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Stability of efficient solutions for semi-infinite vector optimization problems

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Abstract

This paper is devoted to the study of the stability of efficient solutions for semi-infinite vector optimization problems (SIO). We first obtain the closedness, Berge-lower semicontinuity and Painlevé-Kuratowski convergence of constraint set mapping. Then, under the assumption of continuous convergence of the objective function, we establish some sufficient conditions of the upper Painlevé-Kuratowski stability of efficient solution mappings to the (SIO). Some examples are also given to illustrate the results.

Keywords: Upper Painlevé-Kuratowski stability, semi-infinite vector optimization, perturbation, efficient solution, continuous convergence.
2010 MSC: 49K40, 90C29, 90C31.

1. Introduction

Let X be a Hausdorff topological space, Y and Z be real Banach spaces with norms denoted by $\|\cdot\|$. Let D (resp. K) be closed, convex and pointed cone in Y (resp. Z) with nonempty interior int$D$ (resp. int$K$). Let A be a nonempty compact convex subset of X. We denote by $\mathcal{U}[A, Y]$ the set of all vector-valued functions from A to Y. Let T be a nonempty compact subset of a Hausdorff topological space, and denote by $\mathcal{USC}[A \times T, Z]$ we mean the set of all K-upper semicontinuous vector-valued functions with respect to the first variable, where the metric of the function $h \in \mathcal{USC}[A \times T, Z]$ is defined as

$$\rho(h_1, h_2) := \min\{ \sup_{x\in A, t\in T} \|h_1(x, t) - h_2(x, t)\|, \frac{1}{5}\}.$$

Consider parametric semi-infinite vector optimization problems (SIO for brevity), or generalized parametric

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vector optimization problems, under functional perturbations of both objective function and constraint set on the parameter space

$$G_0 := \mathcal{U}[A, Y] \times \mathcal{USC}[A \times T, Z]$$

formulated as follows: for every double of parameter $p := (f, h) \in G_0$, we have the semi-infinite vector optimization problem

$$\text{(SIO)} \qquad \begin{cases} D - \min\ f(x) \\ \text{s.t.}\ \ x \in M(h), \end{cases} \tag{1.1}$$

where

$$\begin{aligned} M(h) &:= \{x \in A : h(x,t) \geqq_K 0, \forall t \in T\}, \\ x \geqq_K y \ &\Leftrightarrow\ x - y \in K. \end{aligned} \tag{1.2}$$

We know that the semi-infinite optimization problem plays a very important role in optimization theory and applications. The models of semi-infinite optimization problems cover, e.g., optimal control, approximation theory, popular semi-definite programming and numerous engineering problems, etc. The semi-infinite optimization problem and its wide range of applications have been an active research area in mathematical programming in recent years. Many paper are published on theory, methods and applications for semi-infinite optimization problem and its extensions; examples of fresh literatures include, the existence results in [5, 6, 19], the optimality and/or characterizations of the solution set in [13, 15, 20], the stability results of solution mappings in [4, 7, 9, 11], etc. Since the semi-infinite vector optimization problem has been acting more and more important role in optimization theory and applications, some new methods and skills will appear gradually.

On the other hand, the stability of solution mappings under certain perturbations, either of the feasible set or the objective function, has been great interest in the optimization theory and related field. There are some stability results for vector optimization problems and related issues with a sequence of sets converging in the sense of Painlevé-Kuratowski. Examples of fresh literatures include, for vector optimization problems, we can see Attouch and Riahi [2], Huang [12], Lucchetti and Miglierina [18], Lalitha and Chatterjee [14]; for vector equilibrium problem, we can refer to Durea [8], Fang and Li [10], Zhao et al.[23], Peng and Yang [21], etc. However, to the best of our knowledge, the Painlevé-Kuratowski stability of efficient solutions set for semi-infinite vector optimization problems has not been found. Thus, it is interesting to investigate the Painlevé-Kuratowski convergence of the efficient solution mapping for semi-infinite vector optimization problems.

