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Bifurcation analysis in a discrete SIR epidemic model with the saturated contact rate and vertical transmission

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Abstract

The aim of paper is dealing with the dynamical behaviors of a discrete SIR epidemic model with the saturated contact rate and vertical transmission. More precisely, we investigate the local stability of equilibriums, the existence, stability and direction of flip bifurcation and Neimark-Sacker bifurcation of the model by using the center manifold theory and normal form method. Finally, the numerical simulations are provided for justifying the validity of the theoretical analysis.

Keywords: Discrete SIR epidemic model, stability, flip bifurcation, Neimark-Sacker bifurcation.
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1. Introduction

Epidemic is caused by the pathogen, which can be spread from human to human, human to animals and animals to animals. Because it can make a range of biological reduce or lose labor, death and spread rapidly in a certain period of time. Therefore, it has caused great attention of scientists and mathematicians. As early as 1927, Kermark and Mckendrick were established the mathematical model of infectious diseases by using the method of dynamics, and constructed the famous SIR bin model [16]. After the middle of the 20th century, the dynamics of infectious disease has been gotten rapid development, and the epidemic

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dynamical models have been widely investigated [2–4, 19]. Hu et al. [14] studied the dynamical behaviors of a discrete predator prey system with nonmonotonic functional response. They obtained the local stability of equilibria of the model, and proved that the model undergoes the flip bifurcation and Hopf bifurcation. He et al. [11] investigated the dynamics of a discrete-time predator prey system, and they proved that the system undergoes the flip bifurcation and Neimark-Sacker bifurcation by using a center manifold theorem and bifurcation theory. The dynamical behaviors of a class of discrete-time SIRS epidemic models are discussed in literature [13], and they obtained the conditions for the existence and local stability of the disease-free equilibrium and endemic equilibrium. Ghaziani et al. [9] studied the resonance and bifurcation in a discrete-time predator-prey system with Holling functional response. Chen et al. [7] applied the forward Euler method to the ratio-dependent predator-prey model, and then investigated the dynamical behaviors of its discrete system by using the center manifold theorem. Elabbasy et al. [8] derived the existence and stability of the fixed points of a discrete reduced Lorenz system by using the center manifold theorem and bifurcation theory. Zhang et al. [22] investigated the dynamical behaviors of the discrete-time predator-prey biological economic system by using the new normal form of differential-algebraic system. Zhao et al. [23] focused on a reaction-diffusion neural network with delays and studied the stability and bifurcation of the networks. Wang and Li [20] revisited a discrete predator-prey model with a non-monotonic functional response and presented a very useful lemma which is a corrected version of a known result, and a key tool to study the local stability and bifurcation of the system.

Although there are many scholars mainly investigated the nonlinear dynamic characteristics of continuous system, but the research about discrete systems is relatively few. Compared with the continuous system, the discrete systems possess their unique dynamic characteristics. And in the real life, many practical problems can be depicted by the discrete systems, we can also discrete the continuous systems. Therefore, the study of discrete system is very important and achieved great development in the field of mathematics, physics and engineering. And due to many infectious diseases' data are collected based on day, week, month or year. So, we need to establish the discrete model for research. At this time, the discrete model is more conform to actual than the continuous model. In recent years, the research and application on discrete infectious disease model has become a very popular topic [1, 5, 6]. Yi et al. [21] obtained a discrete epidemic model with nonlinear incidence rate by using the forward Euler method, and derived the conditions for existence of codimension-1 bifurcations and codimension-2 bifurcation. Li [17] studied a discrete food-limited population model with time delay, and by choosing the time delay as the bifurcation parameter, he proved that Neimark-Sacker bifurcations occur when the delay passes a sequence of critical values. Hu et al. [15] investigated the flip bifurcation and Hopf bifurcation of a discrete-time SIR epidemic model by using the center manifold theorem and bifurcation theory. Wang et al. [20] formulated an easily verified and complete discrimination criterion for the local stability of the two equilibria of a discrete predator-prey model with a non-monotonic functional response, and then studied the stability and bifurcation for the equilibrium point. Hu et al. [12] discussed the globally stability of a SIR epidemic model with the saturated contact rate and vertical transmission by using V function and Dulac function.

We consider the following continuous-time SIR epidemic model with the saturated contact rate and vertical transmission:

$$\begin{cases} \dfrac{dS}{dt} = \mu(1-S) - \dfrac{\beta SI}{1+\alpha S} - \mu(1-\delta)I, \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{1+\alpha S} - \gamma I - \mu\delta I, \\ \dfrac{dR}{dt} = \gamma I - \mu R, \\ N(t) = S(t) + I(t) + R(t), \end{cases} \tag{1.1}$$

where $S(t), I(t)$ and $R(t)$ denote the numbers of susceptible, infective, recovered individuals and total numbers of the individuals at time t, respectively. $\mu > 0$ is the birth rate (death rate) of the population, $\frac{\beta SI}{1+\alpha S}(\beta > 0, \alpha \geq 0)$ is the saturated contact rate, $\gamma > 0$ is the recovery rate of the infective individuals, $1-\delta(0 \leq \delta \leq 1)$ is the proportion of vertical transmission.

Applying the forward Euler scheme to model (1.1), we obtained the following discrete-time SIR epidemic model with the saturated contact rate and vertical transmission:

$$\begin{cases} S_{n+1} = S_n + h[\mu(1-S_n) - \dfrac{\beta S_n I_n}{1+\alpha S_n} - \mu(1-\delta)I_n], \\ I_{n+1} = I_n + h[\dfrac{\beta S_n I_n}{1+\alpha S_n} - \gamma I_n - \mu\delta I_n], \\ R_{n+1} = R_n + h[\gamma I_n - \mu R_n], \\ N_{n+1} = h\mu + (1-h\mu)N_n, \end{cases} \tag{1.2}$$

where h is the step size. $\mu, \alpha, \beta, \gamma$ and δ are defined as model (1.1). Assume that $S(0) > 0, I(0) \geq 0, R(0) \geq 0$, and all the parameters are positive. In this paper, we mainly investigate the dynamical behaviors of model (1.2), such as stability of the equilibrium, flip bifurcation, Neimark-Sacker bifurcation and chaos phenomenon.

Due to the first two equations of model (1.2) is about (S_n, I_n) not including R_n, and the third equation is the linear equation of R_n. Therefore, the dynamical behaviors of model (1.2) is equivalent to the dynamical behaviors of the following model:

$$\begin{cases} S_{n+1} = S_n + h[\mu(1-S_n) - \dfrac{\beta S_n I_n}{1+\alpha S_n} - \mu(1-\delta)I_n], \\ I_{n+1} = I_n + h[\dfrac{\beta S_n I_n}{1+\alpha S_n} - \gamma I_n - \mu\delta I_n]. \end{cases} \tag{1.3}$$

In this paper, we study the dynamical behaviors of model (1.3), such as local stability of equilibria, flip bifurcation and Neimark-Sacker bifurcation.

The paper is organized as follows. We discuss the existence and local stability of equilibria in model (1.3) in Section 2. In Section 3, we study the flip bifurcation and Neimark-Sacker bifurcation for model (1.3) by choosing as a bifurcation parameter. We present the numerical simulations illustrating our results with the theoretical analysis in Section 4. In Section 5, conclusion of the paper is given.

2. Existence and stability of equilibria

In this section, we discuss the existence and stability of the equilibria of model (1.3). The basic reproductive rate of model (1.3) is $R_0 = \frac{\beta}{(1+\alpha)(\gamma+\mu\delta)}$ which is the average number of secondary infections generated by an initial population of infected individuals over their lifetimes. Through a simple calculation, we can get the following results.

