Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3090–3097
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On approximation properties of certain multidimensional nonlinear integrals

Sevgi Esen Almali[a,*], Akif D. Gadjiev[b]

[a] *Faculty of Sciences and Art, Depatment of Mathematics, Kirikkale University, Kirikkale, Turkey.*
[b] *National Academy of Sciences of Azerbaijan, 9 F. Agaev Street, 1141, Baku, Azerbaijan.*

Communicated by J. Brzdek

Abstract

We prove theorems on convergence of multidimensional nonlinear integrals in Lebesgue points of generated function, and show that the main results are applicable to a wide class of exponentially nonlinear integral operators, which may be constructed by using well known positive kernels in approximation theory.

Keywords: Nonlinear integral operators, positive kernels, Lebesgue points, approximation, exponentially nonlinear integrals.
2010 MSC: 42A35, 42A63.

1. Introduction

It is well known that the solution of boundary value problems for elliptic and parabolic differential equations can be expressed as integral operators with positive kernels. The most famous of these are the kernels of Gauss–Weierstrass, Poisson, Abel–Poisson, etc. Sequences or families of integral operators with positive kernels play an important role in approximation of functions and serve as the main model in the creation of the so called theory of approximation of positive linear operators [2, 6].

Approximative properties of the sequences of linear integral operators with positive kernels generated interest to study approximation by nonlinear integrals [3]–[5], [8]–[12]. Related to that, presumably it would be interesting to find a connection between the approximation by nonlinear integrals and boundary value problems for certain nonlinear differential operators.

*Corresponding author
Email addresses: sevgi_esen@hotmail.com (Sevgi Esen Almali), akif_gadjiev@mail.us (Akif D. Gadjiev)

Received 2015-09-30

On the other hand, given that the classical theorems on the convergence of sequences of operators are only valid for linear operators, approximation by using nonlinear integrals is of interest in terms of the convergence of sequences (or families) of these integrals and founding its limit values. The results of the papers [1, 7], on limiting properties of multidimensional integrals, called Riesz and Bessel potentials, can be considered as regarding this. The main goal of our work is to find appropriate expressions of limits of a family of certain nonlinear integrals.

We will study nonlinear multidimensional integrals of the form

$$L_\lambda(u,x) = \frac{\lambda^n}{w_{n-1}} \int_{\mathbb{R}^n} K(\lambda\,|t-x|\,, u(t))dt,$$

where $x \in \mathbb{R}^n$, $|t-x| = \left((t_1-x_1)^2 + (t_2-x_2)^2 + \cdots + (t_n-x_n)^2\right)^{\frac{1}{2}}$ and w_{n-1} is the area of unit sphere S^{n-1} in $\mathbb{R}^n$.

The kernel-function $K(\xi, u)$, depending on two variables $\xi \geq 0$ and u, $|u| \leq M$ satisfy some properties which will be given in Section 2. We want to find the limit of $L_\lambda(u, x_0)$ as $\lambda \to \infty$ where x_0 is the Lebesgue point of the function $u(x)$, $x \in \mathbb{R}^n$. In Section 3 we will show that the kernel-function $K(\xi, u)$, satisfying all our conditions, may be easily obtained by using classical positive kernels of approximation theory, such as Gauss–Weierstrass, Abel–Poisson, Jacson and other kernels. The main theorem, proved in Part 2, allow us to give a definition of a general subclass of nonlinear integrals $L_\lambda(u, x_0)$ exponentially depending of the function $u(x)$, $x \in \mathbb{R}^n$.

2. Main result

First, we recall the well known definition of Lebesgue points in $\mathbb{R}^n$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and let for $x \in \mathbb{R}^n$,

$$f(r, u(x)) = \int_{S^{n-1}} |u(x + r\theta) - u(x)|\, d\theta, \; 0 < r < \infty.$$

Definition 2.1. The point $x \in R^n$ is called the Lebesgue point of function $u \in L_1(R^n)$ if

$$\lim_{r\to 0} \frac{1}{r^n} \int_{|t|\leq r} |u(x-t) - u(x)|\, dt = 0.$$

From this definition, we can find an $\delta > 0$ such that

$$\frac{1}{r^n} \int_{|t|\leq r} |u(x-t) - u(x)|\, dt < \delta$$

provided $r \leq \delta$.