The rest of the paper is organized as follows. In Sect. 2, we recall some basic definitions and preliminaries from set-valued analysis and vector optimization, which will be used in next section. The main result is presented in Sect. 3. In Sect. 3, we first establish the closeness, Berge-lower semicontinuity and Painlevé-Kuratowski convergence of constraint set mapping. Then, under the assumption of continuous convergence of the objective function, we obtain some sufficient conditions of the upper Painlevé-Kuratowski stability of efficient solution mappings to the semi-infinite vector optimization problem (SIO). We also give some examples to illustrate our main results.

2. Preliminaries

In this section, we give some basic definitions and preliminary results which are needed in the sequel.

Throughout this paper, unless specified otherwise, X, Y, Z, D, K and T are as mentioned above. Relations in Y associated with the cone D are defined as follows: for any $y_1, y_2 \in Y$,

$$y_1 \leqq_D y_2 \ \Leftrightarrow\ y_2 - y_1 \in D; \qquad y_1 \nleqq_D y_2 \ \Leftrightarrow\ y_2 - y_1 \notin D;$$

$$y_1 \leq_D y_2 \ \Leftrightarrow\ y_2 - y_1 \in D \setminus \{0\}; \qquad y_1 \nleq_D y_2 \ \Leftrightarrow\ y_2 - y_1 \notin D \setminus \{0\};$$

$$y_1 <_D y_2 \ \Leftrightarrow\ y_2 - y_1 \in \text{int}D; \qquad y_1 \nless_D y_2 \ \Leftrightarrow\ y_2 - y_1 \notin \text{int}D,$$

and the vector ordering relations in Z associated with the cone K are similar as above.

For the semi-infinite vector optimization problem (1.1), we call the set-valued mapping (or multifunction) $M : \mathcal{USC}[A \times T, Z] \rightrightarrows A$ (given in (1.2)) the constraint set mapping of (SIO). A vector $x \in M(h)$ is said to be a strictly efficient solution of (SIO), if and only if for any $y \in M(h), y \neq x$,

$$f(y) - f(x) \notin -D.$$

A vector $x \in M(h)$ is said to be an efficient solution of (SIO), if

$$\{f(x)\} = (f(x) - D) \cap f(M(h)).$$

A vector $x \in M(h)$ is said to be a weakly efficient solution of (SIO), if

$$(f(x) - \mathrm{int}D) \cap f(M(h)) = \emptyset.$$

For each $p = (f, h) \in G_0$, let SSol$(M(h), f)$, ESol$(M(h), f)$ and WESol$(M(h), f)$ denote the sets of strictly efficient solutions, efficient solutions and the set of weakly efficient solutions of (SIO), respectively.

Now, we give Example 2.1 to illustrate efficient solutions of (SIO) in Banach space.

Example 2.1. Let $X = Y = l^1 = \{(x_1, \cdots, x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}, A = clco\{\{\frac{e_n}{n}\}_{n=1}^{\infty} \cup \{0_X\}\}$, where $e_1 = (1, 0, 0, \cdots), e_2 = (0, 1, 0, \cdots), e_3 = (0, 0, 1, 0, \cdots), \cdots$. Let $Z = \mathbb{R}^2, K = \mathbb{R}^2_+, T = [0, 1] \subset \mathbb{R}, D = \{x = (x_1, \cdots, x_n, \cdots) \in l^1 : x_n \geq 0, n = 1, 2, \cdots\}$. Then, we can observe that A is a compact convex set in X. We consider $h : A \times T \to Z, f : A \to Y$ by

$$h(x,t) = \big(\sum_{n=1}^{\infty} |y_n - x_n| + \frac{t}{2} + 1, \sum_{n=1}^{\infty} |x_n| + \frac{1}{2}\big), \text{ for all } x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in A,$$

$$f(x) = \frac{x}{3}, \text{ for all } x = (x_1, \cdots, x_n, \cdots) \in A.$$

From a direct computation, we can get that $M(h) = A$ and ESol$(M(h), f) = \{0_X\}$.

