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Non-Nehari manifold method for a semilinear Schrödinger equation with critical Sobolev exponent

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Abstract

We consider the semilinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u), x \in R^N, \\ u \in H^1(R^N), \end{cases}$$

where $N \geq 4$, $2^* := 2N/(N-2)$ is the critical Sobolev exponent, V, K, f is 1-periodic in x_j for $j = 1, ..., N$, $f(x,u)$ is subcritical growth. We develop a direct approach to find ground state solutions of Nehari-Pankov type for the above problem. The main idea is to find a minimizing Cerami sequence for the energy functional outside the Nehari-Pankov manifold by using the diagonal method.

Keywords: Schrödinger equation, ground state solutions of Nehari-Pankov type, critical Sobolev exponent, non-Nehari-manifold method.
2010 MSC: 35Q55, 35J60.

1. Introduction

Consider the following semilinear Schrodinger equation which also have been studied in [3, 4, 5, 10, 14, 17, 23, 24, 26, 27]

$$\begin{cases} -\triangle u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u), x \in R^N, \\ u \in H^1(R^N), \end{cases} \tag{1.1}$$

where $V : R^N \to R$ and $f : R^N \times R \to R$ satisfy the following standard assumptions, respectively:

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(V0) $V \in C(R^N)$

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] < 0 < \overline{\Lambda} := \inf[\sigma(-\Delta + V) \cap (0, \infty)], \tag{1.2}$$

where σ denotes the spectrum in $L^2(R^N)$, V is 1-periodic in each of $x_1, x_2, ..., x_N$;

(V1) $K \in C(R^N), k_0 := inf_{x \in R^N} K(x) > 0$ and K is 1-periodic in x_j for $j = 1, ..., N$;

(V2) $K(x_0) := max_{x \in R^N} K(x)$ and $K(x) - K(x_0) = o(|x - x_0|^2)$ as $x \to x_0$ and $V(x_0) < 0$;

(F1) $f \in C(R^N \times R)$ is 1-periodic in each of $x_1, x_2, ..., x_N$, $f(x, t) = o(|t|)$, as $|t| \to 0$, uniformly in $x \in R^N$, and $F(x, t) := \int_0^t f(x, s)ds \geq 0$;

(F2) $|f(x, u)| \leq c_0(1 + |u|^{p-1})$ on $R^N \times R$ for some $c_0 \geq 0$ and $p \in (2, 2^*)$;

(F3) $lim_{|t| \to \infty} \frac{|F(x,t)|}{t^2} = \infty, \quad a.e. \ \ x \in R^N$;

(F4) $\exists \ \theta_0 \in (0, 1), \quad s.t. \ \ \frac{1-\theta^2}{2} tf(x, t) \geq \int_{\theta t}^t f(x, s)ds, \quad \forall \ \theta \in [0, \theta_0], \quad (x, t) \in R^N \times R.$

We point out that the condition (F4) is weaker than the following Nehari type assumption:
(Ne) $t \mapsto f(x, t)/|t|$ is strictly increasing on $R - \{0\}$.

The existence of a nontrivial solution of (1.1) has been obtained in [1, 2, 11, 12, 13] under different conditions. But very few people discuss whether the problem (1.1) has a ground state solution of Nehari-Pankov type or not. Indeed solutions of (1.1) correspond to critical points of the functional

$$\Phi(u) = \frac{1}{2} \int_{R^N} (|\nabla u|^2 + V(x)u^2)dx - \frac{1}{2^*} \int_{R^N} K|u|^{2^*} dx - \int_{R^N} F(x, u)dx. \tag{1.3}$$

Note that $2^* = 2N/(N-2)$ is the limiting Sobolev exponent for embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$. Since this embedding is not compact, the functional Φ does not satisfy the $(C)_c$ condition that any sequence u_n such that

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0,$$

have a convergent subsequence. Hence there are serious difficulties when trying to find critical points by standard variational methods. Our main existence result will be based on the following critical point theorem [10]:

Lemma 1.1 ([3]:Theorem 4.5, [9]:Theorem 2.1 in, [8])**.** *Let X be a real Hilbert space with $X = X^- \bigoplus X^+$ (where X^-, X^+ similar to the positive space E^+ and negative space E^- behind the paper) and $X^- \perp X^+$ (where $\perp$ means "orthogonal") and let $\varphi \in C^1(X, R)$ of the form*

$$\varphi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(LS1) $\psi \in C^1(X, R)$ is bounded from below and weakly sequentially lower semi-continuous;

(LS2) ψ' is weakly sequentially continuous;

(LS3) there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$k := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{w + se : w \in X^-, s \geq 0, \|w + se\| \leq r\}.$$

Then for some $c \in [k, sup\Phi(Q)]$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{1.4}$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$.

