\}$. Then C is countable and $C \subseteq M_\mu = M_{\mu_\omega} = \bigcup \mathcal{A}$. Therefore, for each $x \in C$ there exists an $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x} - C_{\alpha_x}$. Set $\mathcal{H} = \{U_\alpha - C_\alpha : \alpha \in \Delta_1\} \cup \{U_{\alpha_x} - C_{\alpha_x} : x \in C\}$. Then $\mathcal{H} \subseteq \mathcal{A}$, $\mathcal{H}$ is countable and $\bigcup \mathcal{H} = M_{\mu_\omega}$.

$\Longleftarrow$) Suppose (X, μ_ω) is Lindelöf. By Theorem 2.7, $M_\mu = M_{\mu_\omega}$ and by Corollary 2.4, $\mu \subseteq \mu_\omega$. It follows that (X, μ) is Lindelöf. □

Theorem 2.18. *Let A be a subset of a generalized topological space (X, μ). Then $(\mu_A)_\omega = (\mu_\omega)_A$.*

Proof. $(\mu_A)_\omega \subseteq (\mu_\omega)_A$. Let $B \in (\mu_A)_\omega$ and $x \in B$. By Corollary 2.3, there exists a $V \in \mu_A$ and a countable subset $C \subseteq M_{\mu_A}$ such that $x \in V - C \subseteq B$. Choose $U \in \mu$ such that $V = U \cap A$. Then $U - C \in \mu_\omega$, $x \in U - C$, and $(U - C) \cap A = V - C \subseteq B$. Therefore, $B \in (\mu_\omega)_A$.

$(\mu_\omega)_A \subseteq (\mu_A)_\omega$. Let $G \in (\mu_\omega)_A$. Then there exists an $H \in \mu_\omega$ such that $G = H \cap A$. If $x \in G$, then $x \in H$ and there exist a $U \in \mu$ and a countable subset $D \subseteq M_\mu$ such that $x \in U - D \subseteq H$. We put $V = U \cap A$. Then $V \in \mu_A$ and $x \in V - D \subseteq G$. It follows that $G \in (\mu_A)_\omega$. □

3. Continuity via ω-open sets in generalized topological spaces

In this section, we introduce ω-(μ_1, μ_2)-continuous functions between generalized topological spaces. We obtain several characterizations of them and we introduce composition and restriction theorems.

Definition 3.1. Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called ω-(μ_1, μ_2)-continuous at a point $x \in X$, if for every μ_2-open set V containing $f(x)$ there is an ω-μ_1-open set U containing x such that $f(U) \subseteq V$. If f is ω-(μ_1, μ_2)-continuous at each point of X, then f is said to be ω-(μ_1, μ_2)-continuous.

Theorem 3.2. *Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. If $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is (μ_1, μ_2)-continuous at $x \in X$, then f is ω-(μ_1, μ_2)-continuous at x.*

Proof. Let V be a μ_2-open set with $f(x) \in V$. Since f is (μ_1, μ_2)-continuous at x, there is a μ_1-open set U containing x such that $f(U) \subseteq V$. By Corollary 2.4, U is ω-μ_1-open. It follows that f is ω-(μ_1, μ_2)-continuous at x. □

It is clear that every (μ_1, μ_2)-continuous function is ω-(μ_1, μ_2)-continuous. The following is an example of ω-(μ_1, μ_2)-continuous function that is not (μ_1, μ_2)-continuous.

Example 3.3. Let $X = Y = \mathbb{R}$, $\mu_1 = \{\emptyset\} \cup \{A \subseteq \mathbb{R} : A \text{ is infinite}\}$, and $\mu_2 = \{\emptyset, \{3\}, \mathbb{R}\}$. Define $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ by $f(x) = x + 2$. Take $V = \{3\}$. Then $V \in \mu_2$ with $f(1) = V$. On the other hand, for each $U \in \mu_1$ with $1 \in U$, U is infinite and so $f(U) \nsubseteq V$. Therefore, f is not (μ_1, μ_2)-continuous at $x = 1$ and hence f is not (μ_1, μ_2)-continuous. To see that f is ω-(μ_1, μ_2)-continuous, let $x \in X$ and $V \in \mu_2$ such that $f(x) \in V$. Since $\{x\} = (\mathbb{Z} \cup \{x\}) - (\mathbb{Z} - \{x\})$, $(\mathbb{Z} \cup \{x\}) \in \mu_1$, and $\mathbb{Z} - \{x\}$ is countable, then $\{x\}$ is ω-μ_1-open. Take $U = \{x\}$. Then U is ω-μ_1-open, $x \in U$ and $f(U) = f(\{x\}) = \{f(x)\} \subseteq V$. It follows that f is ω-(μ_1, μ_2)-continuous.

