Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3036–3047
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Some coincidence point theorems for g-monotone increasing multi-valued mappings in cone metric spaces over Banach algebras

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Communicated by R. Saadati

Abstract

In this paper, in partially ordered cone metric spaces over Banach algebras, we introduce the concept of g-monotone mappings and prove some coincidence point theorems for multi-valued and single-valued g-monotone increasing mappings satisfying certain metric inequalities which are established by an altering distance function. The presented results extend and improve some recent results. An illustrative example is given to support our results.

Keywords: Multi-valued mappings, g-increasing mapping, altering functions.
2010 MSC: 54H25, 47H10.

1. Introduction

The fixed point theory of multi-valued mappings as a vital branch of nonlinear sciences has attracted much attention from many researches, and a large number of existence results of fixed points for these mappings in some partially ordered metric spaces have been proven by many scholars (see [1, 2]). For instance, Choudhury [1] used the control functions on $[0, +\infty)$ to study some fixed point problems of multi-valued mappings in partially ordered metric spaces and obtained some interesting results; Choudhury and Metiya [2], in partially ordered spaces, proved the existence results of fixed point for a class of increasing multi-valued mappings satisfying certain metric inequalities yielded by a control function. Moreover, Khamsi

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Received 2015-10-25

and Khan [4] and Khan et al.[5] obtained some coincidence points results of multi-valued Lipschitzian mappings on a metric space without an order and on a hyperbolic metric space in which the related mappings need to satisfy certain metric inequalities independent of the altering distances, respectively. From the known results, it is not difficult to find that the properties of the involved spaces are fairly essential to get such conclusions.

It is remarkable that in 2007, Huang and Zhang [3] introduced the concept of cone metric spaces and proved that the classical Banach contraction principle remains true in the setting of cone metric spaces. Since then, a great number of fixed point results of the mappings satisfying certain contractive properties on cone metric spaces have been proved on the basis of Huang and Zhang's work (see [6, 7, 8, 9] and the references therein). Among those works, the results of [6] have attracted our much attention since the concept of cone metric spaces over Banach algebras was introduced, and by replacing cone metric spaces with cone metric spaces over Banach algebras, the authors extended the Banach contraction principle to a more general form. Inspired by the works of Khamsi and Khan [4] and Khan et al. [5] and Liu and Xu [6], in this article, we are planning to investigate some fixed point problems of multi-valued mappings in cone metric spaces over Banach algebras. We introduce the concept of g-monotone mappings and prove some fixed point theorems for multi-valued and single-valued g-monotone increasing mappings satisfying certain metric inequalities. First of all, we review some necessary definitions and results from Huang and Zhang [3] and Liu and Xu [6] as following.

Let $\mathcal{A}$ be a real Banach algebra and θ be the null of $\mathcal{A}$ (see [6] for detailed information). For convenience, we assume that e is a unit of the Banach algebra $\mathcal{A}$ satisfying $ex = xe = x$ for all $x \in \mathcal{A}$. We say $x \in \mathcal{A}$ is invertible if there is $y \in \mathcal{A}$ such that $xy = yx = e$, denoted by x^{-1} the inverse of x. Also let the non-empty closed convex subset P of the Banach algebra $\mathcal{A}$ be a cone which determines a partial order '$\preceq$' as follows: $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ denotes $x \preceq y$ and $x \neq y$. $x \ll y$ means that $y - x \in \mathrm{int}(P)$, where $\mathrm{int}(P)$ denotes the set of the interior points of P. P is called a solid cone if $\mathrm{int}(P) \neq \emptyset$.

Definition 1.1 ([6])**.** Let X be a non-empty set and $\mathcal{A}$ be a real Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space over the Banach algebra $\mathcal{A}$.

Some supporting examples of cone metric spaces over Banach algebras can be founded in [6].

Definition 1.2 ([7])**.** Let (X, d) be a cone metric space over the Banach algebra $\mathcal{A}$, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x if for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(2) $\{x_n\}$ is a Cauchy sequence if for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$.

(3) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

The following lemma about spectral radius is indispensable for our proofs.

Lemma 1.3 ([9])**.** *Let x, y be vectors in $\mathcal{A}$. If x and y commute, then the spectral radius r satisfies the following properties:*

(i) $r(xy) \leq r(x)r(y)$;

(ii) $r(x+y) \leq r(x) + r(y)$;

(iii) $|r(x) - r(y)| \leq r(x-y)$.