2. Preliminaries

Let $\mathcal{A} = -\Delta + V$. Then $\mathcal{A}$ is self-adjoint in $L^2(R^N)$ with domain $\mathcal{D}(\mathcal{A}) = H^2(R^N)$ (see [7], Theorem 4.26). Let $\{\mathcal{F}(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = id - \mathcal{F}(0) - \mathcal{F}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$ (see [6], Theorem IV 3.3). Let

$$E = \mathcal{D}(|\mathcal{A}|^{1/2}), \qquad E^- = \mathcal{F}(0)E, \qquad E^+ = [id - \mathcal{F}(0)]E. \tag{2.1}$$

For any $u \in E$, it is easy to see that $u = u^- + u^+$, where

$$u^- := \mathcal{F}(0)u \in E^-, \qquad u^+ := [id - \mathcal{F}(0)]u \in E^+, \tag{2.2}$$

and

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \qquad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall u \in E \cap \mathcal{D}(\mathcal{A}). \tag{2.3}$$

Define an inner product

$$(u, v) = (|\mathcal{A}|^{1/2}u, \mathcal{A}|^{1/2}v)_{L^2}, \quad u, v \in E, \tag{2.4}$$

and the corresponding norm

$$\|u\| = \left\||\mathcal{A}|^{1/2}u\right\|_2, \quad u \in E, \tag{2.5}$$

where$(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(R^N)$, By(V1), E and $H^1(R^N)$ have equivalent norms. Therefore, E embeds continuously in $L^s(R^N)$ for all $2 \le s \le 2^*$. In addition, one has the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

$$\langle \Phi'(u), v\rangle = \int_{R^N} (\nabla u \nabla v + V(x)uv)dx - \int_{R^N} K|u|^{2^*-1}vdx - \int_{R^N} f(x, u)vdx, \quad \forall u, v \in E, \tag{2.6}$$

and

$$\langle \Phi'(u), u\rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{R^N} K|u|^{2^*-1}udx - \int_{R^N} f(x, u)udx, \quad \forall u = u^- + u^+ \in E^- \oplus E^+ = E, \tag{2.7}$$

and

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2^*}\int_{R^N} K|u|^{2^*-1}udx - \int_{R^N} F(x, u)dx, \quad \forall u = u^- + u^+ \in E^- \oplus E^+ = E. \tag{2.8}$$

Now, we are in a position to state the main result of this paper.

Theorem 2.1. *Assume that V and f satisfy (V0), (V1), (F1), (F2), (F3) and (F4). Then problem* (1.1) *has a nontrivial solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}^0} \Phi > 0$, where*

$$\mathcal{N}^0 = \{u \in E\backslash E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0, \forall v \in E^-\}. \tag{2.9}$$

The set $\mathcal{N}^0$ was first introduced by Pankov [15, 16]*, which is a subset of the Nehari manifold*

$$\mathcal{N} = \{u \in E\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\}. \tag{2.10}$$

The remainder of this paper is organized as follows. In Sections 3, 4, some crucial lemmas are presented. The proof of Theorems 2.1 is given in Section 5.

3. Existence of a Palais-Smale sequence

Lemma 3.1. *Suppose that (V1),(F1),(F2) and (F3) are satisfied. Then for $u \in E$,*

$$\begin{aligned}\Phi(u) &\ge \Phi(tu + w) + \frac{1}{2}\|w\|^2 \\ &\quad + \frac{1 - t^2}{2}\langle \Phi'(u), u\rangle - t\langle \Phi'(u), w\rangle, \quad \forall t \ge 0, w \in E^-.\end{aligned} \tag{3.1}$$

Proof. For any $x \in R^N$ and $\tau \neq 0$, (F3) yields

$$\frac{1-t^2}{2}\tau K(x)|\tau|^{2^*-2}u + f(x,\tau) \geq \int_{t\tau}^{\tau}[K(x)|s|^{2^*-2}s + f(x,s)]ds, \quad t \geq 0. \tag{3.2}$$

It follows that

$$\big(\frac{1-t^2}{2}\tau - t\tau\big)K(x)|\tau|^{2^*-2}\tau + f(x,\tau) \geq \int_{t\tau+\sigma}^{\tau}[K(x)|s|^{2^*-2}s + f(x,s)]ds, \quad t \geq 0, \quad \sigma \in R. \tag{3.3}$$

We let $b : E \times E \to R$ denote the symmetric bilinear form given by

$$b(u,v) = \int_{R^N}(\triangledown u \triangledown v + V(x)uv)dx, \qquad \forall u,v \in E. \tag{3.4}$$

By virtue of (1.3) and (2.6), one has

$$\Phi(u) = \frac{1}{2}b(u,u) - \frac{1}{2^*}\int_{R^N} K|u|^{2^*}dx - \int_{R^N} F(x,u)dx, \qquad \forall u \in E. \tag{3.5}$$

and

$$\langle \Phi'(u), v\rangle = b(u,v) - \int_{R^N} K|u|^{2^*-1}vdx - \int_{R^N} f(x,u)vdx, \qquad \forall u,v \in E. \tag{3.6}$$