Let us define a new function by

$$F(\rho) = \int_0^\rho r^{n-1} f(r, u(x_0))dr. \tag{2.1}$$

Definition 2.1 gives that for any $\varepsilon > 0$ there exist a positive number $\delta = \delta(\varepsilon)$ such that

$$F(\rho) < \varepsilon\rho^n, \text{ for all } \rho \leq \delta, \tag{2.2}$$

(see [11]). Moreover, from (2.1) we have

$$dF(\rho) = \rho^{n-1} f(\rho, u(x_0))d\rho. \tag{2.3}$$

Consider now the following nonlinear multidimensional integral

$$L_\lambda(u,x) = \frac{\lambda^n}{w_{n-1}} \int_{\mathbb{R}^n} K(\lambda\,|t-x|\,, u(t))dt, \tag{2.4}$$

where, as we noted above, w_{n-1} is the area of unit sphere S^{n-1} in $\mathbb{R}^n$ and λ is a positive parameter.

Note that for partial derivatives of the function $K(\xi,u)$ with respect to the variable u, we will write for brevity

$$K_u^m(\xi,0) = \frac{\partial^m K(\xi,u)}{\partial u^m}\,|_{u=0}, \quad m=1,2,3,\dots\ .$$

We assume that the kernel function $K(\xi,u)$ satisfy the following conditions:

(a) For any $\xi \geq 0$, $K(\xi,u)$ is an infinitely differentiable function of variables u and ξ and $K(\xi,0)=0$.

(b) For any natural m, $K_u^{(m)}(\xi,0)$ is nonnegative monotonically decreasing function of ξ on semiaxis $[0,\infty)$.

(c)
$$K_u^m(\xi,0) \leq K_u^{(m-1)}(\xi,0), \qquad m=1,2,3,\dots\ .$$

(d)
$$\int_0^\infty K_u^m(\xi,0)\xi^{n-1}d\xi = c_m < \infty, \qquad m=1,2,\dots\ . \tag{2.5}$$

From the conditions(a)-(d), it follows that the kernel $K(\xi,u)$ may be expanded in Maclaurin series by the powers of function u. Moreover, (c) gives that the first partial derivative $K_u^1(\xi,0)$ is a majorant function for other derivatives, that is

$$K_u^{(m.)}(\xi,0) \leq K_u^1(\xi,0), \quad m=1,2,\dots\ .$$

From (d) we can infer the inequality

$$\begin{aligned}\int_{\frac{p}{2}}^p K_u^{(m)}(\xi,0)\xi^{n-1}d\xi &\geq K_u^{(m)}(p,0)\int_{\frac{p}{2}}^p \xi^{n-1}d\xi \\ &= K_u^{(m)}(p,0)p^n\left(1-\frac{1}{2^n}\right)\frac{1}{n}, \quad m=1,2,\dots\ ,\end{aligned}$$

for any positive number.

Using (d) we obtain

$$\lim_{p\to\infty} p^n K_u^{(m)}(p,0) = 0,$$

and from this it follows that for any $m=1,2,\dots,$

$$\lim_{p\to\infty} K_u^{(m)}(p,0) = 0. \tag{2.6}$$

The properties (d) gives also

$$\lim_{p\to\infty}\int_p^\infty K_u^{(m)}(\xi,0)\xi^{n-1}d\xi = 0. \tag{2.7}$$

We will prove now the main theorem on convergence of the family $L_\lambda(u,x)$ as $\lambda\to\infty$ at a fixed point x_0, being the Lebesgue point of the function $u(x)$.