Proposition 2.1.

(1) *If the basic reproductive rate $R_0 < 1$, then model (1.3) has only a disease-free equilibrium $E_1 = (1, 0)$;*

(2) *if the basic reproductive rate $R_0 > 1$, then model (1.3) has two equilibria: disease-free equilibrium $E_1 = (1, 0)$ and endemic equilibrium $E_2 = (S^*, I^*)$, where*

$$S^* = \frac{\gamma+\mu\delta}{\beta-\alpha(\gamma+\mu\delta)}, \qquad I^* = \frac{\mu}{\mu+\gamma}(1 - \frac{\gamma+\mu\delta}{\beta-\alpha(\gamma+\mu\delta)}).$$

Lemma 2.2 ([21]). *Let $F(\lambda) = \lambda^2 + B\lambda + C$. Suppose that $F(1) > 0$ and λ_1, λ_2 be the two roots of $F(\lambda) = 0$, then*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;

(ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) if and only if* $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$*;*

(iv) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $B \neq 0, 2$*;*

(v) λ_1 *and* λ_2 *are complex numbers with* $|\lambda_1| = 1$ *and* $|\lambda_2| = 1$ *if and only if* $B^2 - 4C < 0, 2$ *and* $C = 1$.

Let λ_1 and λ_2 be two roots of the characteristic equation of Jacobian matrix J. The fixed point (x, y) is called a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and the sink is locally asymptotic stable. (x, y) is called a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and the source is locally unstable. (x, y) is called a *saddle* if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). And (x, y) is called *non-hyperbolic* if either $|\lambda_1| = 1$ and $|\lambda_2| = 1$.

Now, we study the local stability of equilibria E_1 and E_2.

Theorem 2.3. *If the basic reproductive rate* $R_0 < 1$*, then we have*

(1) $E_1 = (1, 0)$ *is asymptotically stable if* $0 < h < min\{\frac{2}{\mu}, \frac{2(1+\alpha)}{(1+\alpha)(\gamma-\mu\delta)-\beta}\}$*;*

(2) $E_1 = (1, 0)$ *is unstable if* $h > max\{\frac{2}{\mu}, \frac{2(1+\alpha)}{(1+\alpha)(\gamma-\mu\delta)-\beta}\}$ *or* $\frac{2(1+\alpha)}{(1+\alpha)(\gamma-\mu\delta)-\beta} < h < \frac{2}{\mu}$, *or* $\frac{2}{\mu} < h < \frac{2(1+\alpha)}{(1+\alpha)(\gamma-\mu\delta)-\beta}$*;*

(3) $E_1 = (1, 0)$ *is non-hyperbolic if* $h = \frac{2}{\mu}$ *or* $h = \frac{2(1+\alpha)}{(1+\alpha)(\gamma-\mu\delta)-\beta}$.

Proof. The Jacobian matrix of model (1.3) evaluated at equilibrium $E_1 = (1, 0)$ is

$$J(E_1) = \begin{pmatrix} 1 - h\mu & h[-\frac{\beta}{1+\alpha} - \mu(1-\delta)] \\ 0 & 1 + h[\frac{\beta}{1+\alpha} - \gamma - \mu\delta] \end{pmatrix}.$$

The $J(E_1)$ has the following two eigenvalues:

$$\lambda_1 = 1 - h\mu, \qquad \lambda_2 = 1 + h[\frac{\beta}{1+\alpha} - \gamma - \mu\delta].$$

According to Lemma 2.2, we know Theorem 2.3 follows. □

Theorem 2.4. *If the basic reproductive rate* $R_0 > 1$*, then we have*

(1) $E_2 = (S^*, I^*)$ *is asymptotically stable if one of the following conditions holds:*

(a) $\xi_1^2 \geq 4\xi_2, \quad \xi_1 < -\Theta_1, \quad 0 < h < \Theta_2$;

(b) $\xi_1^2 < 4\xi_2, \quad \xi_1\xi_2 < 0, \quad 0 < h < -(\xi_1/\xi_2)$;

(2) $E_2 = (S^*, I^*)$ *is unstable if one of the following conditions holds:*

(a) $\xi_1^2 \geq 4\xi_2, \quad \xi_1 > \Theta_1$ *or* $\xi_1 < -\Theta_1$ *and* $h > \Theta_2$;

(b) $\xi_1^2 < 4\xi_2, \quad \xi_2 \neq 0, \quad h > -(\xi_1/\xi_2)$;

(c) $\xi_1^2 \geq 4\xi_2, \quad \xi_1 > \Theta_1, \quad \Theta_2 < h < \Theta_3$;

(3) $E_2 = (S^*, I^*)$ *is non-hyperbolic if one of the following conditions holds:*

(a) $\xi_1^2 \geq 4\xi_2, \quad \xi_2 \neq 0, \quad \xi_1 = -\Theta_1$ *or* $h = \Theta_2, \quad \xi_1 = \Theta_1$ *or* $h = \Theta_3$;

(b) $\xi_1^2 < 4\xi_2, \quad h = -(\xi_1/\xi_2)$, *where*

$$\Theta_1 = \sqrt{\xi_1^2 - 4\xi_2}, \quad \Theta_2 = \frac{-\xi_1 + \sqrt{\xi_1^2 - 4\xi_2}}{\xi_2}, \quad \Theta_3 = \frac{-\xi_1 - \sqrt{\xi_1^2 - 4\xi_2}}{\xi_2}.$$

Proof. The Jacobian matrix of model (1.3) at equilibrium $E_2 = (S^*, I^*)$ is

$$J(E_2) = \begin{pmatrix} 1 + h(\frac{\beta S^* I^*}{(1+\alpha S^*)^2} - \frac{\beta I^*}{1+\alpha S^*} - \mu) & h(\mu\delta - \frac{\beta S^*}{1+\alpha S^*} - \mu) \\ h(\frac{\beta I^*}{1+\alpha S^*} - \frac{\beta S^* I^*}{(1+\alpha S^*)^2}) & 1 + h(\frac{\beta S^*}{1+\alpha S^*} - \gamma - \mu\delta) \end{pmatrix}.$$

The characteristic polynomial of the Jacobian matrix of model (1.3) evaluated at $E_2 = (S^*, I^*)$ has the form

$$f(\lambda) = \lambda^2 - (2 + h\xi_1)\lambda + 1 + h\xi_1 + h^2\xi_2,$$

where

$$\xi_1 = \frac{\beta S^* I^*}{(1+\alpha S^*)^2} - \frac{\beta(S^* - I^*)}{1+\alpha S^*} - \mu\delta - \mu - \gamma,$$
$$\xi_2 = \frac{\beta(\mu+\gamma)}{1+\alpha S^*}(1 - \frac{S^*}{1+\alpha S^*}) - \frac{\mu\beta S^*}{1+\alpha S^*} + \mu(\gamma + \mu\delta).$$

So, the eigenvalues of $J(E_2)$ are:

$$\lambda_{1,2} = 1 + \frac{h}{2}(\xi_1 \pm \sqrt{\xi_1^2 - 4\xi_2}).$$

According to Lemma 2.2, we can get the conclusions of Theorem 2.4. □

3. Bifurcation analysis

The endemic equilibrium $E_2 = (S^*, I^*)$ is more meaningful in SIR epidemic model with the saturated contact rate and vertical transmission. So, we will study the local stability of the equilibrium E_2. We choose the step size h as the bifurcation parameter to investigate the flip bifurcation and Neimark-Sacker bifurcation of the model (1.3) by using the bifurcation theory and the center manifold theorem.