Thus, by (1.3),(3.3)–(3.6), one has

$$\begin{aligned}
\Phi(u) - \Phi(tu+w) &= \frac{1}{2}[b(u,u) - b(tu+w,tu+w)] \\
&\quad + \frac{1}{2^*}\int_{R^N} K(|tu+w|^{2^*} - |u|^{2^*})dx + \int_{R^N}[F(x,tu+w) - F(x,u)]dx \\
&= \frac{1-t^2}{2}b(u,u) - tb(u,w) - \frac{1}{2}b(w,w) \\
&\quad + \frac{1}{2^*}\int_{R^N} K(|tu+w|^{2^*} - |u|^{2^*})dx + \int_{R^N}[F(x,tu+w) - F(x,u)]dx \\
&= -\frac{1}{2}b(w,w) + \frac{1-t^2}{2}\langle \Phi'(u),u\rangle - t\langle \Phi'(u),w\rangle \\
&\quad + \int_{R^N}\Big[(\frac{1-t^2}{2}u - tw)[K(x)|u|^{2^*-2}u + f(x,u)] - \int_{tu+w}^{u}[K(x)|s|^{2^*-2}s + f(x,s)]ds\Big]dx \\
&= \frac{1}{2}\|w\|^2 + \frac{1-t^2}{2}\langle \Phi'(u),u\rangle - t\langle \Phi'(u),w\rangle \\
&\quad + \int_{R^N}\Big[(\frac{1-t^2}{2}u - tw)[K(x)|u|^{2^*-2} - \int_{tu+w}^{u}[K(x)|s|^{2^*-2}s + f(x,s)]ds\Big]dx \\
&\geq \frac{1}{2}\|w\|^2 + \frac{1-t^2}{2}\langle \Phi'(u),u\rangle - t\langle \Phi'(u),w\rangle, \quad \forall t \geq 0, \quad w \in E^-.
\end{aligned}$$

This shows that (3.1) holds. □

Lemma 3.2.

(i) Let $e \in E^+$, then there exist $\alpha, \rho > 0$ and $R > \rho$ (R depending on e), such that

$$m = \inf_{\mathcal{N}^0}\Phi \geq \kappa := \inf\{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0,$$

and $\quad \|u^+\| \geq \max\{\|u^-\|, \sqrt{2m}\}$ *for all* $u \in \mathcal{N}^0$.

(ii) $\Phi(u) \le 0$ *for all* $u \in \partial Q$, *there*

$$Q = \{w + se : w \in E^-, s \ge 0, \|w + se\| \le r\}.$$

(iii) We set

$$\Psi(u) := (2^*)^{-1} \int_{R^N} K|u|^{2^*} dx + \int_{R^N} F(x,u)dx, \ u \in E. \tag{3.7}$$

We have

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi(u), \quad u \in E. \tag{3.8}$$

Then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.

Proof. (i) Let $u \in E^+, \|u\| = \rho$, then

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_{R^N} K(x)|u|^{2^*} dx - \int_{R^N} F(x,u)dx.$$

It follows from (F2) and (F3), that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|F(x,s)| \le \varepsilon s^2 + C_\varepsilon |s|^p$$

for all $s \in R$. Applying the Sobolev embedding theorem we get that

$$\int_{R^N} F(x,u)dx \le C(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p),$$

for some constant $C > 0$. Consequently,

$$\Phi(u) \ge \frac{1}{2}\|u\|^2 - \frac{K(x_0)}{2^*}\|u\|^2 - C(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^p).$$

Choosing $\varepsilon > 0$ and $\rho > 0$ sufficiently small, the result

$$m = \inf_{\mathcal{N}^0} \Phi \ge \kappa := \inf\{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0,$$

readily follows.

From Lemma 3.1, $\forall u \in \mathcal{N}^0, \quad w \in E^-, \quad t \ge 0$ we have

$$\Phi(u) \ge \Phi(tu + w),$$

so

$$\|u^+\| \ge \|u^-\|, \qquad u = u^- + u^+ \ \in \ \mathcal{N}^0,$$

and when $u \in E^+$, we have

$$\|u^+\|^2 = 2\Phi(u) + \frac{2}{2^*}\int_{R^N} K(x)|u|^{2^*} dx + 2\int_{R^N} F(x,u)dx \ge 2m.$$

(ii) (V1), (F1) yields that $K \ge 0$ and $F(x,t) \ge 0$ for all $(x,t) \in R^N \times R$, and when $u \in \ E^-$, from (2.8) we have:

$$\Phi(u) = -\frac{1}{2}\|u^-\|^2 - \frac{1}{2^*}\int_{R^N} K|u|^{2^*} dx - \int_{R^N} F(x,u)dx \le 0.$$

Next, it is sufficient to show that $\Phi(u) \to -\infty$ as $u \in E^- \oplus Re$. Arguing indirectly, assume that for some sequence $\{w_n + s_n e\} \subset E^- \oplus Re$ with $\|w_n + s_n e\| \to \infty$, there is $M > 0$ such that $\Phi(w_n + s_n e) \ge -M$ for all $n \in N$. Set $v_n = (w_n + s_n e)/\|w_n + s_n e\| = v_n^- + t_n e$, then $\|v_n^- + t_n e\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, then $v_n \to v$ a.e. on R^N, $v_n^- \rightharpoonup v^-$ in E, $t_n \to \bar{t}$, and

$$\begin{aligned} -\frac{M}{\|w_n + s_n e\|^2} &\leq \frac{\Phi(w_n + s_n e)}{\|w_n + s_n e\|^2} \\ &= \frac{t_n^2}{2} - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2^*}\int_{R^N} K\|w_n + s_n e\|^{2^*-2} dx - \int_{R^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx. \end{aligned} \tag{3.9}$$

If $\bar{t} = 0$, then it follows from (3.9) that

$$0 \leq \frac{1}{2}\|v_n^-\|^2 + \frac{1}{2^*}\int_{R^N} K\|w_n + s_n e\|^{2^*-2} dx + \int_{R^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx \leq \frac{t^2}{2} + \frac{M}{\|w_n + s_n e\|^2} \to 0,$$

which yields $\|v_n^-\| \to 0$, and so $1 = \|v_n\| \to 0$, a contradiction.