Definition 2.2. Let A be a nonempty convex subset of X, and let f be a mapping from A to Y. We say that f is D-convex on A, if for any $x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \in \lambda f(x_1) + (1 - \lambda)f(x_2) - D.$$

Definition 2.3 ([17])**.** Let A be a nonempty convex subset of X, and f be a mapping from A to Y. We say that

- (i) f is properly quasi D-convex on A, if for any $x_1, x_2 \in A$ and $\lambda \in [0, 1]$, either $f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_1) - D$ or $f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_2) - D$.
- (ii) f is semistrictly (strictly) properly quasi D-convex on A, if for any $x_1, x_2 \in A$ with $f(x_1) \neq f(x_2)$ $(x_1 \neq x_2)$ and $\lambda \in (0, 1)$, either $f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - \mathrm{int}D$ or $f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - \mathrm{int}D$.

In [17], Luc gave the following definition of C-upper semicontinuity.

Definition 2.4. Let E be a nonempty subset of X. Let f be a mapping from E to Y. f is said to be D-upper semicontinuous at $x_0 \in E$, if for any neighborhood W of 0 in Y, there is a neighborhood U of x_0 such that for each $x \in U \cap E$,

$$f(x) \in f(x_0) + W - D.$$

Definition 2.5. Let E be a nonempty convex subset of X. Let f be a mapping from E to Z. We say that f is K-quasiconvex on E, if for any $z \in Z, x_1, x_2 \in E$ with $x_1 \neq x_2$ and $\lambda \in [0, 1]$,

$$f(x_1), f(x_2) \in z - K \text{ implies } f(\lambda x_1 + (1 - \lambda)x_2) \in z - K.$$

Remark 2.6. We call f is K-quasiconcave on E if $-f$ is K-quasiconvex on E.

Definition 2.7 ([1, 3])**.** Let X and Y be topological vector spaces, $F : X \to 2^Y$ be a set-valued mapping.

(i) F is said to be Berge-lower semicontinuous at $x_0 \in X$, if for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

(ii) F is said to be Berge-lower semicontinuous on X, iff it is Berge-lower semicontinuous at each $x \in X$;

(iii) F is closed if $\text{Graph}(F)$ is a closed set in $X \times Y$.

Now, we recall the well known notion of set-convergence, namely Painlevé-Kuratowski set-convergence.

A sequence of sets $\{B_n \subset X : n \in \mathbb{N}\}$ is said to converge in the sense (see also [8, 22]) of Painlevé-Kuratowski (P.K.) to B (denoted as $B_n \xrightarrow{P.K.} B$) if

$$\limsup_{n\to\infty} B_n \subset B \subset \liminf_{n\to\infty} B_n$$

with

$$\liminf_{n\to\infty} B_n := \{x \in X | \exists (x_n), x_n \in B_n, \forall n \in N, x_n \to x\},$$
$$\limsup_{n\to\infty} B_n := \{x \in X | \exists (n_k), \exists (x_{n_k}), x_{n_k} \in B_{n_k}, \forall k \in N, x_{n_k} \to x\}.$$

When $\limsup_{n\to\infty} B_n \subset B$ holds, the relation is referred as upper Painlevé-Kuratowski convergence (u.P.K, for briefness). When $K \subset \liminf_{n\to\infty} K_n$ holds, the relation is referred as lower Painlevé-Kuratowski convergence (l.P.K, for briefness).