Theorem 2.2. *Let $u(x)$ be a bounded function in $\mathbb{R}^n$, belonging to $L_1(\mathbb{R}^n)$ and let the kernel $K(\xi,u)$ satisfy conditions (a)–(d). Then at each Lebesgue point x_0 of the function $u(x)$ we have*

$$\lim_{\lambda\to\infty} L_\lambda(u,x_0) = \sum_{m=1}^\infty \frac{c_m}{m!}u^m(x_0),$$

where c_m are defined in (2.5).

Proof. First we write the Maclurin expansion of the function $K(\xi, u)$ by the powers of u. We have [1]

$$K(\lambda|t-x|, u(t)) = \sum_{m=1}^{\infty} \frac{1}{m!} K_u^{(m)}(\lambda|t-x_0|, 0)u^m(t).$$

By condition (c) and boundedness of the function $u(t)$, the series in right-hand side is majorating and therefore

$$L_\lambda(u, x_0) = \frac{\lambda^n}{w_{n-1}} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^n} K_u^{(m)}(\lambda|t-x_0|, 0)u^m(t)dt.$$

Firstly, let us transform $t - x_0 = p$ and then take $p = t$

$$L_\lambda(u, x_0) = \frac{\lambda^n}{w_{n-1}} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^n} K_u^{(m)}(\lambda|t|, 0)u^m(t+x_0)dt.$$

From binomial expansion of $u^m(x_0+t)$ we can write

$$L_\lambda(u, x_0) = \frac{\lambda^n}{w_{n-1}} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^n} K_u^{(m)}(\lambda|t|, 0) \sum_{k=0}^{m} C_m^k \left(u(x_0+t) - u(x_0)\right)^{m-k} u^k(x_0)dt,$$

where C_m^k are binomial coefficients. From this we obtain

$$L_\lambda(u, x_0) - \sum_{m=1}^{\infty} \frac{c_m}{m!} u^m(x_0) = \frac{\lambda^n}{w_{n-1}} \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^n} K_u^{(m)}(\lambda|t|, 0) \sum_{k=0}^{m-1} C_m^k \left(u(x_0+t) - u(x_0)\right)^{m-k} u^k(x_0)dt.$$

Since $u(x)$ is bounded, say, $|u(x)| \leq M < \infty$, we have

$$\begin{aligned}\left|\sum_{k=0}^{m-1} C_m^k \left(u(x_0+t) - u(x_0)\right)^{m-k} u^k(x_0)\right| &\leq |u(x_0+t) - u(x_0)| \sum_{k=0}^{m-1} C_m^k (2M)^{m-1-k} M^k \\ &\leq (4M)^m |u(x_0+t) - u(x_0)|.\end{aligned}$$

Therefore, using the property (c), we can write

$$\left|L_\lambda(u, x_0) - \sum_{m=1}^{\infty} \frac{c_m}{m} u^m(x_0)\right| \leq \frac{\lambda^n}{w_{n-1}} \sum_{m=1}^{\infty} \frac{(4M)^m}{m!} \int_{\mathbb{R}^n} |u(x_0+t) - u(x_0)| K_u^1(\lambda|t|, 0)dt.$$

So, for any fixed δ, which will be chosen below, we have

$$\begin{aligned}\left|L_\lambda(u, x_0) - \sum_{m=1}^{\infty} \frac{c_m}{m} u^m(x_0)\right| &\leq \frac{\lambda^n}{w_{n-1}} \left\{\int_{t<\delta} + \int_{t\geq\delta}\right\} |u(x_0+t) - u(x_0)| K_u^1(\lambda|t|, 0)dt \\ &= \frac{1}{w_{n-1}} \left(e^{4M} - 1\right) \left\{\lambda^n I_\lambda^\imath + \lambda^n I_\lambda^{\imath\imath}\right\}.\end{aligned} \tag{2.8}$$