In the rest of the section, we always assume that (X, d) is a partially ordered cone metric space over the Banach algebra $\mathcal{A}$, in which the partially order is denoted by '$\leqslant$'. Denote by $B(X)$ the class of nonempty bounded subsets of X. For $A, B \in B(X)$, from Zorn's Lemma, $D(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ and $\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}$ can be exactly defined. We write $D(A, B) = D(x, B)$ and $\delta(A, B) = \delta(x, B)$ for $A = \{x\}$. In addition, if $B = \{y\}$, then $D(A, B) = d(x, y)$ and $\delta(A, B) = d(x, y)$. Obviously, $D(A, B) \leq \delta(A, B)$. For any $A, B, C \in B(X)$, from the definition of $\delta(A, B)$, we have

(1) $\delta(A, B) = \delta(B, A)$;

(2) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$;

(3) $\delta(A, B) = 0$ if and only if $A = B = \{x\}$;

(4) $\delta(A, A) = \mathrm{diam}(A)$, where $\mathrm{diam}(A)$ denotes the diameter of A.

For the sake of convenience, let us recall some notions as follows.

Definition 1.4 ([2])**.** Let A and B be two nonempty subsets of X. The relation between A and B is denoted and defined as follows: $A < B$, if for every $x \in A$, there exists $y \in B$ such that $x \leqslant y$.

Definition 1.5. Let P be a cone of the Banach algebra $\mathcal{A}$. $\psi : P \to P$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous;

(ii) $\psi(t) = \theta$ if and only if $t = \theta$.

Definition 1.6. Let $T : X \to B(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. If there exists $x \in X$ such that $g(x) \in Tx$, then x is said to be a coincidence point of g and T. If $T : X \to X$ is a single-valued mapping and there exists $z \in X$ such that $g(z) = Tz$, then z is said to be a coincidence point of g and T.

Remark 1.7. The similar concepts as Definitions 1.5 and 1.6 can be found in [2].

Definition 1.8. Let $T : X \to B(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. For any $x, y \in X$, if $g(x) \leqslant g(y)$ implies $Tx < Ty$ ($Ty < Tx$), then T is said to be g-monotone increasing (decreasing).

For $x, y \in X$, if $g(x) \leqslant g(y)$ or $g(y) \leqslant g(x)$ holds, then we call that x, y are g-comparable.

Definition 1.9. Let $S : X \to X$ and $g : X \to X$. If for any sequence $\{x_n\}$ in X, $g(x_n) \to g(x)$ implies $S(x_n) \to S(x)$ as $n \to \infty$, then S is said to be g-continuous.

Example 1.10. Let $\mathcal{A} = \mathbb{R}^3$. For each $x = (x_1, x_2, x_3) \in \mathcal{A}$, let $\|x\| = |x_1| + |x_2| + |x_3|$. For $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathcal{A}$, the multiplication is defined by

$$xy = (x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1y_1 + x_2y_3 + x_3y_2, x_1y_2 + x_2y_1 + x_3y_3, x_1y_3 + x_2y_2 + x_3y_1).$$

Then one can easily verify that $\mathcal{A}$ is a Banach algebra with unit $e = (1, 0, 0)$. Set $X = \mathbb{R}^3$ and $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$. Obviously, P is a cone in $\mathcal{A}$. A metric d on X is defined by

$$d(x, y) = d((x_1, x_2, x_3), (y_1, y_2, y_3)) = (|x_1 - y_1|,\ |x_2 - y_2|,\ |x_3 - y_3|) \in P.$$

Under the metric d, (X, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$.

Suppose that $g : X \to X$ is a mapping defined as follows: for any $(x_1, x_2, x_3) \in X$,

$$g(x_1, x_2, x_3) = \begin{cases} \left(\sin\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right) & \text{if } x_1x_2x_3 \neq 0, \\ (0, 0, 0) & \text{if } x_1x_2x_3 = 0, \end{cases}$$

and assume $S : X \to X$ as $S(x_1, x_2, x_3) = 2g(x_1, x_2, x_3)$ for any $(x_1, x_2, x_3) \in X$. Clearly, S is g-continuous but not continuous.

2. Main results

In this section, we assume that (X, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$, and there is a partial order $\leqslant$ on X such that $(X, \leqslant)$ is a partial order set, P is a cone of $\mathcal{A}$ which gives a partial order on $\mathcal{A}$, denoted by '$\preceq$', and the altering distance function $\psi : P \to P$ is a homeomorphism. Let $Q \subset \mathcal{A}$. $\bigvee\{Q\}$ in this work means that there exists an element in Q. For example, let $x \in \mathcal{A}$ and $Q \subset \mathcal{A}$, then $x \preceq \bigvee\{Q\}$ means that there is $y \in Q$ such that $x \preceq y$.

Next, we present our first result as follows.