3.1. Flip bifurcation

Now, we use the center manifold theorem and Theorem 2.4 to analyze the flip bifurcation of the model (1.3). Define the following parameter sets:

$$S_1 = \{(h,\mu,\alpha,\beta,\delta,\gamma) | h = \Theta_2,\ R_0 > 1,\ \xi_1^2 \geq 4\xi_2,\ \xi_1 < -\Theta_1,\ \xi_2 \neq 0,$$
$$\xi_1^2 - \xi_1\Theta_1 \neq 2\xi_2, 4\xi_2,\ h,\mu,\alpha,\beta,\delta,\gamma > 0\},$$

$$S_2 = \{(h,\mu,\alpha,\beta,\delta,\gamma) | h = \Theta_3,\ R_0 > 1,\ \xi_1^2 \geq 4\xi_2,\ \xi_1 > \Theta_1 \text{ or } \xi_1 < -\Theta_1,$$
$$\xi_1^2 - \xi_1\Theta_1 \neq 2\xi_2, 4\xi_2,\ h,\mu,\alpha,\beta,\delta,\gamma > 0\}.$$

We discuss the flip bifurcation of model (1.3) at $E_2 = (S^*, I^*)$ when h varies in the small neighborhood of h^* and $(h,\mu,\alpha,\beta,\delta,\gamma) \in S_1$. And the similar arguments can be applied to the other case $(h,\mu,\alpha,\beta,\delta,\gamma) \in S_2$.

Theorem 3.1. *If $\alpha_2 \neq 0$, then the model (1.3) undergoes a flip bifurcation at $E_2 = (S^*, I^*)$ when the parameter h^* changes in the small neighborhood. Moreover, if $\alpha_2 > 0$ (respectively, $\alpha_2 < 0$), then period-2 points that bifurcate from $E_2 = (S^*, I^*)$ are stable (respectively, unstable).*

Proof. When the parameters $(h,\mu,\alpha,\beta,\delta,\gamma) \in S_1$, the model (1.3) can be described as follows:

$$\begin{cases} S_{n+1} = S_n + h_1[\mu(1 - S_n) - \dfrac{\beta S_n I_n}{1+\alpha S_n} - \mu(1-\delta)I_n], \\ I_{n+1} = I_n + h_1[\dfrac{\beta S_n I_n}{1+\alpha S_n} - \gamma I_n - \mu\delta I_n]. \end{cases} \tag{3.1}$$

Obviously, the model (3.1) exists the equilibrium $E_2 = (S^*, I^*)$. The corresponding eigenvalues of model (3.1) at E_2 are $\lambda_1 = -1,\ \lambda_2 = 3 + \xi_1\Theta_2$, and $|\lambda_2| \neq 1$.

Because $(h, \mu, \alpha, \beta, \delta, \gamma) \in S_1$ and $h_1 = \Theta_2$, we choose h^* as the bifurcation parameter, and consider a perturbation of model (3.1) as follows:

$$\begin{cases} S_{n+1} = S_n + (h_1 + h^*)[\mu(1 - S_n) - \dfrac{\beta S_n I_n}{1 + \alpha S_n} - \mu(1 - \delta)I_n], \\ I_{n+1} = I_n + (h_1 + h^*)[\dfrac{\beta S_n I_n}{1 + \alpha S_n} - \gamma I_n - \mu\delta I_n], \end{cases}$$

where $|h^*| \ll 1$, which is a small perturbation parameter.

Let $u_n = S_n - S^*, v_n = I_n - I^*$, then we transform equilibrium $E_2 = (S^*, I^*)$ into the origin $(0, 0)$. Through the complex calculation, we obtain

$$\begin{cases} u_{n+1} = a_{11}u_n + a_{12}v_n + a_{13}u_n^2 + a_{14}u_n v_n + b_{11}v_n h^* + b_{12}v_n h^* + c_{11}u_n^3 + c_{12}u_n^2 v_n \\ \qquad d_{11}u_n^2 h^* + d_{12}u_n v_n h^* + e_1 h^* + O((|u_n| + |v_n| + |h^*|)^4), \\ v_{n+1} = a_{21}u_n + a_{22}v_n + a_{23}u_n^2 + a_{24}u_n v_n + b_{21}v_n h^* + b_{22}v_n h^* + c_{21}u_n^3 + c_{22}u_n^2 v_n \\ \qquad d_{21}u_n^2 h^* + d_{22}u_n v_n h^* + e_2 h^* + O((|u_n| + |v_n| + |h^*|)^4), \end{cases} \tag{3.2}$$

where

$$a_{11} = 1 - h\mu - \frac{h\beta I^*}{1 + \alpha S^*} + \frac{h\alpha\beta S^* I^*}{(1 + \alpha S^*)^2}, \quad a_{12} = -h\mu(1 - \delta) - \frac{h\beta S^*}{1 + \alpha S^*}, \quad a_{13} = \frac{h\alpha\beta I^*}{(1 + \alpha S^*)^3},$$

$$a_{14} = \frac{h\alpha\beta S^*}{(1 + \alpha S^*)^2} - \frac{h\beta}{1 + \alpha S^*}, \quad b_{11} = -\mu - \frac{\beta I^*}{1 + \alpha S^*} - \frac{\alpha\beta S^* I^*}{(1 + \alpha S^*)^2}, \quad b_{12} = -\mu(1 - \delta) - \frac{\beta S^*}{1 + \alpha S^*},$$

$$c_{11} = -\frac{h\alpha^2\beta I^*}{(1 + \alpha S^*)^4}, \quad c_{12} = -\frac{h\alpha\beta}{(1 + \alpha S^*)^3}, \quad d_{11} = \frac{\alpha\beta I^*}{(1 + \alpha S^*)^3}, \quad d_{12} = \frac{\alpha\beta S^*}{(1 + \alpha S^*)^2} - \frac{\beta}{1 + \alpha S^*},$$

$$e_1 = \mu - \frac{\beta S^* I^*}{(1 + \alpha S^*)} - \mu S^* - \mu(1 - \delta)I^*, \quad a_{21} = \frac{h\beta I^*}{1 + \alpha S^*} + \frac{h\alpha\beta S^* I^*}{(1 + \alpha S^*)^2},$$

$$a_{22} = 1 - h(\gamma + \mu\delta - \frac{\beta S^*}{1 + \alpha S^*}), \quad a_{23} = -\frac{h\alpha\beta I^*}{(1 + \alpha S^*)^3}, \quad a_{24} = \frac{h\beta}{1 + \alpha S^*} - \frac{h\alpha\beta S^*}{(1 + \alpha S^*)^2},$$

$$b_{21} = \frac{\beta I^*}{1 + \alpha S^*} - \frac{\alpha\beta S^* I^*}{(1 + \alpha S^*)^2}, \quad b_{22} = -\gamma - \mu\delta - \frac{\beta S^*}{1 + \alpha S^*}, \quad c_{21} = -\frac{h\alpha^2\beta I^*}{(1 + \alpha S^*)^4}, \quad c_{22} = -\frac{h\alpha\beta}{(1 + \alpha S^*)^3},$$

$$d_{21} = -\frac{\alpha\beta I^*}{(1 + \alpha S^*)^3}, \quad d_{22} = \frac{\beta}{1 + \alpha S^*} - \frac{\alpha\beta S^*}{(1 + \alpha S^*)^2}, \quad e_2 = \frac{\beta S^* I^*}{(1 + \alpha S^*)} - \gamma I^* - \mu\delta I^*.$$

Define the invertible matrix T as follows:

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}.$$

Using the following translation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = T \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

then the model (3.2) becomes into the following form

$$\begin{cases} X_{n+1} = -X_n + F(u_n, v_n, h^*), \\ Y_{n+1} = -Y_n + G(u_n, v_n, h^*), \end{cases} \tag{3.3}$$

where

$$F(u_n, v_n, h^*) = \frac{(\lambda_2 - a_{11})a_{13} - a_{12}a_{23}}{a_{12}(\lambda_2 + 1)}u_n^2 + \frac{(\lambda_2 - a_{11})a_{14} - a_{12}a_{24}}{a_{12}(\lambda_2 + 1)}u_n v_n + \frac{(\lambda_2 - a_{11})b_{11} - a_{12}b_{21}}{a_{12}(\lambda_2 + 1)}u_n h^*$$

$$
\begin{aligned}
&+\frac{(\lambda_2-a_{11})b_{12}-a_{12}b_{22}}{a_{12}(\lambda_2+1)}v_nh^*+\frac{(\lambda_2-a_{11})c_{11}-a_{12}c_{21}}{a_{12}(\lambda_2+1)}u_n^3+\frac{(\lambda_2-a_{11})c_{12}-a_{12}c_{22}}{a_{12}(\lambda_2+1)}u_n^2v_n\\
&+\frac{(\lambda_2-a_{11})d_{11}-a_{12}d_{21}}{a_{12}(\lambda_2+1)}u_n^2h^*+\frac{(\lambda_2-a_{11})d_{12}-a_{12}d_{22}}{a_{12}(\lambda_2+1)}u_nv_nh^*+\frac{(\lambda_2-a_{11})e_1-a_{12}e_2}{a_{12}(\lambda_2+1)}h^*\\
&+O((|u_n|+|v_n|+|h^*|)^4),
\end{aligned}
$$

$$
\begin{aligned}
G(u_n,v_n,h^*)=&\frac{(1+a_{11})a_{13}+a_{12}a_{23}}{a_{12}(\lambda_2+1)}u_n^2+\frac{(1+a_{11})a_{14}+a_{12}a_{24}}{a_{12}(\lambda_2+1)}u_nv_n+\frac{(1+a_{11})b_{11}+a_{12}b_{21}}{a_{12}(\lambda_2+1)}u_nh^*\\
&+\frac{(1+a_{11})b_{12}+a_{12}b_{22}}{a_{12}(\lambda_2+1)}v_nh^*+\frac{(1+a_{11})c_{11}+a_{12}c_{21}}{a_{12}(\lambda_2+1)}u_n^3+\frac{(1+a_{11})c_{12}+a_{12}c_{22}}{a_{12}(\lambda_2+1)}u_n^2v_n\\
&+\frac{(1+a_{11})d_{11}+a_{12}d_{21}}{a_{12}(\lambda_2+1)}u_n^2h^*+\frac{(1+a_{11})d_{12}+a_{12}d_{22}}{a_{12}(\lambda_2+1)}u_nv_nh^*+\frac{(1+a_{11})e_2-a_{12}e_1}{a_{12}(\lambda_2+1)}h^*\\
&+O((|u_n|+|v_n|+|h^*|)^4),
\end{aligned}
$$

and

$$
\begin{aligned}
u_n&=a_{12}(X_n+Y_n),\quad v_n=-(1+a_{11})X_n+(\lambda_2-a_{11})Y_n,\\
u_nv_n&=a_{12}[-(1+a_{11})X_n^2+(\lambda_2-1-2a_{11})X_nY_n+(\lambda_2-a_{11})Y_n^2],\\
u_n^2&=a_{12}^2(X_n^2+2X_nY_n+Y_n^2),\quad u_n^3=a_{12}^3(X_n^3+3X_n^2Y_n+3X_nY_n^2+Y_n^3),\\
u_n^2v_n&=-a_{12}^3(1+a_{11})X_n^3+(a_{12}^2(\lambda_2-a_{11})-2a_{12}^2(1+a_{11}))X_n^2Y_n\\
&\quad+(2a_{12}^2(\lambda_2-a_{11})-a_{12}^2(1+a_{11}))X_nY_n^2+a_{12}^2(\lambda_2-a_{11})Y_n^3.
\end{aligned}
$$

Based on the center manifold theorem, we can get the following approximate representation of the center manifold $W^c(0,0)$ of model (3.3) at the origin in a small neighborhood of $h^*=0$.

$$
W^c(0,0)=\{(X_n,Y_n)\in R^2|Y_n=a_0h^*+a_1X_n^2+a_2X_nh^*+a_3h^*+O((|u_n|+|v_n|+|h^*|)^3)\},
$$

and

$$
\begin{aligned}
a_0&=\frac{e_1a_{12}+e_2(1+a_{11})}{a_{12}(1-\lambda_2^2)},\quad a_1=\frac{a_{12}[(1+a_{11})a_{13}+a_{12}a_{23}]-(1+a_{11})[(1+a_{11})a_{14}+a_{12}a_{24}]}{1-\lambda_2^2},\\
a_2&=\frac{(1+a_{11})[(1+a_{11})b_{12}+a_{12}b_{22}]-a_{12}[(1+a_{11})b_{11}+a_{12}b_{21}]+2a_1[e_1(\lambda_2-a_{11})-a_{12}e_2]}{a_{12}(1-\lambda_2^2)^2},\\
a_3&=\frac{a_2[e_1(\lambda_2-a_{11})-a_{12}e_2]}{a_{12}(1-\lambda_2^2)}-\frac{a_1[e_1(\lambda_2-a_{11})-a_{12}e_2]^2}{a_{12}^2(1-\lambda_2^2)^2(1-\lambda_2)}.
\end{aligned}
$$

We can restrict the model (3.3) to the center manifold $W^c(0,0)$, and obtain

$$
\begin{aligned}
X_{n+1}=&-X_n+h_1X_n^2+h_2h^*+h_3X_nh^*+h_4h^{*2}+h_5X_n^2h^*+h_6X_nh^{*2}\\
&+h_7X_n^3+h_8h^{*3}+O((|X_n|+|h^*|)^4),
\end{aligned}
$$

where

$$
h_1=\frac{a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]-(1+a_{11})[a_{14}(\lambda_2-a_{11})-a_{14}a_{24}]}{1-\lambda_2},
$$

$$
h_2=\frac{e_1(\lambda_2-a_{11})-a_{12}e_2}{a_{12}(1+\lambda_2)},
$$

$$
h_3=\frac{2a_0a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]+a_0(\lambda_2-1-2a_{11})[a_{13}(\lambda_2-a_{11})-a_{12}a_{24}]}{1+\lambda_2}
$$

$$+\frac{a_{12}[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]-(1+a_{11})[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}]}{a_{12}(1+\lambda_2)},$$

$$\begin{aligned}h_4=&\frac{a_0^2a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]+a_0^2(\lambda_2-a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]}{1+\lambda_2}\\&+\frac{a_0a_{12}[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]-a_0(\lambda_2-a_{11})[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}]}{a_{12}(1+\lambda_2)},\end{aligned}$$

$$\begin{aligned}h_5=&\frac{2a_{12}(a_2+a_0a_1)[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]}{1+\lambda_2}-\frac{(1+a_{11})[d_{12}(\lambda_2-a_{11})-a_{12}d_{22}]}{1+\lambda_2}\\&+\frac{a_{12}[d_{11}(\lambda_2-a_{11})-a_{12}d_{21}]}{1+\lambda_2}+\frac{3a_0a_{12}^2[c_{11}(\lambda_2-a_{11})-a_{12}c_{21}]}{1+\lambda_2}\\&+\frac{(3\lambda_2-1-4a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]}{1+\lambda_2}+\frac{a_1[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]}{1+\lambda_2}\\&+\frac{a_0a_{12}(\lambda_2-1-3a_{11})[c_{12}(\lambda_2-a_{11})-a_{12}c_{22}]}{1+\lambda_2}+\frac{a_1(\lambda_2-a_{11})[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}]}{a_{12}(1+\lambda_2)},\end{aligned}$$

