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Differential equations associated with λ-Changhee polynomials

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Abstract

In this paper, we study linear differential equations arising from λ-Changhee polynomials (or called degenerate Changhee polynomials) and give some explicit and new identities for the λ-Changhee polynomials associated with linear differential equations.

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1. Introduction

For $N \in \mathbb{N}$, we define the generalized harmonic numbers as follows:

$$H_{N,0} = 1, \quad \text{for all } N,$$

$$H_{N,1} = H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}, \quad (\text{see [7]}), \tag{1.1}$$

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j}, \quad (2 \leq j \leq N).$$

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For $k \in \mathbb{N}$ and $N, j \in \mathbb{N} \cup \{0\}$, we define the generalized Changhee power sums $S_{k,j}(N)$ as follows:

$$S_{k,0}(N) = (N+1)^k, \tag{1.2}$$

$$S_{k,j}(N) = \sum_{l=0}^{N} S_{k,j-1}(l), \quad (j \geq 1), \quad (\text{see [7, 9]}). \tag{1.3}$$

In particular, for $k=1$, we also define $S_{1,-1}(N) = 1$.
As is well known, the Euler polynomials are defined by the generating function

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 4, 11, 12]}). \tag{1.4}$$

With the viewpoint of deformed Euler polynomials, the Changhee polynomials are defined by the generating function

$$\frac{2}{t+2}(t+1)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_n(x) \frac{t^n}{n!}, \quad (\text{see [3, 5, 6]}). \tag{1.5}$$

From (1.4) and (1.5), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \mathrm{Ch}_m(x) \frac{1}{m!} \left(e^t - 1\right)^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \mathrm{Ch}_m(x) S_2(n,m) \right) \frac{t^n}{n!}, \end{aligned} \tag{1.6}$$

where $S_2(n,m)$ are the Stirling numbers of the second kind. Thus, by (1.6), we get

$$E_n(x) = \sum_{m=0}^{n} \mathrm{Ch}_m(x) S_2(n,m) \quad (n \geq 0).$$

The Stirling numbers of the first kind $S_1(n,l)$ appear in the expansion of the falling factorial

$$\begin{aligned} (x)_0 = 1, \quad (x)_n &= x(x-1)\cdots(x-n+1) \\ &= \sum_{l=0}^{n} S_1(n,l) x^l, \quad (n \geq 1). \end{aligned} \tag{1.7}$$

From (1.7), we note that the generating function of the Stirling numbers of the first kind is given by

$$(\log(1+t))^n = n! \sum_{m=n}^{\infty} S_1(m,n) \frac{t^m}{m!}, \quad (\text{see [8, 10, 13]}).$$

By (1.1), we easily get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{Ch}_n(x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} E_m(x) \frac{1}{m!} (\log(1+t))^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_1(n,m) E_m(x) \right) \frac{t^m}{m!}. \end{aligned} \tag{1.8}$$

Thus, by (1.8), we have

$$\mathrm{Ch}_n(x) = \sum_{m=0}^{n} S_1(n,m) E_m(x), \quad (n \geq 0), \quad (\text{see [17, 18]}).$$

Recently, λ-Changhee polynomials (or called degenerate Changhee polynomials) are defined by the generating function

$$\frac{2\lambda}{2\lambda+\log(1+\lambda t)}\left(1+\frac{\log(1+\lambda t)}{\lambda}\right)^{x}=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,\lambda}(x)\frac{t^{n}}{n!},\quad (\text{see [14]}).$$

When $x=0$, $\mathrm{Ch}_{n,\lambda}=\mathrm{Ch}_{n,\lambda}(0)$ are called λ-Changhee numbers (or called degenerate Changhee numbers).

In [7], Kim-Kim gave some explicit and new identities for the Bernoulli numbers of the second kind arising from nonlinear differential equations. It is known that some interesting identities and properties of the Frobenius-Euler polynomials are also derived from the non-linear differential equations (see [9, 12]).

Recently, several authors have studied some interesting properties for the Changhee numbers and polynomials (see [1–19]).

In this paper, we develop some new method for obtaining identities related to λ-Changhee polynomials arising from linear differential equations. From our study, we derive some explicit and new identities for the λ-Changhee polynomials.

2. Some identities for the λ-Changhee polynomials arising from linear differential equations

First, we introduce lemma for the generalized Changhee power sum $S_{k,j}(N)$.

Lemma 2.1. *For $2\le r\le N$ and $1\le i\le r-1$, we have*

$$S_{1,i-1}(r-1-i)+S_{1,i-2}(r-i)=S_{1,i-1}(r-i). \tag{A}$$

Proof. From (1.2) and (1.3), we have

$$\begin{aligned}S_{1,i-1}(r-1-i)+S_{1,i-2}(r-i)&=\sum_{l=0}^{r-1-i}S_{1,i-2}(l)+S_{1,i-2}(r-i)\\&=\sum_{l=0}^{r-i}S_{1,i-2}(l)\\&=S_{1,i-1}(r-i).\end{aligned}$$