Theorem 2.1. *Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that $\{g(x_0)\} < Tx_0$;

(ii) if $\{g(x_n)\}$ is a nondecreasing sequence in X and $\{g(x_n)\} \to g(x)$, then $g(x_n) \leqslant g(x)$ for all n;

(iii) $\psi(\delta(Tx, Ty)) \preceq \alpha\psi\left(\bigvee\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), H(x, y)\}\right)$ for all g-comparable $x, y \in X$, where $H(x, y) = \frac{D(g(x),Ty)+D(g(y),Tx)}{2}$, $\alpha \in P$ with the spectral radius $r(\alpha) \in (0, 1)$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

Proof. It is clear that there exists $x_1 \in X$ such that $g(x_1) \in Tx_0$ and $g(x_0) \leqslant g(x_1)$. As T is g-monotone increasing, $Tx_0 < Tx_1$. Similarly, there is $x_2 \in X$ such that $g(x_2) \in Tx_1$ and $g(x_1) \leqslant g(x_2)$. Continuing this process we construct a monotone increasing sequence $\{g(x_n)\}$ in X such that $g(x_{n+1}) \in Tx_n$ for all $n \geq 0$, and

$$g(x_0) \leqslant g(x_1) \leqslant \cdots \leqslant g(x_n) \leqslant \cdots .$$

If there has a positive integer k such that $g(x_k) = g(x_{k+1})$, then $g(x_k) = g(x_{k+1}) \in Tx_k$, which yields x_k is a coincidence point of g and T. Hence we may assume $g(x_n) \neq g(x_{n+1})$ for all $n \geq 0$.

From the monotone property of ψ, for all $n \geq 0$, we have

$$\begin{aligned} &\psi(d(g(x_{n+1}), g(x_{n+2}))) \\ &\quad \preceq \psi(\delta(Tx_n, Tx_{n+1})) \\ &\quad \preceq \alpha\psi\left(\bigvee\{d(g(x_n), g(x_{n+1})), D(g(x_n), Tx_n), D(g(x_{n+1}), Tx_{n+1}), H(x_n, x_{n+1})\}\right) \\ &\quad \preceq \alpha\psi\left(\bigvee\{d(g(x_n), g(x_{n+1})), d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2})), I(x_n, x_{n+1}, x_{n+2})\}\right), \end{aligned}$$

where $I(x_n, x_{n+1}, x_{n+2}) = \frac{d(g(x_n),g(x_{n+2}))+d(g(x_{n+1}),g(x_{n+1}))}{2}$. Note that

$$\frac{d(g(x_n), g(x_{n+2}))}{2} \preceq \bigvee\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2}))\},$$

we have

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \preceq \alpha\psi(\bigvee\{d(g(x_n), g(x_{n+1})), d(g(x_{n+1}), g(x_{n+2}))\}). \tag{2.1}$$

Suppose that there exists some positive integer n such that

$$d(g(x_n), g(x_{n+1})) \preceq d(g(x_{n+1}), g(x_{n+2})),$$

then it follows from (2.1) that $\psi(d(g(x_{n+1}), g(x_{n+2}))) \preceq \alpha\psi(d(g(x_{n+1}), g(x_{n+2})))$. Since $r(\alpha) \in (0, 1)$, by Lemma 1.3, we get $d(g(x_{n+1}), g(x_{n+2})) = 0$ or $g(x_{n+1}) = g(x_{n+2})$, which contradicts the above assumption that $g(x_{n+1}) \neq g(x_{n+2})$ for each n. Thus

$$d(g(x_{n+1}), g(x_{n+2})) \prec d(g(x_n), g(x_{n+1})),$$

for all $n \geq 0$. Noting that ψ is an altering distance function on P, for any positive integer n, we get

$$\begin{aligned}\psi(d(g(x_n), g(x_{n+1}))) &\preceq \alpha\psi(d(g(x_{n-1}), g(x_n)))\\ &\preceq \cdots\\ &\preceq \alpha^n\psi(d(g(x_0), g(x_1))).\end{aligned}$$

Because $\mathcal{A}$ is a Banach algebra,

$$\| \psi(d(g(x_n), g(x_{n+1}))) \| \leq N \| \alpha^n\psi(d(g(x_0), g(x_1))) \| \leq N \| \alpha \|^n \| \psi(d(g(x_0), g(x_1))) \| .$$

Notice $r(\alpha) \in (0, 1)$, we get that $\psi(d(g(x_n), g(x_{n+1}))) \to \theta$ as $n \to \infty$. Since ψ is a homeomorphism,

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = \theta. \tag{2.2}$$

Next we prove that $\{g(x_n)\}$ is a Cauchy sequence. Suppose on the contrary that there exists $c > \theta$ for which there exist two sequences of positive integers $\{m(k)\}$and $\{n(k)\}$ such that for each positive integer k, $n(k) > m(k) > k$ and $d(g(x_{m(k)}), g(x_{n(k)})) \succeq c$. Assuming that $n(k)$ is the smallest such positive integer, we immediately have $n(k) > m(k) > k$, $d(g(x_{n(k)}), g(x_{m(k)})) \succeq c$ and $d(g(x_{n(k)-1}), g(x_{m(k)})) \prec c$. Now

$$c \preceq d(g(x_{n(k)}), g(x_{m(k)})) \preceq d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(x_{n(k)-1}), g(x_{m(k)})),$$

implies that

$$c \preceq d(g(x_{n(k)}), g(x_{m(k)})) \prec d(g(x_{n(k)}), g(x_{n(k)-1})) + c. \tag{2.3}$$