If $\bar{t} \neq 0$, then $v \neq 0$, it follows from (3.9), (F3) and Fatou's lemma that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \Big[\frac{t_n^2}{2} - \frac{1}{2}\|v_n^-\|^2 - \frac{1}{2^*}\int_{R^N} K\|w_n + s_n e\|^{2^*-2} dx - \int_{R^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} d\Big] \\ &\leq \limsup_{n\to\infty} \Big[\frac{t_n^2}{2} - \frac{1}{2}\|v_n^-\|^2 - \int_{R^N} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2} v_n^2 dx\Big] \\ &\leq \frac{\bar{t}^2}{2} - \int_{R^N} \liminf_{n\to\infty} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2} v_n^2 dx \\ &= -\infty, \end{aligned}$$

a contradiction.

(iii) For convenience,

$$\Psi(u) := (2^*)^{-1}\int_{R^N} K|u|^{2^*} dx + \int_{R^N} F(x, u) dx, \ u \in E.$$

By (F_1) and (F_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon|u| + C_\varepsilon|u|^{q-1} \ \text{ and } \ |F(x, u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{q}|u|^q. \tag{3.10}$$

For any $u, v \in E$ and $0 < |t| < 1$, by mean value theorem and (3.10), there exists $0 < \theta < 1$ such that

$$\begin{aligned} \frac{|F(x, u + tv) - F(x, u)|}{|t|} &\leq |f(x, u + \theta tv)v| \\ &\leq \varepsilon|u + \theta tv||v| + C_\varepsilon|u + \theta tv|^{q-1}|v| \\ &\leq \varepsilon|u||v| + \varepsilon|v|^2 + C_\varepsilon|u + \theta tv|^{q-1}|v| \\ &\leq \varepsilon|u||v| + \varepsilon|v|^2 + 2^{q-1}C_\varepsilon(|u|^{q-1}|v| + |v|^q), \end{aligned}$$

and

$$\begin{aligned} \frac{|u + tv|^{2^*} - |u|^{2^*}}{2^*|t|} &\leq |u + \theta tv|^{2^*-1}|v| \\ &\leq (2^* - 1)|u|^{2^*-1}|v|^{2^*}. \end{aligned}$$

The Hölder inequality implies that

$$\varepsilon|u||v| + \varepsilon|v|^2 + 2^{q-1}C_\varepsilon(|u|^{q-1}|v| + |v|^q) + K(2^* - 1)|u|^{2^*-1}|v|^{2^*} \in L^1(\mathbb{R}^N).$$

Consequently, by the Lebesgue's Dominated Theorem, we have

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} K|u|^{2^*-1} v dx + \int_{\mathbb{R}^N} f(|x|, u) v dx, \quad \forall\ u, v \in E.$$

Next, we show that $\Psi' : E \to E^*$ is weak continuous. Assume that $u_n \rightharpoonup u$ in E, by Sobolev embedding theorem, we get

$$u_n \rightharpoonup u \ \text{ in } \ L^p(\mathbb{R}^N), \ \text{for } p \in (2, 2_s^*)$$

and

$$u_n \to u \ \text{ in } \ C_0^\infty(\mathbb{R}^N), \ \text{for } p \in (2, 2_s^*),$$

there $C_0^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$.

By the Hölder inequality, we have

$$\begin{aligned}\|\Psi'(u_n) - \Psi'(u)\|_{E^*} &= \sup_{\|v\|\le 1} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| \\ &\le \sup_{\|v\|\le 1} \int_{\mathbb{R}^N} (|f(|x|, u_n) - f(|x|, u)| + K||u_n|^{2^*-1} - |u|^{2^*-1}|)|v| dx \to 0 \quad \text{ as } \ n \to \infty.\end{aligned}$$

□

Lemma 3.3. *Suppose that (V1),(F1),(F2) (F3) and (F4) are satisfied. Then there exist a constant $c \in [\kappa, \sup \Phi(Q)]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \ \text{as } n \to \infty. \tag{3.11}$$

Proof. Lemma 3.3 is a direct corollary of Lemma 1.1 and Lemma 3.2. □

Lemma 3.4 ([18, 19, 20, 21, 22])**.** *Suppose that (V1),(F1),(F2) (F3) and (F4) are satisfied. Then there exist a constant $c_* \in [\kappa, m]$ (where κ and m are stated as Lemma 3.2) and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \ \text{as } n \to \infty. \tag{3.12}$$