A set-valued mapping $\psi : X \to 2^Y$ is said to be Painlevé-Kuratowski convergent at $x \in \text{dom}\psi := \{x \in X | \psi(x) \neq \emptyset\}$ if and only if for any sequence x_n in $\text{dom}\psi$ converging to x, one has

$$\limsup_{n\to\infty} \psi(x_n) \subset \psi(x) \subset \liminf_{n\to\infty} \psi(x_n).$$

Definition 2.8 ([16, 22])**.** Let $f_n, f : X \to Y$ be vector-valued mappings and $A \subset X$. We say that f_n continuously converges to f (denoted as $f_n \xrightarrow{c} f$), iff for every $x \in A$ and for every sequence $\{x_n\}$ in A, $f_n(x_n) \to f(x)$ for all $x_n \to x$.

In [1], Aubin et al. also gave the following properties for Berge-lower semicontinuous.

Lemma 2.9. *Let X and Y be topological vector spaces, $F : X \to 2^Y$ be a set-valued mapping. F is Berge-lower semicontinuous at $x_0 \in X$ if and only if for any sequence $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in F(x_0)$, there exists $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y_0$.*

Lemma 2.10 ([3])**.** *Let Y be a topological vector space. For each zero neighborhood U in Y, there exist zero neighborhood U_1 and U_2 in Y such that $U_1 + U_2 \subset U$.*

3. Main results

In this section, we aim to establish the Painlevé-Kuratowski stability of efficient solution mappings to the semi-infinite vector optimization problem.

We first give some sufficient conditions for closeness, Berge-lower semicontinuity and Painlevé-Kuratowski convergence of the constraint set mapping $M : \mathcal{USC}[A \times T, Z] \rightrightarrows A$ as follows.

Theorem 3.1. *Let $p := (f, h)$ be any given point in G_0.*

(i) For each $t \in T$, $x \mapsto h(x,t)$ is K-quasiconcave on A, then $M(\cdot)$ is convex at h.

(ii) If $h_n(\cdot, t) \xrightarrow{\rho} h(\cdot, t)$ for any $t \in T$, then the constraint set mapping $M(\cdot)$ is closed at h.

Proof. (i) Getting $x_1, x_2 \in M(h)$, one has

$$h(x_1, t) \geqq_K 0, \text{ for all } t \in T$$

and

$$h(x_2, t) \geqq_K 0, \text{ for all } t \in T.$$

Then, for each $t \in [0,1]$, $tx_1 + (1-t)x_2 \in A$ as A is convex. It follows from the K-quasiconcavity of $h(\cdot, t)$ on A and equations above that

$$h(tx_1 + (1-t)x_2, t) \in K, \ \forall t \in T.$$

This means $tx_1 + (1-t)x_2 \in M(h)$, i.e., $M(h)$ is a convex set.

(ii) Let $\{(h_n, x_n)\} \subset \text{Graph}(M), h_n \xrightarrow{\rho} h, x_n \to x'$. Then $x' \in A$ as A is compact. Since $x_n \in M(h_n)$, for every $n \in \mathbb{N}$,

$$h_n(x_n, t) \geqq_K 0, \text{ for all } t \in T. \tag{3.1}$$

Now, we verify that $x' \in M(h)$. Suppose the contrary is true, that is, there exists $t' \in T$ such that $h(x', t') \notin K$. By the openness of $Y \setminus K$, there exists an open neighborhood U of 0_Y in Y such that

$$h(x', t') + U \subset Y \setminus K. \tag{3.2}$$

From Lemma 2.10, for above U, there exist two neighborhoods U_1 and U_2 of 0_Y in Y such that

$$U_1 + U_2 \subset U. \tag{3.3}$$

By the K-upper semicontinuity of $h(\cdot, t')$ at x' for above U_1, there exists a neighborhood $U(x')$ of x', such that

$$h(x, t') \in h(x', t') + U_1 - K, \forall x \in U(x') \cap A.$$

Since $x_n \to x'$, there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$, one has