Taking limit as $k \to \infty$ in (2.3) and noting (2.2), we get

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)})) = c. \tag{2.4}$$

Also, we have

$$d(g(x_{m(k)}), g(x_{n(k)})) \preceq d(g(x_{m(k)}), g(x_{m(k)+1})) + d(g(x_{m(k)+1}), g(x_{n(k)+1})) + d(g(x_{n(k)+1}), g(x_{n(k)}))$$

and

$$d(g(x_{m(k)+1}), g(x_{n(k)+1})) \preceq d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(x_{m(k)}), g(x_{n(k)})) + d(g(x_{n(k)}), g(x_{n(k)+1})).$$

Taking limit as $k \to \infty$ in the above two inequalities and using (2.2) and (2.4), we get

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)+1})) = c. \tag{2.5}$$

Similarly, we have

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)})) = c \tag{2.6}$$

and

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)+1})) = c. \tag{2.7}$$

As for each positive integer k, $g(x_{m(k)}) \leqslant g(x_{n(k)})$, so $g(x_{m(k)})$ and $g(x_{n(k)})$ are comparable. Then by the monotone property of ψ, we have

$$\begin{aligned}
&\psi(d(g(x_{n(k)+1}), g(x_{m(k)+1})))\\
&\quad \preceq \psi(\delta(Tx_{n(k)}, Tx_{m(k)}))\\
&\quad \preceq \alpha\psi\left(\bigvee\{d(g(x_{n(k)}), g(x_{m(k)})), D(g(x_{n(k)}), Tx_{n(k)}), D(g(x_{m(k)}), Tx_{m(k)}), H(x_{n(k)}, x_{m(k)})\}\right)\\
&\quad \preceq \alpha\psi\left(\bigvee\{d(g(x_{n(k)}), g(x_{m(k)})), d(g(x_{n(k)}), g(x_{n(k)+1})), d(g(x_{m(k)}), g(x_{m(k)+1})), J(n(k), m(k))\}\right),
\end{aligned}$$

where $J(n(k), m(k)) = \frac{d(g(x_{m(k)}), g(x_{n(k)+1})) + d(g(x_{n(k)}), g(x_{m(k)+1}))}{2}$. Letting $k \to \infty$ in the above inequality, from (2.2) and (2.4)–(2.7) and the continuity of ψ, we have $\psi(c) \preceq \alpha\psi(c)$. Notice $0 < r(\alpha) < 1$, by Lemma 1.3, we have $\psi(c) = \theta$ which implies $c = \theta$, which is contrary to $c > \theta$. Hence $\{g(x_n)\}$ is a Cauchy sequence. The completeness of X and the surjective property of g implies that there exists $z \in X$ such that

$$g(x_n) \to g(z) (n \to \infty). \tag{2.8}$$

By the assumption (ii), $g(x_n) \leqslant g(z)$, for all n.

Then from the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}
\psi(\delta(g(x_{n+1}), Tz)) \leq \psi(\delta(Tx_n, Tz))&\\
&\preceq \alpha\psi\left(\bigvee\{d(g(x_n), g(z)), D(g(x_n), Tx_n), D(g(z), Tz), H(x_n, z)\}\right)\\
&\preceq \alpha\psi\left(\bigvee\{d(g(x_n), g(z)), d(g(x_n), g(x_{n+1})), \delta(g(z), Tz), K(x_n, x_{n+1}, z)\}\right),
\end{aligned}$$

where $K(x_n, x_{n+1}, z) = \frac{\delta(g(x_n), Tz) + d(g(z), g(x_{n+1}))}{2}$. Taking limit as $n \to \infty$ in the above inequality, from (2.2), (2.8) and the continuity of ψ, we have

$$\psi(\delta(g(z), Tz)) \preceq \alpha\psi(\delta(g(z), Tz)),$$

which together with Lemma 1.3 implies that $\{g(z)\} = Tz$. Thus z is a coincidence point of g and T.

In addition, suppose that g is an injection. Now we prove the uniqueness of the coincidence point of g and T as follows.