Proof. Choose $v_k \in \mathcal{N}^0$ such that

$$m \le \Phi(v_k) < m + \frac{1}{k}, \quad k \in N. \tag{3.13}$$

By Lemma 3.2 (i) , $\|v_k^+\| \ge \sqrt{2m} > 0$. Set $e_k = v_k^+/\|v_k^+\|$. Then $e_k \in E^+$ and $\|e_k\| = 1$. In view of Lemma 3.2, there exists $r_k > max\{\rho, \|v_k\|\}$ such that $\sup \Phi(\partial Q_k) \le 0$, where

$$Q_k = \{w + se_k : w \in E^-, s \ge 0, \|w + se_k\| \le r_k\}, \quad k \in N. \tag{3.14}$$

Hence, applying Lemma 1.1 to the above set Q_k, there exist a constant $c_k \in [\kappa, \sup \Phi(Q_k)]$ and a sequence $\{u_{k,n}\}_{n\in N} \subset E$ satisfying

$$\Phi(u_{k,n}) \to c_k, \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \ \text{as } n \to \infty, \quad k \in N. \tag{3.15}$$

By virtue of Lemma 3.1, one can get that

$$\Phi(v_k) \ge \Phi(tv_k + w), \quad \forall t \ge 0, \quad w \in E^-. \tag{3.16}$$

Since $tv_k + w \in Q_k$, it follows that $\Phi(v_k) = \sup \Phi(Q_k)$. Hence, by (3.13) and (3.15), one has

$$\Phi(u_{k,n}) \to c_k < m + \frac{1}{k}, \quad \|\Phi'(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \ \text{as } n \to \infty, \quad k \in N. \tag{3.17}$$

Now, we can choose a sequence $\{n_k\} \subset N$ such that

$$\Phi(u_{k,n_k}) < m + \frac{1}{k}, \quad \|\Phi'(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in N. \tag{3.18}$$

Let $u_k = u_{k,n_k}, k \in N$. Then, going if necessary to a subsequence, by using the diagonal method, we have

$$\Phi(u_n) \to c_* \in [\kappa, m], \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \ as\ n \to \infty. \tag{3.19}$$

The proof of Lemma 3.4 is completed.

□

Lemma 3.5. *The Cerami sequence above is bounded.*

This result is essentially contained in [1] (Proposition 3.2), but for the reader's convenience we choose to write it in detail.

Proof. It follows from (F1)–(F3) that for each $\varepsilon > 0$ there exists $c_1(\varepsilon)$ such that $|f(x,u)| \le \varepsilon|u| + c_1(\varepsilon)|u|^{2^*-1}$. By (F4),

$$c + 1 + \|u_n\| \ge \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \ge \frac{1}{N}\int_{R^N} K|u_n|^{2^*} dx,$$

for almost all n, and since $K(x)$ is bounded below by a positive constant,

$$\|u_n\|_{2^*}^{2^*} \le c_2 + c_3\|u_n\|, \tag{3.20}$$

Using the Hölder and Sobolev inequalities we obtain, for large n,

$$\begin{aligned}\|u_n^+\|^2 &= \langle \Phi'(u_n), u_n^+\rangle + \textstyle\int_{R^N} K|u_n|^{2^*-2} u_n u_n^+ dx + \int_{R^N} f(x,u_n)u_n^+ dx \\ &\le \|u_n^+\| + c_4\|u_n\|_{2^*}^{2^*-1}\|u_n^+\| + c_5(\varepsilon\|u_n\| + c_1\varepsilon\|u_n\|_{2^*}^{2^*-1})\|u_n^+\|.\end{aligned}$$

Hence by (3.4),

$$\|u_n^+\| \le c_6(\varepsilon) + c_7(\varepsilon)\|u_n\|^{(2^*-1)/2^*} + c_5\varepsilon\|u_n\|$$

and a similar inequality holds for $\|u_n^-\|$. Choosing ε sufficiently small, we see that (u_n) must be bounded. □

Lemma 3.6 ([20, 21, 22])**.** *Suppose that (V0)–(V2) and (F1)–(F4) are satisfied. Then for any $u \in E\backslash E^-$, there exist $t(u) > 0$ and $w(u) \in E^-$ such that $t(u)u + w(u) \in \mathcal{N}^0$. Consequently, $\mathcal{N}^0 \cap (E^- \oplus R^+u) \neq \varnothing$. where R^+u means the space $\{ru : \ \ r \in R^+, \ \ u \in E\backslash E^-\}$.*

Proof. By view of Lemma 3.2, there exists a constant $R > 0$ such that

$$\Phi(v) \le 0 \ \ \forall v \in (E^- \oplus R^+u)\backslash B_R(0),$$

where $B_R(0)$ is the ball center of 0 and it's radius is R.

By Lemma 3.2 (i) , $\Phi(tu) > 0$ for small $t > 0$. Thus we have, $0 < \sup \Phi(E^- \oplus R^+u) < \infty$. It is easy see that Φ is weakly upper semicontinuous on $E^- \oplus R^+u$; therefore, $\Phi(u_0) = \sup \Phi(E^- \oplus R^+u)$ for some $u_0 \in E^- \oplus R^+u$. This u_0 is a critical point of $\Phi|_{E^- \oplus R^+u}$, so $\langle \Phi'(u_0), u_0\rangle = \langle \Phi'(u_0), v\rangle = 0$ for all $v \in E^- \oplus R^+u$. Consequently, $u_0 \in \mathcal{N}^0 \cap (E^- \oplus R^+u)$. □

4. Estimates for critical levels

Lemma 4.1. *Let*

$$S := \inf_{E\backslash\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

If $0 < c < d := \frac{S^{N/2}}{N\|K\|_\infty^{(N-2)/2}}$, then the Cerami sequence (u_n) cannot be vanishing.