From (b), for $\lambda^n I_\lambda^{\imath\imath}$, we immediately obtain

$$\begin{aligned}\lambda^n I_\lambda^{\imath\imath} &\leq \lambda^n K_u^1(\lambda\delta, 0) \int_\delta^\infty \int_{S^{n-1}} |u(x_0+\rho\theta| \rho^{n-1} d\theta d\rho + |u(x_0)| \lambda^n \int_{t\geq\delta} K_u^1(\lambda|t|, 0)dt \\ &\leq \lambda^n \|u\|_{L_1(\mathbb{R}^n)} K_u^1(\lambda\delta, 0) + \lambda^n u(x_0) w_{n-1} \int_\delta^\infty K_u^1(\lambda\rho, 0) \rho^{n-1} d\rho.\end{aligned}$$

The first term in right-hand side tends to zero as $\lambda \to \infty$ by (2.6), and the second term by (2.7). Therefore

$$\lim_{\lambda\to\infty} \lambda^n I_\lambda^{\imath\imath} = 0. \tag{2.9}$$

Consider $\lambda^n I_\lambda^\imath$. As we noted above, assuming that x_0 is a Lebesgue point of the function u, we obtain, by Definition 2.1, that for $\varepsilon > 0$ there exist a number $\delta > 0$ such that the function $F(\rho)$, defined in (2.1) satisfies the inequality (2.2). Fixing this $\delta > 0$, we can write

$$\lambda^n I_\lambda^\imath = \lambda^n \int_0^\delta \int_{S^{n-1}} |u(x_0+\rho\theta) - u(x_0)| \, K_u^1 (\lambda\rho, 0) \, \rho^{n-1} d\theta d\rho,$$

or, using (2.3) and integrating by parts

$$\begin{aligned}\lambda^n I_\lambda^\imath &= \lambda^n \int_0^\delta K_u^1 (\lambda\rho, 0) \, dF(\rho) \\ &= \lambda^n K_u^1 (\lambda\rho, 0) \, F(\delta) + \lambda^n \int_0^\delta F(\rho) d \left[-K_u^1 (\lambda\rho, 0) \right].\end{aligned}$$

Using (2.2), we have

$$\begin{aligned}\lambda^n I_\lambda^\imath &< \varepsilon (\lambda\delta)^n \, K_u^1 (\lambda\rho, 0) + \varepsilon\lambda^n \int_0^\delta \rho^n d \left[-K_u^1 (\lambda\rho, 0) \right] \\ &= n\varepsilon\lambda^n \int_0^\delta K_u^1 (\lambda\rho, 0) \, \rho^{n-1} d\rho \\ &= n\varepsilon \int_0^\delta K_u^1 (\rho, 0) \, \rho^{n-1} d\rho.\end{aligned}$$

By (2.5), the limit of the integral in the right-hand side is finite and therefore

$$\lim_{\lambda\to\infty} \lambda^n I_\lambda^\imath = 0, \tag{2.10}$$

since ε is an arbitrary small number. Using (2.9) and (2.10) in (2.8), proves the theorem. □

We give the following simple example of an application of Theorem 2.2. Lets consider the kernel-function

$$K(\xi, u) = \frac{1}{\sqrt{2\pi}} \left[\exp e^{-\xi^2} u - 1 \right]. \tag{2.11}$$

Obviously $K(\xi, 0) = 0$. We have

$$K_u^{(m)}(\xi, 0) = \frac{1}{\sqrt{2\pi}} e^{-m\xi^2},$$

and all the conditions of Theorem 2.2 are satisfied. Therefore we use the following well known formula,

$$\int_0^\infty \rho^{n-1} e^{-m\rho^2} d\rho = \frac{\Gamma\left(\frac{n}{2}\right)}{2} \frac{1}{m^{\frac{n+1}{2}}}.$$

3. Exponentially nonlinear integrals

The example of kernel $K(\xi, u)$, given in formula (2.11), shows that Theorem 2.2 covers the subclass of nonlinear integrals which may be described by the following definition.