The proof of the following theorem is obvious and left to the reader.

Theorem 3.4. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a function. Then the following conditions are equivalent:*

a. *The function f is ω-(μ_1, μ_2)-continuous.*

b. *For each μ_2-open set $V \subseteq Y$, $f^{-1}(V)$ is ω-μ_1-open in X.*

c. *For each μ_2-closed set $M \subseteq Y$, $f^{-1}(M)$ is ω-μ_1-closed in X.*

The following theorem is an immediate consequence of Theorem 3.4.

Theorem 3.5. *A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-continuous if and only if $f : (X, (\mu_1)_\omega) \longrightarrow (Y, \mu_2)$ is $((\mu_1)_\omega, \mu_2)$-continuous.*

Theorem 3.6. *If $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-continuous and $g : (Y, \mu_2) \longrightarrow (Z, \mu_3)$ is (μ_1, μ_2)-continuous, then $g \circ f : (X, \mu_1) \longrightarrow (Z, \mu_3)$ is ω-(μ_1, μ_2)-continuous.*

Proof. Let $V \in \mu_3$. Since g is a (μ_1, μ_2)-continuous function, then $g^{-1}(V) \in \mu_2$. Since f is ω-(μ_1, μ_2)-continuous, then $f^{-1}\left(g^{-1}(V)\right)$ is ω-μ_1-open in X. Thus $(g \circ f)^{-1}(V) = f^{-1}\left(g^{-1}(V)\right)$ is ω-μ_1-open and hence $(g \circ f)$ is ω-(μ_1, μ_2)-continuous. □

Theorem 3.7. *If A is a subset of a generalized topological space (X, μ_1) and $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-continuous, then the restriction of f to A, $f|_A : (A, (\mu_1)_A) \longrightarrow (Y, \mu_2)$ is an ω-$((\mu_1)_A, \mu_2)$-continuous function.*

Proof. Let V be any μ_2-open set in Y. Since f is ω-(μ_1, μ_2)-continuous, then $f^{-1}(V) \in \mu_\omega$ and so $(f|_A)^{-1}(V) = f^{-1}(V) \cap A \in (\mu_\omega)_A$. Therefore, by Theorem 2.18, $(f|_A)^{-1}(V) \in (\mu_A)_\omega$. It follows that$f|_A$ is ω-$((\mu_1)_A, \mu_2)$-continuous. □

Lemma 3.8. *Let (X, μ) be a strong generalized topological space and A a nonempty subset of X. Then a subset $C \subseteq A$ is μ_A-closed, if and only if there exists a μ-closed set H such that $C = H \cap A$.*

Proof. C is μ_A-closed, if and only if $A - C$ is μ_A-open, which is true if and only if there is a μ-open set U such that $A - C = A \cap U$, but in this case $X - U$ is μ-closed, and $C = (X - U) \cap A$. □

Theorem 3.9. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a function and $X = A \cup B$, where A and B are ω- μ_1-closed subsets of (X, μ_1) and $f|_A : (A, (\mu_1)_A) \longrightarrow (Y, \mu_2)$, $f|_B : (B, (\mu_1)_B) \longrightarrow (Y, \mu_2)$ are ω-(μ_1, μ_2)-continuous functions. Then f is ω-(μ_1, μ_2)-continuous.*

Proof. We will use Theorem 3.4. Let C be a μ_2-closed subset of (Y, μ_2). Then

$$f^{-1}(C) = f^{-1}(C) \cap X = f^{-1}(C) \cap (A \cup B) = \left(f^{-1}(C) \cap A\right) \cup \left(f^{-1}(C) \cap B\right).$$

Since $f|_A : (X, (\mu_1)_A) \longrightarrow (Y, \mu_2)$ and $f|_B : (X, (\mu_1)_B) \longrightarrow (Y, \mu_2)$ are ω-(μ_1, μ_2)-continuous functions, then $(f|_A)^{-1}(C) = f^{-1}(C) \cap A$ is ω-$(\mu_1)_A$-closed in $(A, (\mu_1)_A)$ and $(f|_B)^{-1}(C) = f^{-1}(C) \cap B$ is ω-$(\mu_1)_B$-closed. By Lemma 3.8, it follows that $(f|_A)^{-1}(C)$ and $(f|_B)^{-1}(C)$ are ω-μ_1-closed in (X, μ_1). It follows that f is ω-(μ_1, μ_2)-continuous. □

For any two generalized topological spaces (X, μ_1) and (Y, μ_2), we call the generalized topology on $X \times Y$ having the family $\{A \times B : A \in \mu_1 \text{ and } B \in \mu_2\}$ as a base, the product of (X, μ_1) and (Y, μ_2) and denote it by μ_{prod} [17].