$$x_n \in U(x') \cap A.$$

It follows that

$$h(x_n, t') \in h(x', t') + U_1 - K. \tag{3.4}$$

As $h_n \xrightarrow{\rho} h$, there exists $n_2 \in \mathbb{N}$ such that for any $n \geq n_2$,

$$h_n(x_n, t') - h(x_n, t') \in U_2. \tag{3.5}$$

From (3.2)-(3.5), for $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned} h_n(x_n, t') &= h_n(x_n, t') - h(x_n, t') + h(x_n, t') \\ &\in U_2 + h(x', t') + U_1 - K \\ &\subset -K + Y \setminus K \subset Y \setminus K, \end{aligned}$$

which contradicts (3.1). Then $x' \in M(h)$. This implies that $M(\cdot)$ is closed at h. □

Theorem 3.2. *Let $p := (f, h)$ be any given point in G_0, for each $t \in T$, $x \mapsto h(x, t)$ is K-quasiconcave on A, then the constraint set mapping $M(\cdot)$ is Berge-lower semicontinuous at h.*

Proof. Let W be an open convex set such that $W \cap M(h) \neq \emptyset$. Since $M(h) \neq \emptyset$, there exists an element $\tilde{x} \in M(h)$ satisfying

$$h(\tilde{x}, t) \geqq_K 0, \text{ for all } t \in T.$$

Taking any $x_0 \in W \cap M(h)$, there exists $r \in (0, 1]$ such that $x_r := x_0 + r(\tilde{x} - x_0) \in W$, then $x_r \in W \cap M(h)$ as $M(h)$ is convex by Theorem 3.1. Since $x_0 \in M(h)$, we have

$$h(x_0, t) \geqq_K 0, \text{ for all } t \in T,$$

and $x_r := x_0 + r(\tilde{x} - x_0) \in A$ by the convexity of A. It follows from two equations above and the K-quasiconcavity of $h(\cdot, t)$ that

$$h(x_r, t) \in K.$$

This means that

$$h(x_r, t) \geqq_K 0, \text{ for all } t \in T. \tag{3.6}$$

For $\bar{h} \in \mathcal{USC}[A \times T, Z]$ satisfies $\rho(\bar{h}, h) < \frac{\delta}{2}$ ($\delta > 0$ is small enough), we clarify that $x_r \in M(\bar{h})$. On the contrary, there exists $\bar{t} \in T$ such that

$$\bar{h}(x_r, \bar{t}) \ngeqq_K 0.$$

By the openness of $Y \setminus K$, there exists a zero neighborhood U in Y such that

$$\bar{h}(x_r, \bar{t}) + U \subset Y \setminus K. \tag{3.7}$$

It follows from $\rho(\bar{h}, h) < \frac{\delta}{2}$ that for above U,

$$\bar{h}(x_r, \bar{t}) - h(x_r, \bar{t}) \in U. \tag{3.8}$$

Combining (3.7)-(3.8), we obtain

$$\begin{aligned} h(x_r, \bar{t}) &= h(x_r, \bar{t}) - \bar{h}(x_r, \bar{t}) + \bar{h}(x_r, \bar{t}) \\ &\in U + \bar{h}(x_r, \bar{t}) \\ &\subset Y \setminus K. \end{aligned}$$

This contradicts to (3.6). Then we have $x_r \in M(\bar{h})$ and $W \cap M(\bar{h}) \neq \emptyset$. This means $M(\cdot)$ is Berge-lower semicontinuous at h and the proof is complete. □

Theorem 3.3. *Let $p := (f, h)$ be any given point in G_0. Suppose that*

(i) for each $t \in T$, $x \mapsto h(x, t)$ is K-quasiconcave on A.