$$\begin{aligned}h_6=&\frac{2a_{12}(a_3+2a_0a_1)[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]}{1+\lambda_2}+\frac{a_1(\lambda_2-a_{11})[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}]}{a_{12}(1+\lambda_2)}\\&+\frac{a_2[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]}{1+\lambda_2}\frac{a_0(\lambda_2-1-2a_{11})[d_{12}(\lambda_2-a_{11})-a_{12}d_{22}]}{1+\lambda_2}\\&+\frac{a_0^2a_{12}(2\lambda_2-1-3a_{11})[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}]}{1+\lambda_2}+\frac{2a_0a_{12}[d_{11}(\lambda_2-a_{11})-a_{12}d_{21}]}{1+\lambda_2}\\&+\frac{[4a_0a_2(\lambda_2-a_{11})+\lambda_2-1-2a_{11}][a_4(\lambda_2-a_{11})-a_{12}a_{24}]}{1+\lambda_2},\end{aligned}$$

$$\begin{aligned}h_7=&\frac{2a_1a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]}{1+\lambda_2}+\frac{a_1(\lambda_2-1-2a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]}{1+\lambda_2}\\&+\frac{a_{12}^2[c_{11}(\lambda_2-a_{11})-a_{12}c_{21}]}{1+\lambda_2}+\frac{a_{12}^2(1+a_{11})[c_{12}(\lambda_2-a_{11})-a_{12}c_{22}]}{a_{12}(1+\lambda_2)},\end{aligned}$$

$$\begin{aligned}h_8=&\frac{2a_0a_2a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]}{1+\lambda_2}+\frac{2a_0a_2(\lambda_2-a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]}{a_{12}(1+\lambda_2)}\\&+\frac{a_3[b_{11}(\lambda_2-a_{11})-a_{12}b_{21}]}{1+\lambda_2}+\frac{a_0^2a_{12}[d_{11}(\lambda_2-a_{11})-a_{12}d_{21}]}{1+\lambda_2}\\&+\frac{a_0^2(\lambda_2-a_{11})[d_{12}(\lambda_2-a_{11})-a_{12}d_{22}]}{(1+\lambda_2)}+\frac{a_0^3a_{12}^2[c_{11}(\lambda_2-a_{11})-a_{12}c_{21}]}{1+\lambda_2}\\&+\frac{(a_3+a_0^3a_{12}^2)(\lambda_2-a_{11})[b_{12}(\lambda_2-a_{11})-a_{12}b_{22}]}{a_{12}(1+\lambda_2)}.\end{aligned}$$

Therefore, we define the map G^* with model (3.3) restricted to the center manifold $W^c(0,0)$ has the following form:

$$\begin{aligned}G^*(X_n)=&-X_n+h_1X_n^2+h_2h^*+h_3X_nh^*+h_4h^{*2}+h_5X_n^2h^*+h_6X_nh^{*2}\\&+h_7X_n^3+h_8h^{*3}+O((|X_n|+|h^*|)^4).\end{aligned}$$

In order to undergo a flip bifurcation, we require that two discriminatory quantities α_1 and α_2 are not equal to zero, where

$$\begin{cases} \alpha_1 = (\dfrac{\partial^2 G^*(X_n)}{\partial X_n \partial h^*} + \dfrac{1}{2}\dfrac{\partial G^*(X_n)}{\partial h^*}\dfrac{\partial^2 G^*(X_n)}{\partial X_n^2}) \mid_{0,0} = h_3 + h_1 h_2, \\ \alpha_2 = (\dfrac{1}{6}\dfrac{\partial^3 G^*(X_n)}{\partial X_n^3} + (\dfrac{\partial^2 G^*(X_n)}{\partial X_n^2})^2) \mid_{0,0} = h_7 + h_1^2. \end{cases}$$

□

From the above analysis and the theorem in [18], we proved the accuracy of the Theorem 3.1. In Section 4 we will choose some values of parameters $\mu, \alpha, \beta, \delta$ and γ to show the process of flip bifurcation of model (1.3).

3.2. Neimark-Sacker bifurcation

Now, we discuss the Neimark-Sacker bifurcation of $E_2 = (S^*, I^*)$ if h varies in the small neighborhood of N, and the parameter set N defined as follows:

$$N = \{(h, \mu, \alpha, \beta, \delta, \gamma) | h = -\frac{\xi_1}{\xi_2}, \quad R_0 > 1, \quad \xi_1^2 < 4\xi_2, \quad \xi_2 \neq 0, \quad h, \mu, \alpha, \beta, \delta, \gamma > 0\}.$$

Theorem 3.2. *If $l > 0, \xi_1^2 \neq 2\xi_2, 3\xi_2$ and $a \neq 0$, then model (3.5) undergoes Neimark-Sacker bifurcation at equilibrium $E_2 = (S^*, I^*)$ when the parameter $\overline{h}^*$ changes in the small neighborhood. Moreover, if $a < 0$ (respectively, $a > 0$), then an attracting (respectively, repelling) invariant closed curve bifurcates from E_2 for $\overline{h}^* > 0$ (respectively, $\overline{h}^* < 0$).*

Proof. By taking the parameters $(h_2, \mu, \alpha, \beta, \delta, \gamma) \in N$, the model (1.3) becomes into:

$$\begin{cases} S_{n+1} = S_n + h_2[\mu(1 - S_n) - \dfrac{\beta S_n I_n}{1 + \alpha S_n} - \mu(1 - \delta) I_n], \\ I_{n+1} = I_n + h_2[\dfrac{\beta S_n I_n}{1 + \alpha S_n} - \gamma I_n - \mu\delta I_n], \end{cases} \tag{3.4}$$

and the model (3.4) has an equilibrium (S^*, I^*). Because $(h_2, \mu, \alpha, \beta, \delta, \gamma) \in N$ and $(h_2 = -\frac{\xi_1}{\xi_2}$, we choose $\overline{h}^*$ as the bifurcation parameter. Giving parameter h_1 a perturbation $\overline{h}^*$, we consider a perturbation of model (3.4) as follows:

$$\begin{cases} S_{n+1} = S_n + (h_2 + \overline{h}^*)[\mu(1 - S_n) - \dfrac{\beta S_n I_n}{1 + \alpha S_n} - \mu(1 - \delta) I_n], \\ I_{n+1} = I_n + (h_2 + \overline{h}^*)[\dfrac{\beta S_n I_n}{1 + \alpha S_n} - \gamma I_n - \mu\delta I_n], \end{cases} \tag{3.5}$$

where $|\overline{h}^*| \ll 1$.