Proof. see [1] (Proposition 4.1), we also give the proof as follow. If (u_n) is vanishing, then it follows from P. L. Lions' lemma ([23]:Lemma 1.21) that $u_n \to 0$ in L^r whenever $2 < r < 2^*$. Let (z_n) be a bounded sequence in E. Since for each $\varepsilon > 0$ there is $c_1(\varepsilon)$ such that $|f(x,u)| \le \varepsilon|u| + c_1(\varepsilon)|u|^{p-1}$,

$$\int_{R^N} |f(x,u_n)||z_n|dx \le c_2\varepsilon\|u_n\|\|z_n\| + c_3(\varepsilon)\|u_n\|_p^{p-1}\|z_n\|.$$

Using this and a similar argument for F we see that

$$\begin{aligned} &\textstyle\int_{R^N} f(x,u_n)z_n dx \to 0, \quad n \to \infty, \\ &\textstyle\int_{R^N} F(x,u_n)dx \to 0, \qquad n \to \infty. \end{aligned} \tag{4.1}$$

Hence

$$\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \frac{1}{N}\int_{R^N} K|u_n|^{2^*}dx + o(1) \to c, \quad n \to \infty. \tag{4.2}$$

Let r be such that $(2^* - 1)/r + 1/q = 1$. Then $2 < r < 2^*$. Since $\|u_n^-\|_q$ is bounded and $u_n \to 0$ in L^r, we obtain using (4.1), (4.2) and the Hölder inequality that

$$\begin{aligned} \|u_n^-\|^2 &= -\langle \Phi'(u_n), u_n^-\rangle - \textstyle\int_{R^n} K|u_n|^{2^*-2}u_n u_n^- dx - \int_{R^N} f(x,u_n)u_n^- dx \\ &\le K(x_0)\|u_n\|_r^{2^*-1}\|u_n^-\|_q + o(1) \to 0. \end{aligned}$$

Similarly,

$$\|w_n\|^2 = \int_{R^N} K|u_n|^{2^*-2}u_n w_n dx + o(1) \to 0.$$

Hence

$$u_n - z_n = w_n + u_n^- \to 0, \tag{4.3}$$

and therefore

$$\begin{aligned} \|z_n\|^2 &= \textstyle\int_{R^N}(|\nabla z_n|^2 + Vz_n^2)dx = \int_{R^N} K|u_n|^{2^*-2}u_n z_n dx + o(1) \\ &= \textstyle\int_{R^N} K|u_n|^{2^*}dx + o(1). \end{aligned} \tag{4.4}$$

Furthermore, for each $\delta > 0$ we may find $\mu > 0$ such that

$$(1-\delta)\int_{R^N} |\nabla z_n|^2 dx \le \int_{R^N}(|\nabla z_n|^2 + Vz_n^2)dx. \tag{4.5}$$

Indeed, since $z_n \in (I - E(\mu))L^2 \cap E$, we have $\int_{R^N}(|\nabla z_n|^2 + Vz_n^2)dx \ge \mu\|z_n\|_2^2$ and

$$\delta\int_{R^N}(|\nabla z_n|^2 dx \ge \delta(\mu - \|V\|_\infty)\|z_n\|_2^2 \ge -\int_{R^N} Vz_n^2 dx,$$

whenever μ is large enough. Combining (4.1), (4.3), (4.4) and (4.5) gives

$$\begin{aligned} &(1-\delta)S\|K\|_\infty^{-2/2^*}\Big(\int_{R^N} K|u_n|^{2^*}dx\Big)^{2/2^*} \\ &\le (1-\delta)S\|u_n\|_{2^*}^2 \\ &= (1-\delta)S\|z_n\|_{2^*}^2 + o(1) \\ &\le (1-\delta)\int_{R^N} |\nabla z_n|^2 dx + o(1) \\ &\le \int_{R^N} K|u_n|^{2^*}dx + o(1). \end{aligned} \tag{4.6}$$

Passing to the limit and using (4.6) we obtain

$$(1-\delta)S\|K\|_\infty^{-2/2^*}(cN)^{2/2^*} \le cN.$$

Hence either $c = 0$ which is impossible or $(1-\delta)^{N/2}c^* \le c < c^*$ which is also impossible because δ may be chosen arbitrarily small. Let

$$\varphi_\varepsilon(x) := \frac{c_N \psi(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}},$$

where $c_N = (N(N-2))^{(N-2)/4}, \varepsilon > 0$ and $\psi \in C_0^\infty(R^N, [0,1])$ with $\psi(x) = 1$ if $|x| \le r/2$; $\psi(x) = 0$ if $|x| \ge r$, r small enough (cf. e.g. pp. 35 and 52 of [25]). We need the following asymptotic estimates as $\varepsilon \to 0^+$ (see e.g. pp.35 and 52 in [23]):