Definition 3.1. We will say that a nonlinear integral operator contains the exponential nonlinearity if its kernel-function $K(\xi, u)$ has the form

$$K(\xi, u) = e^{A(\xi)u} - 1, \tag{3.1}$$

where $A(\xi)$ is a given function, defined for $\xi \geq 0$.

Definition 3.2. Nonlinear integral, having exponential nonlinearity, will be called exponentially nonlinear integral.

As an example, consider the exponentially nonlinear integral

$$B_\lambda(u,x) = \frac{\lambda^n}{w_{n-1}} \int_{\mathbb{R}^n} \left[e^{A(\lambda|t-x|)u(t)} - 1 \right] dt. \tag{3.2}$$

Following Theorem 2.2, we can see that the following statement holds.

Theorem 3.3. *Let a nonnegative function $A(\xi)$ satisfy the following conditions:*

(a$_1$) $A(\xi) < 1$ for all $\xi \geq 0$,

(b$_1$) $A(\xi)$ is monotonically decreasing function in semiaxis $[0,\infty)$,

(c$_1$) $\int_0^\infty A(\xi)\xi^{n-1}d\xi < \infty$.

Then at each Lebesgue point x_0 of a bounded function $u \in L_1(\mathbb{R}^n)$, we have

$$\lim_{\lambda\to\infty} B_\lambda(u,x_0) = \sum_{m=1}^{\infty} \frac{d_m}{m!} u^m(x_0), \tag{3.3}$$

where $d_m = \int_0^\infty A^m(\xi)\xi^{n-1}d\xi$.

Proof. It is clear that for function (3.1) we have $K(\xi,0) = 0$, $K_u^{(m)}(\xi,0) = A^m(\xi)$, $m = 1,2,\ldots$, and by (a$_1$)

$$K_u^{(m)}(\xi,0) < K_u^{(m-1)}(\xi,0),$$

and (b$_1$) implies (b). Therefore, the conditions (a)–(c) of Theorem 2.2 are satisfied. Moreover, using (c$_1$), we see that all conditions (a)–(d) of Theorem 2.2 are satisfied, which gives the proof. □

Remark 3.4. It is clear that Theorem 3.3 is a corollary of Theorem 2.2. We give this statement in the form of special Theorem since it covers many applicable cases.

Now we consider the case of exponential nonlinear integrals where the function $A(\xi)$ does not satisfy the condition of monotony.

Theorem 3.5. *Let $A(\xi)$ be a nonnegative function such that there exist a monotonically decreasing majorant function $D(\xi)$ on $[0,\infty)$ such that*

$$A^m(\xi) \leq D(\xi), \quad m = 1,2,\ldots\ , \tag{3.4}$$

and

$$\int_0^\infty \xi^{n-1} D(\xi) d\xi < \infty. \tag{3.5}$$

Then (3.3) holds at every Lebesgue point x_0 of bounded function $u \in L_1(\mathbb{R}^n)$.

Proof. As in the proof of Theorem 2.2 we can write the inequality (2.8) for integral (3.2) in the following form

$$\left| B_\lambda(u,x_0) - \sum_{m=1}^{\infty} \frac{d_m}{m!} u^m(x_0) \right| \leq \frac{\lambda^n}{w_{n-1}} \left(e^{4M} - 1\right) \left\{ \lambda^n I_\lambda^\imath + \lambda^n I_\lambda^n \right\},$$

where

$$\lambda^n I_\lambda^\imath = \int_{t<\delta} \left| u(x_0+t) - u(x_0) \right| D(\lambda|t|) dt,$$

and

$$\lambda^n I_\lambda^n = \int_{t\geq\delta} |u(x_0+t)-u(x_0)|\, D(\lambda\,|t|)dt.$$

Since $D(\xi)$ is monotonically decreasing, we obtain that

$$\lim_{\lambda\to\infty} \lambda^n I_\lambda^i = \lim_{\lambda\to\infty} \lambda^n I_\lambda^n = 0.$$