Let $u_n = S_n - S^*, v_n = I_n - I^*$, then we transform equilibrium $E_2 = (S^*, I^*)$ into the origin $(0, 0)$, and by calculating we get

$$\begin{cases} u_{n+1} = a_{11}u_n + a_{12}v_n + a_{13}u_n^2 + a_{14}u_n v_n + c_{11}u_n^3 + c_{12}u_n^2 v_n + O((|u_n| + |v_n|)^4), \\ v_{n+1} = a_{21}u_n + a_{22}v_n + a_{23}u_n^2 + a_{24}u_n v_n + c_{21}u_n^3 + c_{22}u_n^2 v_n + O((|u_n| + |v_n|)^4). \end{cases} \tag{3.6}$$

The characteristic equation associated with the linearization equation of model (3.6) at $(0, 0)$ is given by

$$\lambda^2 + p(\overline{h}^*)\lambda + q(\overline{h}^*) = 0,$$

where

$$p(\overline{h}^*) = -2 - \xi_1(h_2 + \overline{h}^*), \quad q(\overline{h}^*) = 1 + \xi_1(h_2 + \overline{h}^*) + \xi_2(h_2 + \overline{h}^*)^2.$$

Since $(h_2, \mu, \alpha, \beta, \delta, \gamma) \in N$, the model (3.6) has a pair of conjugate complex roots $\lambda, \overline{\lambda}$ with modulus 1 at $E_2 = (S^*, I^*)$, where

$$\begin{aligned}\lambda, \overline{\lambda} &= -\tfrac{p(\overline{h}^*)}{2} \pm \sqrt{4q(\overline{h}^*) - p^2(\overline{h}^*)} \\ &= 1 + \tfrac{\xi_1(h_2+\overline{h}^*)}{2} \pm \tfrac{i(h_2+\overline{h}^*)}{2}\sqrt{4\xi_2 - \xi_1^2}.\end{aligned}$$

So we have

$$|\lambda| = \sqrt{q(\overline{h}^*)}, \quad l = \frac{d\,|\,\lambda\,|}{d\overline{h}^*}\,|_{\overline{h}^*=0} = -\frac{\xi_1}{2}.$$

In addition, it is required that $\lambda^m, \overline{\lambda}^m \neq 1, (m = 1, 2, 3, 4)$ when $\overline{h}^* = 0$, which is equivalent to $p(0) \neq -2, 0, 1, 2$. Note that $(h_2, \mu, \alpha, \beta, \delta, \gamma) \in N$, then $p(0) \neq -2, 2$. We just need to require that $p(0) \neq 0, 1$, that is

$$\xi_1^2 \neq 2\xi_2, 3\xi_2. \tag{3.7}$$

Therefore, the eigenvalues $\lambda, \overline{\lambda}$ of equilibrium $(0, 0)$ of model (3.6) do not lay in the intersection of the unit circle with the coordinate axes when $\overline{h}^* = 0, \xi_1 < 0$ and the condition (3.7) holds.

Let $\overline{h}^* = 0, \omega_1 = 1 + \frac{\xi_1 h_2}{2}, \omega_2 = \frac{h_2}{2}\sqrt{4\xi_2 - \xi_1^2}$, and the invertible matrix

$$T = \begin{pmatrix} a_{12} & 0 \\ \omega_1 - a_{11} & -\omega_2 \end{pmatrix}.$$

Using the following translation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = T \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

the model (3.6) becomes into the following form

$$\begin{cases} X_{n+1} = \omega_1 X_n - \omega_2 Y_n + \overline{F}(X_n, Y_n), \\ Y_{n+1} = \omega_2 X_n + \omega_1 Y_n + \overline{G}(X_n, Y_n), \end{cases}$$

where

$$\begin{aligned}
\overline{F}(X_n, Y_n) &= \frac{a_{13}}{a_{12}}u_n^2 + \frac{a_{14}}{a_{12}}u_n v_n + \frac{c_{11}}{a_{12}}u_n^3 + \frac{c_{12}}{a_{12}}u_n^2 v_n + O((|X_n| + |Y_n|)^4), \\
\overline{G}(X_n, Y_n) &= \frac{a_{13}(\omega_1 - a_{11}) - a_{12}a_{23}}{a_{12}\omega_2}u_n^2 + \frac{a_{14}(\omega_1 - a_{11}) - a_{12}a_{24}}{a_{12}\omega_2}u_n v_n + \frac{c_{11}(\omega_1 - a_{11}) - a_{12}c_{21}}{a_{12}\omega_2}u_n^3 \\
&\quad + \frac{c_{12}(\omega_1 - a_{11}) - a_{12}c_{22}}{a_{12}\omega_2}u_n^2 v_n + O((|X_n| + |Y_n|)^4),
\end{aligned}$$

and

$$\begin{aligned}
u_n^2 &= a_{12}^2 X_n^2, u_n v_n = a_{12}(\omega_1 - a_{11})X_n^2 - a_{12}\omega_2 X_n^2 Y_n, \\
u_n^3 &= a_{12}^3 X_n^3, u_n^2 v_n = a_{12}^2(\omega_1 - a_{11})X_n^3 - a_{12}^2\omega_2 X_n^2 Y_n.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\overline{F}_{X_nX_n} &= 2a_{12}a_{13} + 2a_{12}(\omega_1 - a_{11}), \quad \overline{F}_{Y_nY_n} = -\omega_2 a_{11}, \quad \overline{F}_{X_nY_n} = 0, \\
\overline{F}_{X_nX_nX_n} &= 6c_{11}a_{12}^2 + 6c_{12}a_{12}(\omega_1 - a_{11}), \quad \overline{F}_{X_nX_nY_n} = -2c_{12}a_{12}\omega_2, \quad \overline{F}_{X_nY_nY_n} = \overline{F}_{Y_nY_nY_n} = 0, \\
\overline{G}_{X_nX_n} &= \frac{2}{\omega_2}[a_{14}(\omega_1 - a_{11})^2 + (\omega_1 - a_{11})(a_{13} - a_{12}a_{24}) - a_{12}a_{23}]2a_{12}a_{13} + 2a_{12}(\omega_1 - a_{11}), \\
\overline{G}_{X_nY_n} &= a_{12}a_{24} - a_{14}(\omega_1 - a_{11}), \quad \overline{G}_{Y_nY_n} = 0, \\
\overline{G}_{X_nX_nX_n} &= \frac{6}{\omega_2}[a_{12}c_{12}(\omega_1 - a_{11})^2 + (\omega_1 - a_{11})(a_{13} - a_{12}^2 c_{22}) - a_{12}a_{23}], \\
\overline{G}_{X_nX_nY_n} &= 2a_{12}[a_{12}c_{22} - c_{12}(\omega_1 - a_{11})], \quad \overline{G}_{X_nY_nY_n} = \overline{G}_{Y_nY_nY_n} = 0.
\end{aligned}$$

In order to undergo Neimark-Sacker bifurcation for model (3.8), we require that the following discriminatory quantity is not equal to zero:

$$a = [-Re(\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}\xi_{20}\xi_{11}) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\overline{\lambda}\xi_{21})]|_{\overline{h}^*=0}, \tag{3.8}$$

where

$$\xi_{02} = \frac{1}{8}[\overline{F}_{X_nX_n} - \overline{F}_{Y_nY_n} + 2\overline{G}_{X_nY_n} + i(\overline{G}_{X_nX_n} - \overline{G}_{Y_nY_n} + 2\overline{F}_{X_nY_n})],$$
$$\xi_{11} = \frac{1}{4}[\overline{F}_{X_nX_n} - \overline{F}_{Y_nY_n} + i(\overline{G}_{X_nX_n} - \overline{G}_{Y_nY_n})],$$
$$\xi_{20} = \frac{1}{8}[\overline{F}_{X_nX_n} - \overline{F}_{Y_nY_n} + 2\overline{G}_{X_nY_n} + i(\overline{G}_{X_nX_n} - \overline{G}_{Y_nY_n} - 2\overline{F}_{X_nY_n})],$$
$$\xi_{21} = \frac{1}{16}[\overline{F}_{X_nX_nX_n} - \overline{F}_{X_nX_nY_n} + \overline{G}_{X_nX_nY_n} + \overline{G}_{Y_nY_nY_n} + i(\overline{G}_{X_nX_nX_n} - \overline{G}_{Y_nY_nY_n} - \overline{F}_{X_nX_nY_n} - \overline{F}_{Y_nY_nY_n})].$$

Therefore, from the above analysis and the theorem in [10], we proved the accuracy of the Theorem 3.2. □

In Section 4 we will choose some values of parameters $\mu, \alpha, \beta, \delta$ and γ to show the process of Neimark-Sacker bifurcation for model (3.5).