□

Let

$$F=F(t;x,\lambda)=\frac{2\lambda}{2\lambda+\log(1+\lambda t)}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^{x}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned}F^{(1)}&=\frac{d}{dt}F(t;x,\lambda)\\&=\lambda(1+\lambda t)^{-1}\left(-(2\lambda+\log(1+\lambda t))^{-1}+x(\lambda+\log(1+\lambda t))^{-1}\right)F.\end{aligned} \tag{2.2}$$

$$\begin{aligned}F^{(2)}&=\frac{dF^{(1)}}{dt}\\&=\lambda^{2}(1+\lambda t)^{-2}\Big\{(2\lambda+\log(1+\lambda t))^{-1}-x(\lambda+\log(1+\lambda t))^{-1}\\&\quad+2(2\lambda+\log(1+\lambda t))^{-2}-2x(2\lambda+\log(1+\lambda t))^{-1}(\lambda+\log(1+\lambda t))^{-1}\\&\quad+(x)_{2}(\lambda+\log(1+\lambda t))^{-2}\Big\}F.\end{aligned} \tag{2.3}$$

So, we are led to put

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t;x,\lambda) \\ &= \lambda^N (1+\lambda t)^{-N} \left(\sum_{1\le i+j\le N} a_{i,j}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{-j} \right) F, \end{aligned} \tag{2.4}$$

where $N = 1, 2, \ldots$, and the sum is over all nonnegative integers i, j with $1 \le i + j \le N$.

On the one hand, by (2.4), we get

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= \lambda^{N+1}(1+\lambda t)^{-(N+1)} \\ &\quad \times \Bigg\{ (-N) \sum_{\substack{1\le i+j\le N \\ i\ge 0, j\ge 0}} a_{i,j}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{-j} \\ &\quad - \sum_{\substack{2\le i+j\le N+1 \\ i\ge 1,\, j\ge 0}} i a_{i-1,j}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{-j} \\ &\quad + \sum_{\substack{2\le i+j\le N+1 \\ i\ge 0, j\ge 1}} (x-j+1)\, a_{i,j-1}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i} \\ &\quad \times \left(\lambda+\log(1+\lambda t)\right)^{-j} \Bigg\} F. \end{aligned}$$

On the other hand, by replacing N by $N + 1$ in (2.4), we have

$$\begin{aligned} F^{(N+1)} &= \lambda^{N+1}(1+\lambda t)^{-(N+1)} \\ &\quad \times \Bigg(\sum_{\substack{1\le i+j\le N+1 \\ i,j\ge 0}} a_{i,j}^{(\lambda)}(N+1,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i} \\ &\quad \times \left(\lambda+\log(1+\lambda t)\right)^{-j} \Bigg) F. \end{aligned}$$

Let $i + j = r$. Then $1 \le r \le N + 1$. Comparing the terms with $r = 1$, we get

$$\begin{aligned} a_{1,0}^{(\lambda)}(N+1,x) &= -N a_{1,0}^{(\lambda)}(N,x) \\ a_{0,1}^{(\lambda)}(N+1,x) &= -N a_{0,1}^{(\lambda)}(N,x). \end{aligned} \tag{2.5}$$

Comparing the terms with $i + j = r$ $(2 \le r \le N)$,

$$\begin{aligned} &\sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N+1,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{i-r} \\ &= -N \sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{i-r} \\ &\quad - \sum_{i=1}^{r} i a_{i-1,r-i}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{i-r} \\ &\quad + \sum_{i=0}^{r-1} (x+i-r+1)\, a_{i,r-i-1}^{(\lambda)}(N,x)\left(2\lambda+\log(1+\lambda t)\right)^{-i}\left(\lambda+\log(1+\lambda t)\right)^{i-r}. \end{aligned} \tag{2.6}$$

Thus, by (2.6), we get

$$a_{0,r}^{(\lambda)}(N+1,x) = -Na_{0,r}^{(\lambda)}(N,x) + (x-r+1)\,a_{0,r-1}^{(\lambda)}(N,x), \tag{2.7}$$

$$a_{r,0}^{(\lambda)}(N+1,x) = -Na_{r,0}^{(\lambda)}(N,x) - ra_{r-1,0}^{(\lambda)}(N,x), \tag{2.8}$$

and

$$\begin{aligned} a_{i,r-i}^{(\lambda)}(N+1,x) &= -Na_{i,r-i}^{(\lambda)}(N,x) - ia_{i-1,r-i}^{(\lambda)}(N,x) \\ &\quad + (x+i-r+1)\,a_{i,r-i-1}^{(\lambda)}(N,x), \end{aligned} \tag{2.9}$$

where $1 \le i \le r-1$.