Lemma 3.10. *Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. Then the projection functions $\pi_x : (X \times Y, \mu_{prod}) \longrightarrow (X, \mu_1)$ on X and $\pi_y : (X \times Y, \mu_{prod}) \longrightarrow (Y, \mu_2)$ on Y are (μ_{prod}, μ_1)-continuous and (μ_{prod}, μ_2)-continuous, respectively.*

Proof. Let U be a μ_1-open set in (X, μ_1). Then $\pi_x^{-1}(U) = U \times Y$ and $U \times Y$ is μ_{prod}-open in $(X \times Y, \mu_{prod})$. It follows that the projection function π_x is (μ_{prod}, μ_1)-continuous. Similarly, we can show that π_y is (μ_{prod}, μ_2)-continuous. □

Theorem 3.11. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ and $g : (X, \mu_1) \longrightarrow (Z, \mu_3)$ be two functions. If the function $h : (X, \mu_1) \longrightarrow (Y \times Z, \mu_{prod})$ defined by $h(x) = (f(x), g(x))$ is ω-(μ_1, μ_{prod})-continuous, then f is ω-(μ_1, μ_2)-continuous and g is ω-(μ_2, μ_3)-continuous.*

Proof. Assume that h is ω-(μ_1, μ_{prod})-continuous. Since $f = \pi_y \circ h$, where $\pi_y : (Y \times Z, \mu_{prod}) \longrightarrow (Y, \mu_2)$ is the projection function on Y, by Lemma 3.10 and Theorem 3.6, it follows that f is ω-(μ_1, μ_2)-continuous. Similarly we can show that g is ω-(μ_1, μ_3)-continuous. □

Theorem 3.12. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a function and let $H \subseteq X$ such that $(\mu_1)_H \subseteq \mu_1$. If there is an $x \in H$ such that the restriction of f to H, $f|_H : (H, (\mu_1)_H) \longrightarrow (Y, \mu_2)$ is ω-$((\mu_1)_H, \mu_2)$-continuous at x, then f is ω-(μ_1, μ_2)-continuous at x.*

Proof. Let V be any set in (Y, μ_2) containing $f(x)$. Since $f|_H$ is ω-$((\mu_1)_H, \mu_2)$-continuous at x, it follows that there is a $G \in (\mu_1)_H$ such that $x \in G$ and $f(G) \subseteq V$. Since by assumption $(\mu_1)_H \subseteq \mu_1$, then $G \in \mu_1$. It follows that f is ω-(μ_1, μ_2)-continuous. □

Corollary 3.13. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a function. Let $\{H_\alpha : \alpha \in \Delta\}$ be a cover of X such that for each $\alpha \in \Delta, (\mu_1)_{H_\alpha} \subseteq \mu_1$ and $f|_{H_\alpha}$ is ω-(μ_1, μ_2)-continuous at each point of H_α. Then f is ω-(μ_1, μ_2)-continuous.*

Proof. Let $x \in X$. We show that $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-continuous at x. Since $\{H_\alpha : \alpha \in \Delta\}$ is a μ_1-open cover of X, then there exists an $\alpha_\circ \in \Delta$ such that $x \in H_{\alpha_\circ}$. Therefore, by Theorem 3.12, it follows that f is ω-(μ_1, μ_2)-continuous at x. Then f is ω-(μ_1, μ_2)-continuous. □

Lemma 3.14. *Every μ-closed subspace of a Lindelöf generalized topological space is Lindelöf.*

Proof. Let (X, μ) be a Lindelöf generalized topological space and A a μ-closed subset of (X, μ). Let $\mathcal{A}$ be a μ-open cover of A. Then $\mathcal{B} = \mathcal{A} \cup \{X - A\}$ is a μ-open cover of (X, μ). Since (X, μ) is Lindelöf, then there exists a countable subfamily $\mathcal{C}$ of $\mathcal{B}$ such that $X = \bigcup \mathcal{C}$. Put $\mathcal{D} = \mathcal{C} - \{X - A\}$. Then $\mathcal{D}$ is countable and $A \subseteq \bigcup \mathcal{D}$. This shows that A is a Lindelöf subset of (X, μ). □

Theorem 3.15. *Any ω-μ-closed subset of a Lindelöf generalized topological space is Lindelöf.*

Proof. Let (X, μ) be a Lindelöf generalized topological space and A an ω-μ-closed subset. By Theorem 2.17, (X, μ_ω) is Lindelöf. Since A is μ-closed in the Lindelöf generalized topological space (X, μ_ω), by Lemma 3.14, A is Lindelöf subset of (X, μ_ω). Since $\mu \subseteq \mu_\omega$, then A is Lindelöf subset of (X, μ). □