Assume that z_1 is another coincidence point of g and T, that is $g(z_1) \in Tz_1$, then

$$\begin{aligned}
\psi(d(g(z), g(z_1))) \preceq \psi(\delta(Tz, Tz_1))&\\
&\preceq \alpha\psi\left(\bigvee\left\{d(g(z), g(z_1)), D(g(z), Tz), D(g(z_1), Tz_1), \frac{D(g(z), Tz_1) + D(g(z_1), Tz)}{2}\right\}\right)\\
&\preceq \alpha\psi\left(\bigvee\left\{d(g(z), g(z_1)), \frac{D(g(z), Tz_1) + D(g(z_1), Tz)}{2}\right\}\right)\\
&\preceq \alpha\psi\left(\bigvee\left\{d(g(z), g(z_1)), \frac{d(g(z), g(z_1)) + d(g(z_1), g(z))}{2}\right\}\right)\\
&\preceq \alpha\psi(d(g(z), g(z_1))).
\end{aligned}$$

The assumption $r(\alpha) \in (0, 1)$ implies $d(g(z), g(z_1)) = \theta$ or $g(z) = g(z_1)$. As g is an injection, $z = z_1$. □

Let ψ be the identity function in Theorem 2.1, then we get the following corollary.

Corollary 2.2. *Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping such that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\{g(x_0)\} < Tx_0$;

(ii) *if* $g(x_n) \to g(x)$ *is a nondecreasing sequence in* X*, then* $g(x_n) \leqslant g(x)$ *for all* n*;*

(iii) $\delta(Tx, Ty) \preceq \alpha \bigvee\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), H(x, y)\}$ *for all* g*-comparable* $x, y \in X$*, where* $\alpha \in P$ *with* $r(\alpha) \in (0, 1)$*.*

Then there exists a coincidence point of g *and* T *in* X*. Moreover, if* g *is an injection, then the coincidence point of* g *and* T *is unique.*

Taking T as a single-valued mapping, we get a special case of Theorem 2.1 as the following corollary.

Corollary 2.3. *Let* $g : X \to X$ *be a surjection and* $T : X \to X$ *be a* g*-monotone increasing mapping satisfying:*

(i) *there exists* $x_0 \in X$ *such that* $g(x_0) < Tx_0$*;*

(ii) *if* $g(x_n) \to g(x)$ *is a nondecreasing sequence in* X*, then* $g(x_n) \leqslant g(x)$*, for all* n*;*

(iii) $\psi(d(T(x), T(y))) \preceq \alpha\psi\left(\bigvee\{d(g(x), g(y)), d(g(x), T(x)), d(g(y), T(y)), H(x, y)\}\right)$*, for all* g*-comparable* $x, y \in X$*, where* $\alpha \in P$ *with* $r(\alpha) \in (0, 1)$*.*

Then there exists a coincidence point of g *and* T *in* X*, and the coincidence point of* g *and* T *is unique if* g *is an injection.*

Replace the condition (ii) of the above corollary by "T is g- continuous", we have the following result.

Theorem 2.4. *Let* $g : X \to X$ *be a surjection and* $T : X \to X$ *be* g*-monotone increasing and* g*-continuous such that the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $g(x_0) < Tx_0$*;*

(ii) $\psi(d(T(x), T(y))) \preceq \alpha\psi\left(\bigvee\{d(g(x), g(y)), d(g(x), T(x)), d(g(y), T(y)), L(x, y)\}\right)$*, for all* g*-comparable* $x, y \in X$*, where* $L(x, y) = \frac{d(g(x),T(y))+d(g(y),T(x))}{2}$ *and* $\alpha \in P$ *with* $r(\alpha) \in (0, 1)$*.*

Then there exists a coincidence point of g *and* T *in* X*. Moreover the coincidence point of* g *and* T *is unique if* g *is an injection.*

Proof. We can consider T as a multi-valued mapping in which case $T(x)$ is a singleton set for every $x \in X$. Then considering the sequence $\{g(x_n)\}$ in the proof of Theorem 2.1 and arguing exactly as in the proof of Theorem 2.1, we get that $\{g(x_n)\}$ is a Cauchy sequence and $\lim_{n\to\infty} g(x_n) = g(z)$. Then, the g- continuity of T implies that

$$g(z) = \lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} T(x_n) = T(z).$$

This proves that z is a coincidence point of g and T. The uniqueness of the coincidence point of g and T can be proved as in Theorem 2.1 provided g is an injection. □

Before presenting the next result, we introduce a property of the cone P in the Banach algebra $\mathcal{A}$ as follows: P is said to have the semi monotone bounded property if every monotone bounded non-increasing sequence in P has a limit. In the rest of this section, we always assume that cone P satisfies the semi monotone bounded property.

Theorem 2.5. *Let* $g : X \to X$ *be a surjection and* $T : X \to B(X)$ *be a* g*-monotone increasing multi-valued mapping such that the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $\{g(x_0)\} < Tx_0$*;*

(ii) *if* $g(x_n) \to g(x)$ *is a nondecreasing sequence in* X*, then* $g(x_n) \leqslant g(x)$*, for all* n*;*

(iii)

$$\begin{aligned}\psi(\delta(Tx,Ty)) \preceq &\psi\left(\bigvee\{d(g(x),g(y)),D(g(x),Tx),D(g(y),Ty),H(x,y)\}\right)\\ &-\phi(\bigvee\{d(g(x),g(y)),\delta(g(y),Ty)\})\end{aligned}$$

for all g-comparable $x,y\in X$, where ψ is an altering distance function and $\phi: P\to P$ is any continuous function with $\phi(t)=\theta$ if and only if $t=\theta$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

Proof. We consider the sequence $\{g(x_n)\}$ of Theorem 2.1. If there exists a positive integer k such that $g(x_k)=g(x_{k+1})$, then x_k is a coincidence point of g and T. Hence we assume that $g(x_n)\neq g(x_{n+1})$ for all $n\geq 0$.