Comparing the terms with $i+j = N+1$, we get

$$\begin{aligned} &\sum_{i=0}^{N+1} a_{i,N+1-i}^{(\lambda)}(N+1,x)\,(2\lambda+\log(1+\lambda t))^{-i}\,(\lambda+\log(1+\lambda t))^{i-(N+1)} \\ &= -\sum_{i=1}^{N+1} ia_{i-1,N+1-i}^{(\lambda)}(N,x)\,(2\lambda+\log(1+\lambda t))^{-i}\,(\lambda+\log(1+\lambda t))^{i-(N+1)} \\ &\quad + \sum_{i=0}^{N}(x+i-N)\,a_{i,N-i}^{(\lambda)}(N,x)\,(2\lambda+\log(1+\lambda t))^{-i}\,(\lambda+\log(1+\lambda t))^{i-(N+1)}. \end{aligned} \tag{2.10}$$

From (2.10), we note that

$$\begin{aligned} a_{0,N+1}^{(\lambda)}(N+1,x) &= (x-N)\,a_{0,N}^{(\lambda)}(N,x), \\ a_{N+1,0}^{(\lambda)}(N+1,x) &= -(N+1)\,a_{N,0}^{(\lambda)}(N,x), \end{aligned} \tag{2.11}$$

and

$$a_{i,N+1-i}^{(\lambda)}(N+1,x) = -ia_{i-1,N+1-i}^{(\lambda)}(N,x) + (x+i-N)\,a_{i,N-i}^{(\lambda)}(N,x), \tag{2.12}$$

where $1 \le i \le N$.

From (2.2) and (2.4), we have

$$\begin{aligned} &\lambda(1+\lambda t)^{-1}\left(-(2\lambda+\log(1+\lambda t))^{-1} + x(\lambda+\log(1+\lambda t))^{-1}\right)F \\ &= F^{(1)} \\ &= \lambda(1+\lambda t)^{-1}\Big(a_{1,0}^{(\lambda)}(1,x)\,(2\lambda+\log(1+\lambda t))^{-1} \\ &\quad + a_{0,1}^{(\lambda)}(1,x)\,(\lambda+\log(1+\lambda t))^{-1}\Big)F. \end{aligned} \tag{2.13}$$

By comparing the coefficients on both sides of (2.13), we have

$$a_{1,0}^{(\lambda)}(1,x) = -1, \quad a_{0,1}^{(\lambda)}(1,x) = x.$$

From (2.5), we note that

$$a_{1,0}^{(\lambda)}(N+1,x) = (-1)^{N+1}N!, \quad a_{0,1}^{(\lambda)}(N+1,x) = (-1)^N N!x.$$

By (2.11), we easily get

$$a_{N+1,0}^{(\lambda)}(N+1,x) = (-1)^{N+1}(N+1)!, \quad a_{0,N+1}^{(\lambda)}(N+1,x) = (x)_{N+1}.$$

For $i = 1$ in (2.12), we have

$$\begin{aligned}
&a^{(\lambda)}_{1,N}(N+1,x)\\
&= -a^{(\lambda)}_{0,N}(N,x) + (x+1-N)\,a^{(\lambda)}_{1,N-1}(N,x)\\
&= -(x)_N + (x+1-N)\,a^{(\lambda)}_{1,N-1}(N,x)\\
&= -(x)_N + (x+1-N)\Big(-(x)_{N-1} + (x+2-N)\,a^{(\lambda)}_{1,N-2}(N-1,x)\Big)\\
&= -2(x)_N + (x+1-N)(x+2-N)\,a^{(\lambda)}_{1,N-2}(N-1,x)\\
&\vdots\\
&= -N(x)_N + (x+1-N)(x+2-N)\cdots(x+N-N)\,a^{(\lambda)}_{1,0}(1,x)\\
&= -(N+1)(x)_N\\
&= -S_{1,0}(N)(x)_N.
\end{aligned}$$

For $i = 2$ in (2.12), we note that

$$\begin{aligned}
&a^{(\lambda)}_{2,N-1}(N+1,x)\\
&= -2a^{(\lambda)}_{1,N-1}(N,x)\\
&\quad + (x+2-N)\,a^{(\lambda)}_{2,N-2}(N,x)\\
&= (-1)^2\,2N(x)_{N-1}\\
&\quad + (x+2-N)\Big\{(-1)^2\,2(N-1)(x)_{N-2} + (x+3-N)\,a^{(\lambda)}_{2,N-3}(N-1,x)\Big\}\\
&= (-1)^2\,2\{N+(N-1)\}(x)_{N-1} + (x+2-N)(x+3-N)\,a^{(\lambda)}_{2,N-3}(N-1,x)\\
&\vdots\\
&= (-1)^2\,2\{N+(N-1)+(N-2)+\cdots+2\}(x)_{N-1}\\
&\quad + (x+2-N)(x+3-N)\cdots x a^{(\lambda)}_{2,0}(2,x)\\
&= (-1)^2\,2\{N+(N-1)+\cdots+2+1\}(x)_{N-1}\\
&= (-1)^2\,2!S_{1,1}(N-1)(x)_{N-1}.
\end{aligned}$$

Let $i = 3$ in (2.12). Then we have

$$\begin{aligned}
&a^{(\lambda)}_{3,N-2}(N+1,x)\\
&= -3a^{(\lambda)}_{2,N-2}(N,x) + (x+3-N)\,a^{(\lambda)}_{3,N-3}(N,x)\\
&= (-1)^3\,3!S_{1,1}(N-2)(x)_{N-2} + (x+3-N)\,a^{(\lambda)}_{3,N-3}(N,x)\\
&\vdots\\
&= (-1)^3\,3!\{S_{1,1}(N-2)+\cdots+S_{1,1}(1)\}(x)_{N-2}\\
&\quad + (x+3-N)(x+4-N)\cdots x a^{(\lambda)}_{3,0}(3,x)\\
&= (-1)^3\,3!\{S_{1,1}(N-2)+\cdots+S_{1,1}(1)+S_{1,1}(0)\}(x)_{N-2}\\
&= (-1)^3\,3!S_{1,2}(N-2)(x)_{N-2}.
\end{aligned}$$