Theorem 3.16. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be (μ_1, μ_2)-continuous and surjective. If (X, μ_1) is Lindelöf, then (Y, μ_2) is Lindelöf.*

Proof. Suppose (X, μ_1) is Lindelöf and let $\mathcal{A}$ be a μ_2-open cover of (Y, μ_2). Since f is (μ_1, μ_2)-continuous, $\left\{f^{-1}(A) : A \in \mathcal{A}\right\} \subseteq \mu_1$, then $\left\{f^{-1}(A) : A \in \mathcal{A}\right\}$ is a μ_1-open cover of (X, μ_1). Since (X, μ_1) is Lindelöf, there exists a countable subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \left\{f^{-1}(A) : A \in \mathcal{B}\right\} = X$. Thus $\bigcup \{f(A) : A \in \mathcal{B}\} = f(X)$. Since f is surjective, then $f(X) = Y$.

□

Corollary 3.17. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be ω-(μ_1, μ_2)-continuous and surjective. If (X, μ_1) is Lindelöf then (Y, μ_2) is Lindelöf.*

Proof. Since $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-continuous, then by Theorem 3.6, $f : (X, (\mu_1)_\omega) \longrightarrow (Y, \mu_2)$ is $((\mu_1)_\omega, \mu_2)$-continuous. Also, since (X, μ_1) is a Lindelöf, then by Theorem 2.17, $(X, (\mu_1)_\omega)$ is Lindelöf. Theorem 3.16, ends the proof. □

Definition 3.18. A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called ω-(μ_1, μ_2)-closed function if it maps μ_1-closed sets onto ω-μ_2-closed sets.

Theorem 3.19. *If $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-closed function such that for each $y \in Y$, $f^{-1}(\{y\})$ is a Lindelöf subset of (X, μ_1), and (Y, μ_2) is Lindelöf, then (X, μ_1) is Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a μ_1-open cover of (X, μ_1). For each $y \in Y$, $f^{-1}(\{y\})$ is a Lindelöf subset of (X, μ_1) and there exists a countable subset $\Delta_1(y)$ of Δ such that $f^{-1}(\{y\}) \subseteq \bigcup \{U_\alpha : \alpha \in \Delta_1(y)\}$. For each $y \in Y$, put $U(y) = \bigcup \{U_\alpha : \alpha \in \Delta_1(y)\}$ and $V(y) = Y - f(X - U(y))$. Since f is ω-(μ_1, μ_2)-closed, then for each $y \in Y$, $V(y)$ is ω-μ_2-open in (Y, μ_2) with $y \in Y$ and $f^{-1}(V(y)) \subseteq U(y)$. Since $V(y)$ is ω-μ_2-open in (Y, μ_2), there exists a μ_2-open set $W(y)$ such that $y \in W(y)$ and $W(y) - V(y)$ is countable. For each $y \in Y$, we have $W(y) \subseteq (W(y) - V(y)) \cup V(y)$ and so

$$f^{-1}(W(y)) \subseteq f^{-1}(W(y) - V(y)) \cup f^{-1}(V(y)) \subseteq f^{-1}(W(y) - V(y)) \cup U(y).$$

Since $W(y) - V(y)$ is countable and $f^{-1}(\{y\})$ is a Lindelöf subset of (X, μ_1), there exists a countable subset $\Delta_2(y)$ of Δ such that $f^{-1}(W(y) - V(y)) \subseteq \bigcup \{U_\alpha : \alpha \in \Delta_2(y)\}$ and hence

$$f^{-1}(W(y)) \subseteq \left[\bigcup \{U_\alpha : \alpha \in \Delta_2(y)\}\right] \cup [U(y)].$$

Since $\{W(y) : y \in Y\}$ is μ_2-open cover of the Lindelöf generalized topological space (Y, μ_2), there exists a countable points $y_1, y_2, y_3, \ldots$ such that $Y = \bigcup \{W(y_i) : i \in \mathbb{N}\}$. Therefore,

$$\begin{aligned} X &= \bigcup \left\{f^{-1}(W(y_i)) : i \in \mathbb{N}\right\} = \bigcup_{i \in \mathbb{N}} \left[\bigcup \{U_\alpha : \alpha \in \Delta_2(y_i)\}\right] \cup \left[\bigcup \{U_\alpha : \alpha \in \Delta_1(y_i)\}\right] \\ &= \bigcup \{U_\alpha : \alpha \in \Delta_1(y_i) \cup \Delta_2(y_i) : i \in \mathbb{N}\}. \end{aligned}$$

This shows that (X, μ_1) is Lindelöf. □

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