$$\begin{aligned}&\|\nabla\varphi_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \quad \|\nabla\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}),\\ &\|\varphi_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^{N-2}), \quad \|\varphi_\varepsilon\|_{2^*-1}^{2^*-1} = O(\varepsilon^{(N-2)/2}), \quad \|\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}),\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\|\varphi_\varepsilon\|_2^2 = b\varepsilon^2|log\varepsilon| + O(\varepsilon^2), \quad if N = 4,\\ &\|\varphi_\varepsilon\|_2^2 = b\varepsilon^2 + O(\varepsilon^{N-2}), \quad\quad if N \ge 5,\end{aligned} \tag{4.8}$$

where b is a positive constant. Finally, Let

$$Z_\varepsilon := E^- \oplus R\varphi_\varepsilon \equiv E^- \oplus R\varphi_\varepsilon^+.$$

We may assume without loss of generality that $K(0) = \|K\|_\infty$ and $V(0) < 0$. Moreover, r in the definition of φ_ε may be chosen so that $V(x) \le -\beta$ for some $\beta > 0$ and all x with $|x| \le r$. □

Lemma 4.2. *If $\varepsilon > 0$ is small enough, then $\sup_{Z_\varepsilon}\Phi < d$. So in particular, if $z_0 = \varphi_\varepsilon^+$ with ε small enough, then $c_* \le m \le sup\Phi(Q) < d$.*

Proof. From Lemma 3.4 and Lemma 3.6, we can see $c_* \le m$ and $m \le sup\Phi(Q)$. Let

$$I(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2^*}\int_{R^N} K|u|^{2^*}dx.$$

Since $I(u) \ge \Phi(u)$ for all u, it suffices to show that $sup_{z_\varepsilon} I < d$.

In what follows we adapt the argument on [25] (pp.52-53). If $u \ne 0$, then

$$\max_{t\ge0} I(tu) = \frac{1}{N}\frac{(\int_{R^N}(|\nabla u|^2 + Vu^2)dx)^{N/2}}{(\int_{R^N} K|u|^{2^*}dx)^{(N-2)/2}}, \tag{4.9}$$

whenever the integral in the numerator above is positive, and the maximum is 0 otherwise. Let $\|u\|_{2^*,K}^{2^*} := \int_{R^N} K|u|^{2^*}dx$. It is easy to see from (4.9) that if

$$m_\varepsilon := \sup_{u\in Z_\varepsilon, \|u\|_{2^*,K}=1}\int_{R^N}(|\nabla u|^2 + Vu^2)dx < \frac{S}{\|K\|_\infty^{(N-2)/N}}, \tag{4.10}$$

then $sup_{Z_\varepsilon}\Phi \le sup_{Z_\varepsilon} I < d$. So it remains to show (4.10) is satisfied for all small $\varepsilon > 0$.

Below we shall repeatedly use (4.7) and (4.8). Since

$$\int_{R^N}(|\nabla\varphi_\varepsilon^-|^2 + V(\varphi_\varepsilon^-)^2)dx \le 0,$$

and

$$\int_{R^N}(|\nabla\varphi_\varepsilon^-|^2dx \le c_1\|\varphi_\varepsilon^-\|_2^2 \le c_1\|\varphi_\varepsilon\|_2^2 \to 0 \quad as \quad \varepsilon\to 0,$$

therefore

$$\|\varphi_\varepsilon^-\|_{2^*} \le c_2\|\varphi_\varepsilon^-\| \to 0,$$

and

$$\|\varphi_\varepsilon^+\|_{2^*}^{2^*} \to S^{N/2}.$$

Suppose $\|u\|_{2^*,K}=1$ and write

$$u = u^- + s\varphi_\varepsilon = (u^- + s\varphi_\varepsilon^-) + s\varphi_\varepsilon^+,$$

We have $\|u^-\|_{2^*} \le c_3$ and $|s|\le c_3$ for some constant c_3 independent of ε. By convexity of $\|\cdot\|_{2^*,K}$, we obtain

$$\begin{aligned}1 = \|u\|_{2^*,K}^{2^*} &\ge \|s\varphi_\varepsilon\|_{2^*,K}^{2^*} + 2^*\int_{R^N}(s\varphi_\varepsilon)^{2^*-1}u^-dx \\ &\ge \|s\varphi_\varepsilon\|_{2^*,K}^{2^*} - c_4\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1}\|u^-\|_2.\end{aligned} \tag{4.11}$$

Moreover,

$$\begin{aligned}\textstyle\int_{R^N}(\nabla\varphi_\varepsilon\cdot\nabla u^- + V\varphi_\varepsilon u^-)dx &\le c_5(\|\nabla\varphi_\varepsilon\|_1 + \|\varphi_\varepsilon\|_1)\|u^-\|_2 \\ &= O(\varepsilon^{(N-2)/2})\|u^-\|_2.\end{aligned} \tag{4.12}$$