Continuing this process, we get

$$a^{(\lambda)}_{i,N+1-i}(N+1,x) = (-1)^i\,i!S_{1,i-1}(N-i+1)(x)_{N-i+1}, \quad (1 \le i \le N).$$

Let $2 \leq r \leq N$. Then, by (2.7), we get

$$\begin{aligned} a_{0,r}^{(\lambda)}(N+1,x) &= (x-r+1)\, a_{0,r-1}^{(\lambda)}(N,x) - N a_{0,r}^{(\lambda)}(N,x) \\ &= (x-r+1)\, a_{0,r-1}^{(\lambda)}(N,x) \\ &\quad -N\left\{(x-r+1)\, a_{0,r-1}^{(\lambda)}(N-1,x) - (N-1)\, a_{0,r}^{(\lambda)}(N-1,x)\right\} \\ &= (x-r+1)\left\{a_{0,r-1}^{(\lambda)}(N,x) - N a_{0,r-1}^{(\lambda)}(N-1,x)\right\} \\ &\quad +(-1)^2 N(N-1)\, a_{0,r}^{(\lambda)}(N-1,x) \\ &\vdots \\ &= (x-r+1)\sum_{i=0}^{N-r}(-1)^i (N)_i\, a_{0,r-1}^{(\lambda)}(N-i,x) \\ &\quad +(-1)^{N-r+1} N(N-1)\cdots r a_{0,r}^{(\lambda)}(r,x) \\ &= (x-r+1)\sum_{i=0}^{N-r+1}(-1)^i (N)_i\, a_{0,r-1}^{(\lambda)}(N-i,x). \end{aligned} \tag{2.14}$$

Now, we give an explicit expression for $a_{0,r}^{(\lambda)}(N+1,x)$ $(2 \leq r \leq N)$.
For $r = 2$ in (2.14), we have

$$\begin{aligned} a_{0,2}^{(\lambda)}(N+1,x) &= (x-1)\sum_{i=0}^{N-1}(-1)^i (N)_i\, a_{0,1}^{(\lambda)}(N-i,x) \\ &= (x)_2 (-1)^{N-1}\sum_{i=0}^{N-1}(N)_i (N-i-1)! \\ &= (x)_2 (-1)^{N-1} N! \sum_{i=0}^{N-1}\frac{1}{N-i} \\ &= (x)_2 (-1)^{N-1} N! \sum_{i=1}^{N}\frac{1}{i} \\ &= (x)_2 (-1)^{N-1} N! H_{N,1}. \end{aligned} \tag{2.15}$$

Let us consider $r = 3$ in (2.14). From (2.15), we note that

$$\begin{aligned} a_{0,3}^{(\lambda)}(N+1,x) &= (x-2)\sum_{i=0}^{N-2}(-1)^i (N)_i\, a_{0,2}^{(\lambda)}(N-i,x) \\ &= (x-2)\sum_{i=0}^{N-2}(-1)^i (N)_i (x)_2 (-1)^{N-i-2}(N-i-1)! H_{N-i-1} \\ &= (x)_3 (-1)^{N-2} N! \sum_{i=0}^{N-2}\frac{H_{N-i-1}}{N-i} \\ &= (x)_3 (-1)^{N-2} N! H_{N,2}. \end{aligned}$$

For $r = 4$ in (2.14), we have

$$a_{0,4}^{(\lambda)}(N+1,x) = (x-3)\sum_{i=0}^{N-3}(-1)^i (N)_i\, a_{0,3}^{(\lambda)}(N-i,x)$$

$$
\begin{aligned}
&= (x-3) \sum_{i=0}^{N-3} (-1)^i (N)_i (x)_3 (-1)^{N-i-3} (N-i-1)! H_{N-i-1,2} \\
&= (x)_4 (-1)^{N-3} N! \sum_{i=0}^{N-3} \frac{H_{N-i-1,2}}{N-i} \\
&= (x)_4 (-1)^{N-3} N! \left\{ \frac{H_{N-1,2}}{N} + \frac{H_{N-2,2}}{N-1} + \cdots + \frac{H_{2,2}}{3} \right\} \\
&= (x)_4 (-1)^{N-3} N! H_{N,3}.
\end{aligned}
$$

Continuing this process, we get

$$
a_{0,r}^{(\lambda)} (N+1, x) = (x)_r (-1)^{N-r+1} N! H_{N,r-1}.
$$

For $2 \leq r \leq N$, by (2.8), we get

$$
\begin{aligned}
a_{r,0}^{(\lambda)} (N+1, x) &= -r a_{r-1,0}^{(\lambda)} (N, x) - N a_{r,0}^{(\lambda)} (N, x) \\
&\vdots \\
&= -r \sum_{i=0}^{N-r} (-1)^i (N)_i a_{r-1,0}^{(\lambda)} (N-i, x) + (-1)^{N-r+1} N (N-1) \cdots r a_{r,0}^{(\lambda)} (r, x) \\
&= -r \sum_{i=0}^{N-r} (-1)^i (N)_i a_{r-1,0}^{(\lambda)} (N-i, x) + (-1)^{N-r+1} (N)_{N-r+1} (-1)^r r! \\
&= -r \sum_{i=0}^{N-r+1} (-1)^i (N)_i a_{r-1,0}^{(\lambda)} (N-i, x).
\end{aligned} \tag{2.16}
$$