Using the increasing property of ψ, we have for all $n\geq 0$,

$$\begin{aligned}&\psi(d(g(x_{n+1}),g(x_{n+2})))\preceq\psi(\delta(Tx_n,Tx_{n+1}))\\ &\quad\preceq\psi\left(\bigvee\{d(g(x_n),g(x_{n+1})),D(g(x_n),Tx_n),D(g(x_{n+1}),Tx_{n+1}),H(x_n,x_{n+1})\}\right)\\ &\qquad-\phi(\bigvee\{d(g(x_n),g(x_{n+1})),\delta(g(x_{n+1}),Tx_{n+1})\})\\ &\quad\preceq\psi\left(\bigvee\left\{d(g(x_n),g(x_{n+1})),d(g(x_{n+1}),g(x_{n+2})),\frac{d(g(x_n),g(x_{n+2}))+d(g(x_{n+1}),g(x_{n+1}))}{2}\right\}\right)\\ &\qquad-\phi(\bigvee\{d(g(x_n),g(x_{n+1}),d(g(x_{n+1}),g(x_{n+2}))\}).\end{aligned}$$

Since $\frac{d(g(x_n),g(x_{n+2}))}{2}\preceq\bigvee\{d(g(x_n),g(x_{n+1})),d(g(x_{n+1}),g(x_{n+2}))\}$, it follows that

$$\begin{aligned}\psi(d(g(x_{n+1}),g(x_{n+2})))\preceq&\psi(\bigvee\{d(g(x_n),g(x_{n+1})),d(g(x_{n+1}),g(x_{n+2}))\}\\ &-\phi(\bigvee\{d(g(x_n),g(x_{n+1}),d(g(x_{n+1}),g(x_{n+2}))\}).\end{aligned}$$

Assume that $d(g(x_k),g(x_{k+1}))\preceq d(g(x_{k+1}),g(x_{k+2}))$ for some positive integer k. Then we have

$$\psi(d(g(x_{k+1}),g(x_{k+2})))\preceq\psi(d(g(x_{k+1}),g(x_{k+2}))-\phi(d(g(x_{k+1}),g(x_{k+2})),$$

which yields $\phi(d(g(x_{k+1}),g(x_{k+2})))\preceq\theta$, and so $d(g(x_{k+1}),g(x_{k+2}))=\theta$, that is $g(x_{k+1})=g(x_{k+2})$, it contradicts the original assumption that $g(x_n)\neq g(x_{n+1})$ for all $n\geq 0$. Therefore

$$d(g(x_{n+1}),g(x_{n+2}))\prec d(g(x_n),g(x_{n+1})),\ \text{for all } n\geq 0.$$

By the semi monotone bounded property of P, there exists $r\in P$ such that

$$\lim_{n\to\infty}d(g(x_n),g(x_{n+1}))=r. \tag{2.9}$$

From the above facts, we have, for all $n\geq 0$,

$$\psi(d(g(x_{n+1}),g(x_{n+2})))\preceq\psi(d(g(x_n),g(x_{n+1}))-\phi(d(g(x_n),g(x_{n+1}))).$$

Taking limit as $n\to\infty$ in the above inequality, and from the continuities of ψ and ϕ and by (2.9), we get $\psi(r)\preceq\psi(r)-\phi(r)$, which is a contradiction unless $r=\theta$. Thus

$$\lim_{n\to\infty}d(g(x_n),g(x_{n+1}))=\theta. \tag{2.10}$$

Next we show that $\{g(x_n)\}$ is a Cauchy sequence. If $\{g(x_n)\}$ is not a Cauchy sequence, then using an argument similar to that given in Theorem 2.1, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ and $\varepsilon \in P$, for which

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)})) = \varepsilon, \tag{2.11}$$