By the same way as above (2.9) and (2.10). The proof is complete. □

Consider the following Fejer-type kernel

$$F(t) = \left(\frac{\sin t}{t}\right)^{2n},$$

and corresponding exponentially nonlinear integral (3.2)

$$F_\lambda(u,x) = \frac{\lambda^n}{w_{n-1}} \int_{\mathbb{R}^n} \left[e^{F(\lambda|t-x|)u(t)} - 1\right] dt.$$

We have

$$F_\lambda(u,x) = \frac{\lambda^n}{w_{n-1}} \int_{\mathbb{R}^n} \sum_{m=1}^{\infty} \left(\frac{\sin(\lambda\,|t-x|)}{\lambda\,|t-x|}\right)^{2nm} u^m(t)dt.$$

Obviously for any natural number n and m

$$\left(\frac{\sin(\lambda\,|t-x|)}{\lambda\,|t-x|}\right)^{2nm} < \left(\frac{\sin(\lambda\,|t-x|)}{\lambda\,|t-x|}\right)^{2n} < \frac{2^n}{\left(1+\lambda^2\,|t-x|^2\right)^n}.$$

Therefore, we have

$$F(\lambda\,|t-x|) < \frac{2^n}{\left(1+\lambda^2\,|t-x|^2\right)^n},$$

which means that the majorant function $D(\xi)$ has the form

$$D(\xi) = \frac{2^n}{(1+\xi^2)^n}, \quad \xi \in [0,\infty).$$

It is easy to see that $D(\xi)$ is a monotonically decreasing function on $[0,\infty)$ and

$$\int_0^\infty \xi^{n-1} D(\xi)\, d\xi \leq \int_0^1 2^n d\xi + 2^n \int_1^\infty \frac{d\xi}{\xi^{n+1}} < \infty.$$

This show that all the conditions of Theorem 3.5 are satisfied.

Acknowledgements

The authors would like to thank the referee for valuable comments.

References

[1] I. A. Aliiev, A. D. Gadjiev, A. Aral, *On approximation properties of a family of linear operators at critical value of parameter*, J. Approx. Theory, **38** (2006), 242–253. 1, 2

[2] F. Altomare, M. Campiti, *Korovkin-Type Approximation Theory And Its Applications*, Walter de Gruyter, Berlin, (1994). 1

[3] L. Angeloni, G. Vinti, *Convergence in variation and rate of approximation for nonlinear integral operators of convolution type*, Results Math., **49** (2006), 1–23. 1

[4] C. Bardaro, J. Musielak, G. Vinti, *Nonlinear integral operators and applications*, Walter de Gruyter, Berlin, (2003).

[5] C. Bardaro, G. Vinti, H. Karsli, *Nonlinear integral operators with homogeneous kernels: pointwise approximation theorems*, Appl. Anal., **90** (2011), 463–474. 1

[6] P. L. Butzer, R. J. Nessel, *Fourier Analysis And Approximation*, Academic Press, New York-London, (1971). 1

[7] A. D. Gadjiev, A. Aral, I. A. Aliev, *On behaviour of the Riesz and generalized Riesz potentials as order tends to zero*, Math. Ineq. Appl., **10** (2007), 875–888. 1

[8] H. Karsli, *On approximation properties of non-convolution type nonlinear integral operators*, Anal. Theory Appl., **26** (2010), 140–152. 1

[9] J. Musielak, *Approximation by nonlinear singular integral operators in generalized Orlicz spaces*, Comment. Math. Prace Mat., **31** (1991), 79–88.

[10] L. Rempulska, A. Thiel, *Approximation of functions by certain nonlinear integral operators*, Lith. Math. J., **48** (2008), 451–462.

[11] L. M. Stein, G. Weiss, *Introduction To Fourier Analysis On Euclidean Spaces*, Princeton University Press, New Jersey, (1971). 2

[12] T. Swiderski, E. Wachnicki, *Nonlinear singular integral depending on two parameters*, Comment. Math. Prace Mat., **40** (2000), 181–189. 1