Let $r = 2$ in (2.16). Then, we have

$$
\begin{aligned}
a_{2,0}^{(\lambda)} (N+1, x) &= -2 \sum_{i=0}^{N-1} (-1)^i (N)_i a_{1,0}^{(\lambda)} (N-i, x) \\
&= 2 (-1)^{N+1} \sum_{i=0}^{N+1} (N)_i (N-i-1)! \\
&= 2 (-1)^{N+1} N! \sum_{i=0}^{N-1} \frac{1}{N-i} \\
&= 2 (-1)^{N+1} N! \sum_{i=1}^{N} \frac{1}{i} \\
&= 2 (-1)^{N+1} N! H_{N,1}.
\end{aligned} \tag{2.17}
$$

For $r = 3$ in (2.16), by (2.17), we get

$$
\begin{aligned}
a_{3,0}^{(\lambda)} (N+1, x) &= -3 \sum_{i=0}^{N-2} (-1)^i (N)_i a_{2,0}^{(\lambda)} (N-i, x) \\
&= -3 \sum_{i=0}^{N-2} (-1)^i (N)_i 2 (-1)^{N-i} (N-i-1)! H_{N-i-1,1} \\
&= 3! (-1)^{N+1} N! \sum_{i=0}^{N-2} \frac{H_{N-i-1,1}}{N-i} \\
&= 3! (-1)^{N+1} N! H_{N,2}.
\end{aligned} \tag{2.18}
$$

From (2.18), by $r = 4$ in (2.16), we note that

$$\begin{aligned} a_{4,0}^{(\lambda)}(N+1,x) &= -4\sum_{i=0}^{N-3}(-1)^i (N)_i\, a_{3,0}^{(\lambda)}(N-i,x) \\ &= (-1)^{N+1}4!N!\sum_{i=0}^{N-3}\frac{H_{N-i-1,2}}{N-i} \\ &= (-1)^{N+1}4!N!H_{N,3}. \end{aligned}$$

Continuing this process, we get

$$a_{r,0}^{(\lambda)}(N+1,x) = (-1)^{N+1}N!r!H_{N,r-1}, \quad (2 \le r \le N).$$

Let $2 \le r \le N$, $1 \le i \le r-1$. Then, by (2.9), we get

$$\begin{aligned} a_{i,r-i}^{(\lambda)}(N+1,x) &= (x+i-r+1)\, a_{i,r-i-1}^{(\lambda)}(N,x) - i a_{i-1,r-i}^{(\lambda)}(N,x) - N a_{i,r-i}^{(\lambda)}(N,x) \\ &= (x+i-r+1)\left\{a_{i,r-i-1}^{(\lambda)}(N,x) - N a_{i,r-i-1}^{(\lambda)}(N-1,x)\right\} \\ &\quad -i\left\{a_{i-1,r-i}^{(\lambda)}(N,x) - N a_{i-1,r-i}^{(\lambda)}(N-1,x)\right\} \\ &\quad +(-1)^2 N(N-1)\, a_{i,r-i}^{(\lambda)}(N-1,x) \\ &\ \ \vdots \\ &= (x+i-r+1)\sum_{s=0}^{N-r}(-1)^s (N)_s\, a_{i,r-i-1}^{(\lambda)}(N-s,x) \\ &\quad -i\sum_{s=0}^{N-r}(-1)^s (N)_s\, a_{i-1,r-i}^{(\lambda)}(N-s,x) \\ &\quad +(-1)^{N-r+1}(N)_{N-r+1}\, a_{i,r-i}^{(\lambda)}(r,x) \\ &= (x+i-r+1)\sum_{s=0}^{N-r}(-1)^s (N)_s\, a_{i,r-i-1}^{(\lambda)}(N-s,x) \\ &\quad -i\sum_{s=0}^{N-r}(-1)^s (N)_s\, a_{i-1,r-i}^{(\lambda)}(N-s,x) \\ &\quad +(-1)^{N-r+1}(N)_{N-r+1}(-1)^i i! S_{1,i-1}(r-i)(x)_{r-i}. \end{aligned} \tag{2.19}$$