Since $V(x)\le -\beta<0$ for $x\in supp\varphi_\varepsilon$ and $K(x)-K(0)=o(|x|^2)$ as $x\to 0$,

$$\begin{aligned}&\textstyle\int_{R^N} V\varphi_\varepsilon^2dx \le (-d\varepsilon^2) \qquad (if N\ge 5),\\ &\textstyle\int_{R^N} V\varphi_\varepsilon^2dx \le (-d\varepsilon^2|log\varepsilon|) \qquad (if N=4)\end{aligned} \tag{4.13}$$

for some $d>0$ and

$$\begin{aligned}\|\varphi_\varepsilon\|_{2^*,K}^{2^*} &= \|K\|_\infty \textstyle\int_{R^N}\varphi_\varepsilon^{2^*}dx + \int_{R^N}(K(x)-K(0))\varphi_\varepsilon^{2^*}dx \\ &= \|K\|_\infty S^{N/2} + o(\varepsilon^2).\end{aligned} \tag{4.14}$$

Let $N\ge 5$. Using (4.11)–(4.14) and the fact that

$$-\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \le O(\varepsilon^{N-2}),$$

we obtain

$$\begin{aligned} m_\varepsilon &\le -\|u^-\|^2 + \frac{\int_{R^N}(|\nabla\varphi_\varepsilon|^2 + V\varphi_\varepsilon^2)dx}{\|\varphi_\varepsilon\|_{2^*,K}^2}\|s\varphi_\varepsilon\|_{2^*,K}^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\le -c_6\|u^-\|_2^2 + \frac{\int_{R^N}(|\nabla\varphi_\varepsilon|^2 + V\varphi_\varepsilon^2)dx}{\|K\|_\infty^{(N-2)/N}S^{(N-2)/2} + o(\varepsilon^2)}(1 + c_4\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1}\|u^-\|_2)^{2/2^*} \\ &\quad + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &= -c_6\|u^-\|_2^2 + \frac{S^{N/2} - d\varepsilon^2 + O(\varepsilon^{(N-2)})}{\|K\|_\infty^{(N-2)/N}S^{(N-2)/2}} + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\le \frac{S}{\|K\|_\infty^{(N-2)/N}} - d_0\varepsilon^2 + o(\varepsilon^2), \end{aligned}$$

where $d_0>0$. If $N=4$, then in a similar way,

$$m_\varepsilon \le \frac{S}{\|K\|_\infty^{(N-2)/N}} - d_0\varepsilon^2|log\varepsilon| + o(\varepsilon^2).$$

Hence (4.10) holds provided ε is sufficiently small. Note that if $K(x)-K(0)=O(|x|^2)$ as $x\to 0$, then (4.14) holds with $O(\varepsilon^2)$ replacing $o(\varepsilon^2)$. □

5. Proof of theorem 2.1

Proof of theorem 2.1. Applying Lemma 3.5, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (3.12). Lemma 4.1 shows that $\{u_n\}$ is a nonvanishing sequence. Passing to a subsequence, we may assume the existence of $k_n \in Z^N$ such that $\int_{B_{1+\sqrt{N}(k_n)}} |u_n|^2 dx > \frac{\delta}{2}$. Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B_{1+\sqrt{N}(0)}} |v_n|^2 dx > \frac{\delta}{2}. \tag{5.1}$$

Since $V(x), K(x)$ and $f(x, u)$ are periodic on x, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n) \to c_*, \quad \|\Phi'(v_n)\|(1 + \|v_n\|) \to 0. \tag{5.2}$$

Passing to a subsequence, we have $v_n \rightharpoonup \overline{v}$ in $L^s_{loc}(R^N)$, $2 \leq s < 2^*$ and $v_n \to \overline{v}$ a.e. on R^N. Obviously, (5.1) and (5.2) implies that $\overline{v} \neq 0$ and $\Phi(\overline{v}) = 0$. This shows that $\overline{v} \in \mathcal{N}^0$ and so $\Phi(\overline{v}) \geq m$. On the other hand, by using (2.7), (3.12) and Fatou's lemma, we have

$$\begin{aligned}
m \geq c_* &= \lim_{n\to\infty} [\Phi(v_n) - \frac{1}{2}\langle \Phi'(v_n), v_n\rangle] \\
&= \lim_{n\to\infty} \int_{R^N} [\frac{1}{2}[K(x)|v_n|^{2^*-2}v_n + f(x, v_n)]v_n - [\frac{1}{2^*}K(x)|v_n|^{2^*} + F(x, v_n)]]dx \\
&\geq \int_{R^N} \lim_{n\to\infty} [\frac{1}{2}[K(x)|v_n|^{2^*-2}v_n + f(x, v_n)]v_n - [\frac{1}{2^*}K(x)|v_n|^{2^*} + F(x, v_n)]]dx \\
&= \int_{R^N} [\frac{1}{2}[K(x)|\overline{v}|^{2^*-2}\overline{v} + f(x, \overline{v})]\overline{v} - [\frac{1}{2^*}K(x)|v|^{2^*} + F(x, v)]]dx \\
&= \Phi(v) - \frac{1}{2}\langle \Phi'(v), v\rangle] = \Phi(v).
\end{aligned}$$

This shows that $\Phi(v) \leq m$ and so $\Phi(v) = m = inf_{\mathcal{N}^0} > 0$. □

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