4. Numerical simulations

In this section, we will present the bifurcation diagrams and phase portraits to illustrate above analytic results.

4.1. The numerical simulation for flip bifurcation

Let $\alpha = 0.52, \beta = 2.2, \mu = 1.2, \delta = 0.01, \gamma = 0.02$, then according to Lemma 2.2, we know that in the model (1.3) exists an endemic equilibrium $E_2(0.0147, 0.9692)$. By calculation we known that, the model (1.3) undergoes the flip bifurcation at E_2 when $h = 0.9984$. At this time, the coefficients $\alpha_1 = 15.1737, \alpha_2 = -26.3378$, and $(\alpha, \beta, \mu, \delta, \gamma) = (0.52, 2.2, 1.2, 0.01, 0.02) \in S_1$. Thus, the above analysis and Fig. 1 verify the correctness of Theorem 3.1.

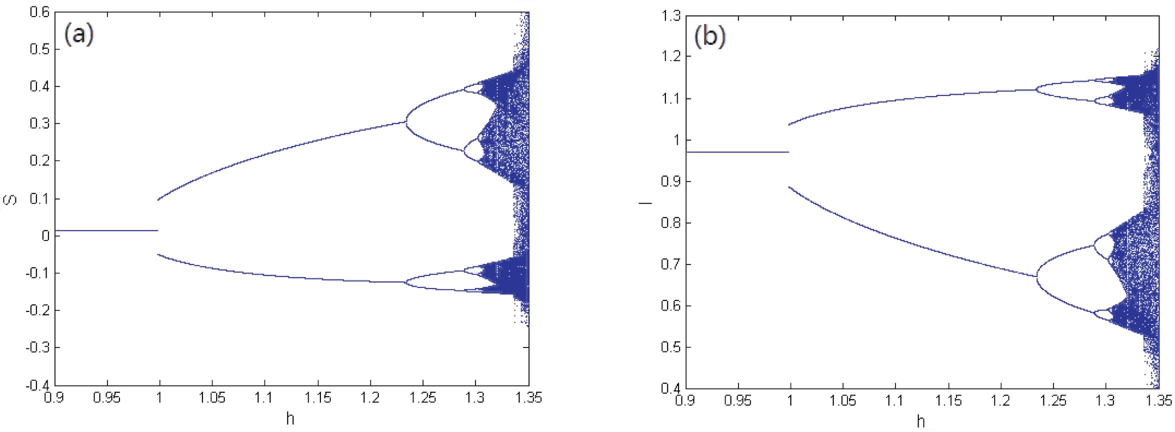

Figure 1: The flip bifurcation of (a) S_n and (b) I_n with $\alpha = 0.52, \beta = 2.2, \mu = 1.2, \delta = 0.01, \gamma = 0.02, h \in [0.9, 1.35]$ and initial values $(S_0, I_0) = (0.06, 0.06)$.

We fix the parameter $\alpha = 0.52, \beta = 2.2, \mu = 1.2, \delta = 0.01, \gamma = 0.02$, and draw the flip bifurcation diagram when $h \in [0.9, 1.35]$, as shown in Fig. 1. And Fig. 2 shows the phase portraits of model (1.3) for different values of corresponding to Fig. 1. From Fig. 1, we see that the equilibrium E_2 is stable when $h < 0.9984$, and loses its stability when $h = 0.9984$. And there is a period-doubling bifurcation when $h > 0.9984$. Further, we can see the period-2 orbits, period-4 orbits and period-8 orbits appear in the range $h \in (0.9984, 1.305)$. Moreover, a chaotic attractor is emerged when $h = 1.33$.

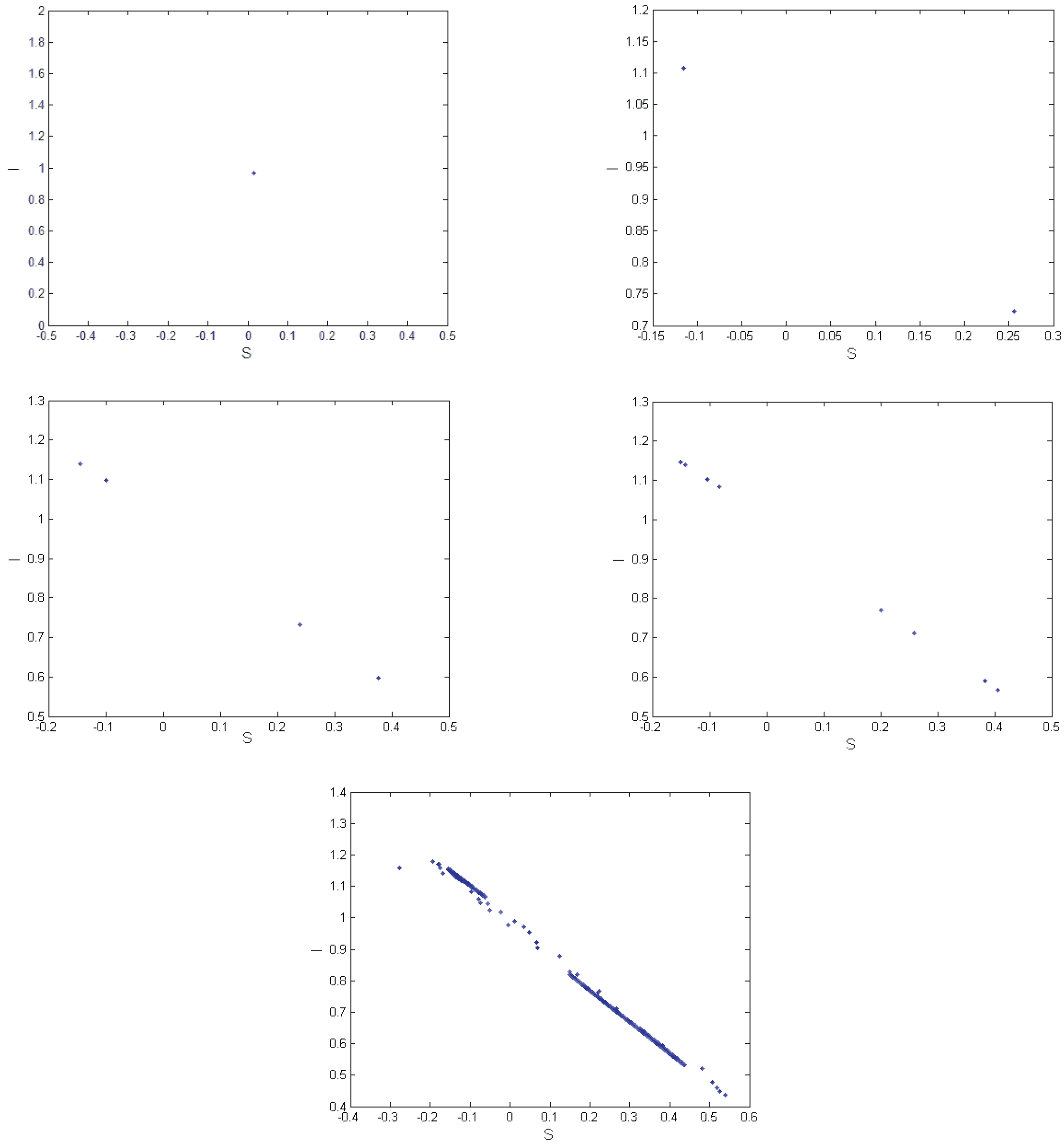

Figure 2: The phase portraits of model (1.3) for different values of h corresponding to Fig. 1.