Let $r = 2$ in (2.19). Then $i = 1$. From (2.19), we note that

$$\begin{aligned} a_{1,1}^{(\lambda)}(N+1,x) &= x\sum_{s=0}^{N-2}(-1)^s (N)_s\, a_{1,0}^{(\lambda)}(N-s,x) \\ &\quad -\sum_{s=0}^{N-2}(-1)^s (N)_s\, a_{0,1}^{(\lambda)}(N-s,x) \\ &\quad +(-1)^{N-1}N!(-1)S_{1,0}(1)\,x \\ &= x\sum_{s=0}^{N-2}(-1)^s (N)_s (-1)^{N-s}(N-s-1)! \\ &\quad -\sum_{s=0}^{N-2}(-1)^s (N)_s (-1)^{N-s-1}(N-s-1)!x + 2(-1)^N N!x \end{aligned}$$

$$
\begin{aligned}
&= 2x\left(-1\right)^{N}\sum_{s=0}^{N-2}\left(N\right)_{s}\left(N-s-1\right)!+2\left(-1\right)^{N}N!x\\
&= 2x\left(-1\right)^{N}N!\sum_{s=0}^{N-2}\frac{1}{N-s}+2\left(-1\right)^{N}N!x\\
&= 2\left(-1\right)^{N}N!x\left\{\frac{1}{N}+\frac{1}{N-1}+\cdots+\frac{1}{2}+\frac{1}{1}\right\}\\
&= 2\left(-1\right)^{N}N!xH_{N}.
\end{aligned}
$$

Let $r=3$. Then $1\le i\le 2$. From (2.19), we note that

$$
\begin{aligned}
a_{i,3-i}^{(\lambda)}\left(N+1,x\right) = \left(x+i-2\right)&\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}a_{i,2-i}^{(\lambda)}\left(N-s,x\right)\\
&-i\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}a_{i-1,3-i}^{(\lambda)}\left(N-s,x\right)\\
&+\left(-1\right)^{N-2}\left(N\right)_{N-2}\left(-1\right)^{i}i!S_{1,i-1}\left(3-i\right)\left(x\right)_{3-i}.
\end{aligned}
\tag{2.20}
$$

For $i=1$ in (2.20), we have

$$
\begin{aligned}
&a_{1,2}^{(\lambda)}\left(N+1,x\right)\\
&= \left(x-1\right)\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}a_{1,1}^{(\lambda)}\left(N-s,x\right)\\
&\quad -\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}a_{0,2}^{(\lambda)}\left(N-s,x\right)\\
&\quad +\left(-1\right)^{N-2}\left(N\right)_{N-2}\left(-1\right)S_{1,0}\left(2\right)\left(x\right)_{2}\\
&= \left(x-1\right)\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}2\left(-1\right)^{N-s-1}\left(N-s-1\right)!H_{N-s-1,1}x\\
&\quad -\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}\left(x\right)_{2}\left(-1\right)^{N-s-2}\left(N-s-1\right)!H_{N-s-1,1}\\
&\quad +3\left(-1\right)^{N-1}\left(N\right)_{N-2}\left(x\right)_{2}\\
&= 3\left(x\right)_{2}\left(-1\right)^{N-1}N!\sum_{s=0}^{N-3}\frac{H_{N-s-1,1}}{N-s}+3\left(x\right)_{2}\left(-1\right)^{N-1}N!\frac{1}{2}\\
&= 3\left(x\right)_{2}\left(-1\right)^{N-1}N!\left\{\frac{H_{N-1,1}}{N}+\frac{H_{N-2,1}}{N-1}+\cdots+\frac{H_{2,1}}{3}+\frac{H_{1,1}}{2}\right\}\\
&= 3\left(x\right)_{2}\left(-1\right)^{N-1}N!H_{N,2}.
\end{aligned}
\tag{2.21}
$$

Let $i=2$ in (2.20). From (2.21), we note that

$$
\begin{aligned}
a_{2,1}^{(\lambda)}\left(N+1,x\right) = x&\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}a_{2,0}^{(\lambda)}\left(N-s,x\right)\\
&-2\sum_{s=0}^{N-3}\left(-1\right)^{s}\left(N\right)_{s}a_{1,1}^{(\lambda)}\left(N-s,x\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{N-2}(N)_{N-2}(-1)^2\,2!S_{1,1}(1)\,x\\
&=x\sum_{s=0}^{N-3}(-1)^s(N)_s\,2(-1)^{N-s}(N-s-1)!H_{N-s-1,1}\\
&\quad-2\sum_{s=0}^{N-3}(-1)^s(N)_s\,2(-1)^{N-s-1}(N-s-1)!H_{N-s-1,1}x\\
&\quad+(-1)^{N-2}(N)_{N-2}(-1)^2\,3!x\\
&=6x(-1)^N N!\sum_{s=0}^{N-3}\frac{H_{N-s-1,1}}{N-s}+6x(-1)^N(N)_{N-2}\\
&=6x(-1)^N N!\left\{\frac{H_{N-1,1}}{N}+\cdots+\frac{H_{2,1}}{3}+\frac{H_{1,1}}{2}\right\}\\
&=6x(-1)^N N!H_{N,2}.
\end{aligned}
$$

Therefore, we obtain the following theorem.

Theorem 2.2. *Let $1\le r\le N+1$, $0\le i\le r$. Then we have*

$$a_{i,r-i}^{(\lambda)}(N+1,x)=(-1)^{N+1+i-r}\,i!S_{1,i-1}(r-i)\,N!H_{N,r-1}(x)_{r-i}\,.$$

Proof. We showed that it is true for $r=1$ and $r=N+1$. Assume that $2\le r\le N$. If $i=0$ or $i=r$, then it is also true. So we prove the assertion by induction on r when $2\le r\le N$, $1\le i\le r-1$. For $r=2$, $i=1$, we showed that

$$a_{1,1}^{(\lambda)}(N+1,x)=2(-1)^N N!H_{N,1}x.$$

Assume now that it is true for $r-1\,(3\le r\le N)$.