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)+1})) = \varepsilon, \tag{2.12}$$

$$\lim_{k\to\infty} d(g(x_{n(k)+1}), g(x_{m(k)})) = \varepsilon, \tag{2.13}$$

$$\lim_{k\to\infty} d(g(x_{n(k)}), g(x_{m(k)+1})) = \varepsilon, \tag{2.14}$$

and for each positive integer k, $g(x_{n(k)})$ and $g(x_{m(k)})$ are comparable. Then from the monotone property of ψ and the condition (iii), we have

$$\begin{aligned}
&\psi(d(g(x_{m(k)+1}), g(x_{n(k)+1})))\\
&\quad \le \psi(\delta(Tx_{m(k)}, Tx_{n(k)}))\\
&\quad \preceq \psi\left(\bigvee\left\{d(g(x_{m(k)}), g(x_{n(k)})), D(g(x_{m(k)}), Tx_{m(k)}), D(g(x_{n(k)}), Tx_{n(k)}), H(x_{m(k)}, x_{n(k)})\right\}\right)\\
&\qquad - \phi(\bigvee\{d(g(x_{m(k)}), g(x_{n(k)}), \delta(g(x_{n(k)}), Tx_{n(k)})\})\\
&\quad \preceq \psi\left(\bigvee\left\{d(g(x_{m(k)}), g(x_{n(k)})), d(g(x_{m(k)}), g(x_{m(k)+1})), d(g(x_{n(k)}), g(x_{n(k)+1}), J(n(k), m(k))\right\}\right)\\
&\qquad - \phi(\bigvee\{d(g(x_{m(k)}), g(x_{n(k)}), d(g(x_{n(k)}), g(x_{n(k)+1}))\}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using the continuities of ψ and ϕ, we have

$$\psi(\varepsilon) \preceq \psi(\varepsilon) - \phi(\varepsilon),$$

which implies $\phi(\varepsilon) \prec \theta$. It is in contradiction with the property of ϕ. Hence $\{g(x_n)\}$ is a Cauchy sequence. From the completeness of X and the surjection of g, there exists $z \in X$ such that $g(x_n) \to g(z)$ as $n \to \infty$. By the assumption (ii), $g(x_n) \preceq g(z)$ for all n. Then the monotone property of ψ together with the condition (iii) implies that

$$\begin{aligned}
&\psi(\delta(g(x_{n+1}), Tz)\\
&\quad \preceq \psi(\delta(Tx_n, Tz)\\
&\quad \preceq \psi\left(\bigvee\left\{d(g(x_n), g(z)), D(g(x_n), Tx_n), D(g(z), Tz), \frac{D(g(x_n), Tz) + D(g(z), Tx_n)}{2}\right\}\right)\\
&\qquad - \phi(\bigvee\{d(g(x_n), g(z), \delta(g(z), Tz)\})\\
&\quad \preceq \psi\left(\bigvee\left\{d(g(x_n), g(z)), d(g(x_n), g(x_{n+1})), D(g(z), Tz), \frac{D(g(x_n), Tz) + d(g(z), g(x_{n+1}))}{2}\right\}\right)\\
&\qquad - \phi(\bigvee\{d(g(x_n), g(z)), \delta(g(z), Tz)\}).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality and using the continuities of ψ and ϕ again, we have

$$\psi(\delta(g(z), Tz)) \preceq \psi(D(g(z), Tz)) - \phi(\delta(g(z), Tz)),$$

which implies that

$$\psi(\delta(g(z), Tz)) \preceq \psi(\delta(g(z), Tz)) - \phi(\delta(g(z), Tz)).$$

This gives a contradiction unless $\delta(g(z), Tz)) = \theta$ or $\{g(z)\} = Tz$, that is $\{g(z)\} = Tz$, so z is a coincidence point of g and T .

In addition, suppose that g is an injection. Now we prove the uniqueness of the coincidence point of g and T .

Assume that z_1 is another coincidence point of g and T, that is $g(z_1) \in Tz_1$, then

$$\begin{aligned}\psi(\delta(g(z), g(z_1))) &\preceq \psi(\delta(Tz, Tz_1)) \\ &\preceq \psi\left(\bigvee\left\{d(g(z), g(z_1)), D(g(z), Tz), D(g(z_1), Tz_1), \frac{D(g(z), Tz_1) + D(g(z_1), Tz)}{2}\right\}\right) \\ &\quad - \phi(\bigvee\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}) \\ &\preceq \psi\left(\bigvee\left\{d(g(z), g(z_1)), \frac{d(g(z), g(z_1)) + d(g(z_1), g(z))}{2}\right\}\right) \\ &\quad - \phi(\bigvee\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}) \\ &\preceq \psi(d(g(z), g(z_1))) - \phi(\bigvee\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}).\end{aligned}$$

Therefore, $\phi(\bigvee\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}) = \theta$. Suppose that $g(z) \neq g(z_1)$, then by the properties of ϕ, we have $\phi(\bigvee\{d(g(z), g(z_1), \delta(g(z_1), Tz_1)\}) \neq \theta$, which yields a contradiction to the above inequality. Thus $g(z) = g(z_1)$. Since g is an injection, $z = z_1$, that is, the coincidence point of g and T is unique. □

Taking ψ as the identity function on X in Theorem 2.5, we have the following result.