(ii) If $h_n(\cdot, t) \xrightarrow{\rho} h(\cdot, t)$ for any $t \in T$,

Then

$$M(h_n) \xrightarrow{P.K.} M(h).$$

Proof. Take an $x \in \limsup_n M(h_n)$. Then, there exists a subsequence $\{x_{n_k}\} \subset M(h_{n_k})$ such that $x_{n_k} \to x$. By Theorem 3.1 (ii), $M(\cdot)$ is closed at h, then we get $x \in M(h)$. Hence, we have $\limsup_n M(h_n) \subset M(h)$.

Next, we prove $M(h) \subset \liminf_n M(h_n)$. Take any $x \in M(h)$, then by Theorem 3.2 ($M(\cdot)$ Berge-lower semicontinuous), there exists $x_n \in M(h_n)$ such that $x_n \to x$. From the definition of lower Painlevé-Kuratowski convergence, we have $x \in \liminf_n M(h_n)$, which means that $M(h) \subset \liminf_n M(h_n)$ as $x \in M(h)$ is arbitrary. This completes the proof. □

Now, we establish the upper Painlevé-Kuratowski stability of solution mappings for the semi-infinite vector optimization problem (SIO).

Theorem 3.4. *Let $p := (f, h)$ be any given point in G_0. Assume that the conditions (i) and (ii) of* Theorem 3.3 *are satisfied and $f_n \xrightarrow{c} f$. Then*

$$\limsup_{n \to \infty} WESol(M(h_n), f_n) \subset WESol(M(h), f).$$

Proof. Take an $x \in \text{limsup}_n \text{WESol}(M(h_n), f_n)$. Then, there exists a subsequence

$$\{x_{n_k}\} \text{ in } \text{WESol}(M(h_{n_k})), f_{n_k})$$

such that $x_{n_k} \to x$. From Theorem 3.3, we conclude $x \in M(h)$. For any $y \in M(h)$, there exists $y_n \in M(h_n)$ such that $y_{n_k} \to y$ since $M(h_n) \xrightarrow{P.K.} M(h)$. It follows from $\{x_{n_k}\} \subset \text{WESol}(M(h_{n_k}), f_{n_k})$ and $y_{n_k} \in M(h_{n_k})$, that

$$f_{n_k}(y_{n_k}) - f_{n_k}(x_{n_k}) \notin -\text{int}D. \tag{3.9}$$

Since $f_n \xrightarrow{c} f$, there exists $N \in \mathbb{N}$ for any $n_k > N$

$$f_{n_k}(y_{n_k}) \to f(y) \text{ and } f_{n_k}(x_{n_k}) \to f(x). \tag{3.10}$$

Now, (3.9), (3.10) and the closedness of $Y \setminus$-int$D$, implies that

$$f(y) - f(x) \notin -\text{int}D.$$

As $y \in M(h)$ is arbitrary, we conclude that $x \in \text{WESol}(M(h), f)$. Thus,

$$\limsup_{n\to\infty} \text{WESol}(M(h_n), f_n) \subset \text{WESol}(M(h), f).$$

This completes the proof. □

Lemma 3.5. *Let $p := (f, h)$ be any given point in G_0.*

(i) If $x \mapsto f(x)$ is semistrictly proper quasi-D-convex on A. Then

$$ESol(M(h), f) = WESol(M(h), f).$$

(ii) If $x \mapsto f(x)$ is strictly proper quasi-D-convex on A. Then

$$SSol(M(h), f) = ESol(M(h), f).$$

Proof. (i) By the definition, $\text{ESol}(M(h), f) \subset \text{WESol}(M(h), f)$. We only need to prove $\text{WESol}(M(h), f) \subset \text{ESol}(M(h), f)$. Suppose to the contrary, there exists $x_0 \in \text{WESol}(M(h), f)$ such that $x_0 \notin \text{ESol}(M(h), f)$. Hence, there exists $y_0 \in M(h)$ such that

$$f(y_0) - f(x_0) \in -D \setminus \{0\}. \tag{3.11}$$

It follows from semistrictly proper quasi-D-convexity of $f(\cdot)$ on A and (3.11), for every $\lambda \in (0, 1)$ that $\lambda x_0 + (1 - \lambda)y_0 \in A$ as A is convex, and

$$f(\lambda x_0 + (1 - \lambda)y_0) \in f(x_0) - \text{int}D,$$

which contradicts $x_0 \in \text{WSol}(M(h), f)$. Then we get $\text{WESol}(M(h), f) \subset \text{ESol}(M(h), f)$.