From 2.19, we note that

$$
\begin{aligned}
a_{i,r-i}^{(\lambda)}(N+1,x)&=(x+i-r+1)\sum_{s=0}^{N-r}(-1)^s(N)_s\,a_{i,r-i-1}^{(\lambda)}(N-s,x)\\
&\quad-i\sum_{s=0}^{N-r}(-1)^s(N)_s\,a_{i-1,r-i}^{(\lambda)}(N-s,x)\\
&\quad+(-1)^{N-r+1}(N)_{N-r+1}(-1)^i\,i!S_{1,i-1}(r-i)(x)_{r-i}\\
&=(x+i-r+1)\sum_{s=0}^{N-r}(-1)^s(N)_s(-1)^{N-s-1+i-r}\,i!\\
&\quad\times S_{1,i-1}(r-1-i)(N-s-1)!H_{N-s-1,r-2}(x)_{r-1-i}\\
&\quad-i\sum_{s=0}^{N-r}(-1)^s(N)_s(-1)^{N-s+i-r}(i-1)!S_{1,i-2}(r-i)\\
&\quad\times(N-s-1)!H_{N-s-1,r-2}(x)_{r-i}\\
&\quad+(-1)^{N-r+1}(N)_{N-r+1}(-1)^i\,i!S_{1,i-1}(r-i)(x)_{r-i}\\
&=(-1)^{N+1+i-r}\,i!N!(x)_{r-i}\\
&\quad\times\left\{(S_{1,i-1}(r-1-i)+S_{1,i-2}(r-i))\sum_{s=0}^{N-r}\frac{H_{N-s-1,r-2}}{N-s}+\frac{S_{1,i-1}(r-i)}{(r-1)!}\right\}.
\end{aligned}
\tag{2.22}
$$

By Lemma 2.1 and (2.22), we get

$$\begin{aligned}a_{i,r-i}^{(\lambda)}(N+1,x) &= (-1)^{N+1+i-r}\, i! S_{1,i-1}(r-i)\, N!\,(x)_{r-i}\\ &\quad\times\left\{\frac{H_{N-1,r-2}}{N}+\cdots+\frac{H_{r-1,r-2}}{r}+\frac{H_{r-2,r-2}}{r-1}\right\}\\ &= (-1)^{N+1+i-r}\, i! S_{1,i-1}(r-i)\, N! H_{N,r-1}(x)_{r-i}.\end{aligned}$$

□

Therefore, we obtain the following theorem.

Theorem 2.3. *The linear differential equations*

$$F^{(N)}=\lambda^N(1+\lambda t)^{-N}\times\left(\sum_{r=1}^{N}\sum_{i=0}^{r}a_{i,r-i}^{(\lambda)}(N,x)\,(2\lambda+\log(1+\lambda t))^{-i}(\lambda+\log(1+\lambda t))^{-(r-i)}\right)F$$

has a solution

$$F=F(t;x,\lambda)=2\lambda(2\lambda+\log(1+\lambda t))^{-1}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^{x},$$

where, for $1\le r\le N$, $0\le i\le r$,

$$a_{i,r-i}^{(\lambda)}(N,x)=(-1)^{N+i-r}\, i! S_{1,i-1}(r-i)\,(N-1)! H_{N-1,r-1}(x)_{r-i}.$$

Recall that the λ-Changhee polynomials, $\mathrm{Ch}_{n,\lambda}(x)$, $(n\ge 0)$, are given by the generating function

$$\begin{aligned}F&=F(t;x,\lambda)\\ &=\frac{2\lambda}{2\lambda+\log(1+\lambda t)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x}\\ &=\sum_{n=0}^{\infty}\mathrm{Ch}_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.23}$$

Thus, by (2.23), we get

$$\begin{aligned}F^{(N)}&=\left(\frac{d}{dt}\right)^{N}F(t;x,\lambda)\\ &=\sum_{k=0}^{\infty}\mathrm{Ch}_{k+N,\lambda}(x)\frac{t^k}{k!}.\end{aligned}\tag{2.24}$$