4.2. The numerical simulation for Neimark-Sacker bifurcation

Let $\alpha = 1.2, \beta = 2.0, \mu = 1.5, \delta = 0.01, \gamma = 0.2$, then according to Lemma 2.2, we know that in the model (1.3) exists an endemic equilibrium $E_2(0.1234, 0.7735)$. Through calculation and analysis, we can get the model (1.3) undergoes the Neimark-Sacker bifurcation at equilibrium E_2 when the parameter $h = 1.342$. And the eigenvalues of equilibrium E_2 are $\lambda_{\pm} = -0.3983 \pm 0.9173i$. So, we have $|\lambda| = 1, l = \frac{d|\lambda|}{d\bar{h}^*}|_{\bar{h}^*=0} = 1.3174 > 0, a = -4.8467, \xi_1^2 = 6.9426, 2\xi_2 = 4.965, 3\xi_2 = 7.4475$, $(\alpha, \beta, \mu, \delta, \gamma) = (1.2, 2.0, 1.5, 0.01, 0.2) \in N$. Thus, combining with the above analysis and Fig. 3 we verify the correctness of Theorem 3.2.

We fix the parameter $\alpha = 1.2, \beta = 2.0, \mu = 1.5, \delta = 0.01, \gamma = 0.2$, and draw the Neimark-Sacker bifurcation diagram when $h \in [1.3, 1.6]$, as shown in Fig. 3. And Fig. 4 shows the phase portraits of model (1.3) for different values of corresponding to Fig. 3. From Fig. 3, we see that the equilibrium E_2 is stable when $h < 1.342$, and loses its stability when $h = 1.342$. And there is an invariant closed curve when $h > 1.342$.

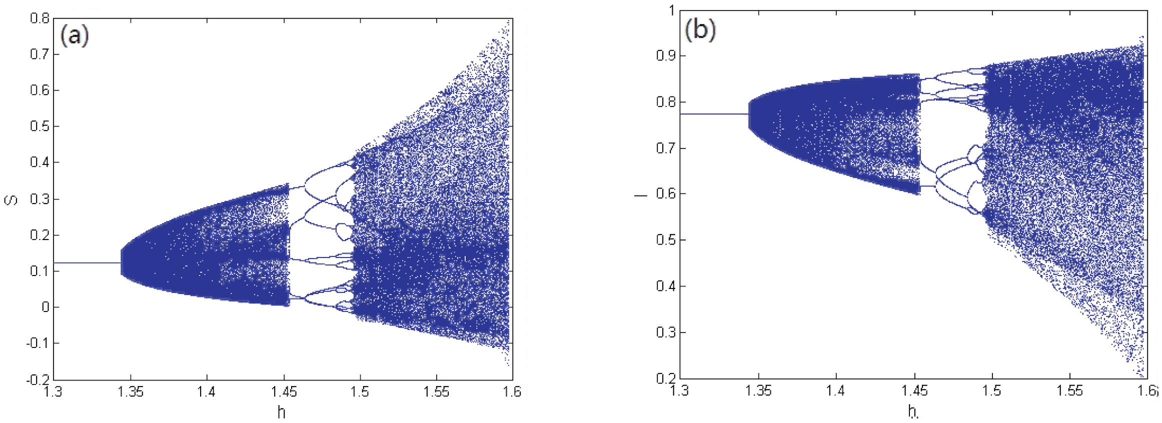

Figure 3: The Neimark-Sacker bifurcation of (a) S_n and (b)I_n with $\alpha = 1.2, \beta = 2.0, \mu = 1.5, \delta = 0.01, \gamma = 0.2$,$h \in [1.3, 1.6]$ and initial values $(S_0, I_0) = (0.06, 0.06)$.

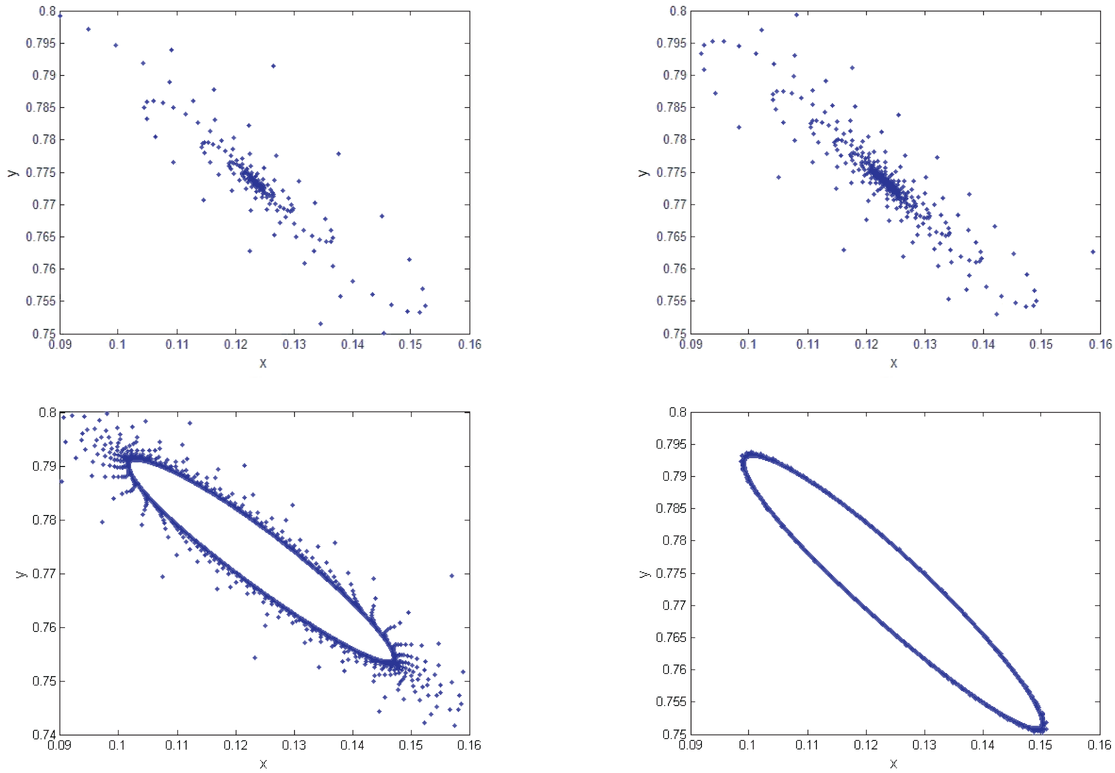

Figure 4: The phase portraits of model (1.3) for different values of h corresponding to Fig. 3.

5. Conclusions

In this paper, we proposed a discrete SIR epidemic model with the saturated contact rate and vertical transmission obtained by Euler method. First, we presented the parameter conditions for the existence of equilibria, and the local stability of the equilibria are also investigated. Moreover, the existence, stability and direction of flip bifurcation and Neimark-Sacker bifurcation are studied by using the center manifold theorem and the norm form theory. The results show that the flip bifurcation and Neimark-Sacker bifurcation occur when the bifurcation parameter crosses some critical values. At last, we presented the numerical simulations illustrating our results with the theoretical analysis. Apparently, there are more interesting problems about this discrete SIR epidemic model which deserve further investigation.

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