Corollary 2.6. *Let $g : X \to X$ be a surjection and $T : X \to B(X)$ be a g-monotone increasing multi-valued mapping such that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\{g(x_0)\} < Tx_0$;

(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \leqslant g(x)$, for all n;

(iii)

$$\begin{aligned}\delta(Tx, Ty) \preceq &\bigvee\{d(g(x), g(y)), D(g(x), Tx), D(g(y), Ty), \frac{D(g(x), Ty) + D(g(y), Tx)}{2}\} \\ &- \phi(\bigvee\{d(g(x), g(y)), \delta(g(y), Ty)\}),\end{aligned}$$

for all g-comparable $x, y \in X$, where $\phi : P \to P$ is any continuous function with $\phi(t) = \theta$ if and only if $t = \theta$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

The following corollary is a special case of Theorem 2.5 when T is a single-valued mapping.

Corollary 2.7. *Let $g : X \to X$ be a surjection and $T : X \to X$ be a g-monotone increasing single-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $g(x_0) < T(x_0)$;

(ii) if $g(x_n) \to g(x)$ is a nondecreasing sequence in X, then $g(x_n) \leqslant g(x)$, for all n;

(iii)

$$\begin{aligned}\psi(d(T(x), T(y))) \preceq &\psi(\bigvee\{d(g(x), g(y)), d(g(x), T(x)), d(g(y), T(y)), L(x, y)\}) \\ &- \phi(\bigvee\{d(g(x), g(y)), d(g(x), T(y))\})\end{aligned}$$

for all g-comparable $x, y \in X$, where ψ is an altering distance function and $\phi : P \to P$ is any continuous function with $\phi(t) = \theta$ if and only if $t = \theta$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

In the following theorem we replace the condition (ii) of the above corollary by "T is g-continuous", we get Theorem 2.8 as follows.

Theorem 2.8. *Let $g : X \to X$ be a surjection and $T : X \to X$ be a g-continuous and g-monotone increasing single-valued mapping such that the following conditions are satisfied:*

(i) there exists $x_0 \in X$ such that $g(x_0) < T(x_0)$;

(ii)
$$\begin{aligned}\psi(d(T(x),T(y))) \preceq &\psi(\bigvee\{d(g(x),g(y)),d(g(x),T(x)),d(g(y),T(y)),L(x,y)\})\\ &-\phi(\bigvee\{d(g(x),g(y)),d(g(y),T(y))\})\end{aligned}$$
for all g-comparable $x,y \in X$, where ψ is an altering distance function and $\phi : P \to P$ is any continuous function with $\phi(t) = \theta$ if and only if $t = \theta$.

Then there exists a coincidence point of g and T in X. In addition, if g is an injection, then the coincidence point of g and T is unique.

Proof. We can treat T as a multi-valued mapping in which case $T(x)$ is a singleton set for every $x \in X$. Then we consider the sequence $\{g(x_n)\}$ as in the proof of Theorem 2.5 and arguing exactly as in the proof of Theorem 2.5, we get that $\{g(x_n)\}$ is a Cauchy sequence and $\lim_{n\to\infty} g(x_n) = g(z)$. Then, the g- continuity of T implies that
$$g(z) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} Tx_n = T(z).$$
This proves that z is a coincidence point of g and T. The uniqueness of z follows as before. □

3. Applications

Example 3.1. Let $\mathcal{A} = \mathbb{R}^2$. For each $x = (x_1, x_2) \in \mathcal{A}$, let $\|x\| = |x_1| + |x_2|$. The multiplication is defined by
$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2).$$
Then $\mathcal{A}$ is a Banach algebra with unit $e = (1, 1)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}$ and $X = \mathbb{R}^2$. A cone metric d on X is defined by
$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$
Then (X, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$.

Now define the mapping $T : X \to X$ by
$$T(x, y) = \left(\ln(e^{x-2} + 1), \tan\left(\frac{2}{\pi}\arctan(y + 1)\right)\right),$$
and $g : X \to X$ by $g((x, y)) = (3x, 4y)$ for each $(x, y) \in X$. Then T and g have a unique coincidence point in X.

In fact, it is clear that g is a bijection and T is g-monotone and g-continuous on X. Taking $x_0 = (0, 0) \in X$, we have $g(x_0) < T(x_0)$, and if let $k = (\frac{2}{e^2}, \frac{1}{\pi}) \in P$ with $0 < r(k) < 1$ and choose the identity mapping I on X as the altering distance function, then for any g-comparable pair $x = (x_1, x_2), y = (y_1, y_2) \in X$, by a simple analysis and calculation, we can verify that the condition (iii) of Theorem 2.4 is satisfied. Hence by Theorem 2.4, T and g have a unique coincidence point in X.

Acknowledgements

The authors thank the editor and the referees for their valuable comments and suggestions. This research was supported by the National Natural Science Foundation of China(11261039) and the Provincial Natural Science Foundation of Jiangxi(20132BAB201009).

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