On the other hand, by Theorem 2.3, we get

$$\begin{aligned}F^{(N)}&=\lambda^N\sum_{l=0}^{\infty}(-1)^l\binom{N+l-1}{l}\lambda^l t^l\sum_{r=1}^{N}\sum_{i=0}^{r}a_{i,r-i}^{(\lambda)}(N,x)\sum_{m=0}^{\infty}(-1)^m\binom{i+m-1}{m}\\ &=(2\lambda)^{-i-m}(\log(1+\lambda t))^m\sum_{n=0}^{\infty}(-1)^n\binom{r+n-i-1}{n}\lambda^{-(r-i)-n}(\log(1+\lambda t))^n\\ &\quad\times\sum_{s=0}^{\infty}\mathrm{Ch}_{s,\lambda}(x)\frac{t^s}{s!}\\ &=\lambda^N\sum_{l=0}^{\infty}(-1)^l(N+l-1)_l\,\lambda^l\frac{t^l}{l!}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{r=1}^{N} \sum_{i=0}^{r} a_{i,N-i}^{(\lambda)}(N,x) \sum_{m=0}^{\infty} (-1)^m (i+m-1)_m (2\lambda)^{-i-m} \\
&\times \sum_{e=0}^{\infty} S_1(e+m,m) \lambda^{e+m} \frac{t^{e+m}}{(e+m)!} \sum_{n=0}^{\infty} (-1)^n (r+n-i-1)_n \\
&\times \lambda^{i-r-n} \sum_{f=0}^{\infty} S_1(f+n,n) \\
&\times \lambda^{f+n} \frac{t^{f+n}}{(f+n)!} \sum_{s=0}^{\infty} \mathrm{Ch}_{s,\lambda}(x) \frac{t^s}{s!} \\
&= \lambda^N \sum_{r=1}^{N} \sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N,x) \sum_{m=0}^{\infty} (-1)^m (i+m-1)_m (2\lambda)^{-i-m} \lambda^m \frac{t^m}{m!} \\
&\times \sum_{n=0}^{\infty} (-1)^n (r+n-i-1)_n \lambda^{i-r-n} \lambda^n \frac{t^n}{n!} \sum_{l=0}^{\infty} (-1)^l (N+l-1)_l \lambda^l \frac{t^l}{l!} \\
&\times \sum_{e=0}^{\infty} S_1(e+m,m) \lambda^e \frac{e!m!}{(e+m)!} \frac{t^e}{e!} \sum_{f=0}^{\infty} S_1(f+n,n) \lambda^f \frac{f!n!}{(f+n)!} \frac{t^f}{f!} \\
&\times \sum_{s=0}^{\infty} \mathrm{Ch}_{s,\lambda}(x) \frac{t^s}{s!} \\
&= \lambda^N \sum_{r=1}^{N} \sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N,x) \sum_{m=0}^{\infty} (-1)^m \\
&\times (i+m-1)_m (2\lambda)^{-i-m} \lambda^m \frac{t^m}{m!} \sum_{n=0}^{\infty} (-1)^n (r+n-i-1)_n \\
&\times \lambda^{i-r-n} \lambda^n \frac{t^n}{n!} \sum_{a=0}^{\infty} \left(\sum_{l+e+f+s=a} (-1)^l \lambda^{a-s} \frac{\binom{a}{l,e,f,s}}{\binom{e+m}{m}\binom{f+n}{n}} (N+l-1)_l \right. \\
&\left. S_1(e+m,m) S_1(f+n,n) \mathrm{Ch}_{s,\lambda}(x) \right) \frac{t^a}{a!} \\
&= \lambda^N \sum_{r=1}^{N} \sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N,x) \lambda^{-r} 2^{-i} \sum_{k=0}^{\infty} \left(\sum_{m+n+a=k} \binom{k}{m,n,a} \left(-\frac{1}{2}\right)^m (-1)^n (i+m-1)_m \right. \\
&\times (r+n-i-1)_n \sum_{l+e+f+s=a} (-1)^l \lambda^{a-s} \frac{\binom{a}{l,e,f,s,}}{\binom{e+m}{m}\binom{f+n}{n}} \\
&\left. \times (N+l-1)_l S_1(e+m,m) \, S_1(f+n,n) \mathrm{Ch}_{s,\lambda}(x) \right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left\{ \lambda^N \sum_{r=1}^{N} \sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N,x) \lambda^{-r} 2^{-i} \right. \\
&\times \sum_{m+n+a=k} \binom{k}{m,n,a} \left(-\frac{1}{2}\right)^m (-1)^n (i+m-1)_m (r+n-i-1)_n \\
&\times \sum_{l+e+f+s=a} (-1)^l \lambda^{a-s} \frac{\binom{a}{l,e,f,s}}{\binom{e+m}{m}\binom{f+n}{n}} (N+l-1)_l \\
&\left. \times S_1(e+m,m) S_1(f+n,n) \mathrm{Ch}_{s,\lambda}(x) \right\} \frac{t^k}{k!}.
\end{aligned}
\tag{2.25}
$$

Therefore, by comparing the coefficients on both sides of (2.24) and (2.25), we obtain the following theorem.

Theorem 2.4. *For $N \in \mathbb{N}$, and $k \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}\mathrm{Ch}_{k+N,\lambda}(x) = \lambda^N \sum_{r=1}^{N} \sum_{i=0}^{r} a_{i,r-i}^{(\lambda)}(N,x)\, \lambda^{-r} 2^{-i} &\sum_{m+n+a=k} \binom{k}{m,n,a} \\ \times \left(-\frac{1}{2}\right)^m &(-1)^n (i+m-1)_m (r+n-i-1)_n \\ \times \sum_{l+e+f+s=a} &(-1)^l \lambda^{a-s} \frac{\binom{a}{l,e,f,s}}{\binom{e+m}{m}\binom{f+n}{n}} \\ \times (N+l-1)_l &S_1(e+m,m) S_1(f+n,n) \mathrm{Ch}_{s,\lambda}(x),\end{aligned}$$

where, for $1 \le r \le N$, $0 \le i \le r$,

$$a_{i,r-i}^{(\lambda)}(N,x) = (-1)^{N+i-r} i! S_{1,i-1}(r-i)(N-1)! H_{N-1,r-1}(x)_{r-i}.$$

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