(ii) From the definition of strictly proper quasi-D-convexity, by using the same method above, with appropriate modification, we can get the result and the proof is complete. □

Theorem 3.6. *Let $p := (f, h)$ be any given point in G_0. Assume that the conditions (i) and (ii) of* Theorem 3.3 *are satisfied, $f_n \xrightarrow{c} f$ and $x \mapsto f(x)$ is semistrictly proper quasi-D-convex on A. Then*

$$\limsup_{n\to\infty} ESol(M(h_n), f_n) \subset ESol(M(h), f).$$

Proof. Combing Theorem 3.4 and Lemma 3.5, we can get the result easily. □

Now, we give an example to illustrate that Theorem 3.6 is applicable.

Example 3.7. Let $X := \mathbb{R}^2, Z := \mathbb{R}, Y := \mathbb{R}^2, T = [0,1] \subset \mathbb{R}$ and

$$A := \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}, K := \mathbb{R}_+, D := \mathbb{R}^2_+.$$

Consider $h, h_n : A \times T \to Z, f_n = f : A \to Z$, which are given by:

$$f(x) := \big(f^1(x), f^2(x)\big), \quad \forall x = (x_1, x_2) \in A,$$

where

$$f^1(x) := \frac{1}{3}x_1 - \frac{1}{4}\ , f^2(x) := \frac{1}{5}x_1 - \frac{1}{8};$$

$$h(x,t) := \frac{1}{2}x_1 + \frac{1}{5},\ h_n(x,t) := \frac{1}{2}x_1 + \frac{1}{5} - \frac{t}{12n^2},\ \text{for all } (x,t) \in A \times T.$$

Let $p := (f,h), p_n := (f_n, h_n) \in G_0$. It is easy to verify that all conditions of Theorem 3.6 are satisfied. By a direct computation, we get

$$\begin{aligned} M(h) &= \Big\{(x_1, x_2) \in \mathbb{R}^2 |\ -\frac{2}{5} \leq x_1 \leq 1, -1 \leq x_2 \leq 1\Big\}, \\ \mathrm{ESol}(M(h), f) &= \Big\{\big(-\frac{2}{5}, x_2\big) \in \mathbb{R}^2 |\ -1 \leq x_2 \leq 1\Big\}, \\ M(h_n) &= \Big\{(x_1, x_2) \in \mathbb{R}^2 |\ -\frac{2}{5} + \frac{t}{6n^2} \leq x_1 \leq 1, -1 \leq x_2 \leq 1\Big\}, \\ \mathrm{ESol}(M(h_n), f_n) &= \Big\{\big(-\frac{2}{5} + \frac{t}{6n^2}, x_2\big) \in \mathbb{R}^2 |\ -1 \leq x_2 \leq 1\Big\}. \end{aligned}$$

Obviously, $\limsup_{n\to\infty} \mathrm{ESol}(M(h_n), f_n) \subset \mathrm{ESol}(M(h), f)$. Thus, Theorem 3.6 is applicable.

Corollary 3.8. *Let $p := (f,h)$ be any given point in G_0. Assume that the conditions (i) and (ii) of* Theorem 3.3 *are satisfied, $f_n \xrightarrow{c} f$ and $x \mapsto f(x)$ is strictly proper quasi-D-convex on A. Then*

$$SSol(M(h_n), f_n) \xrightarrow{u.P.K.} SSol(M(h), f).$$

Proof. By virtue of Lemma 3.5 and Theorem 3.6, we can get the result. □

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