Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3212–3225
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Multiplicity solutions for discrete fourth-order boundary value problem with multiple parameters

Yanxia Wang, Chenghua Gao*, Tianmei Geng

Department of Mathematics, Northwest Normal University, 730070, Lanzhou, P. R. China.

Communicated by C. Park

Abstract

In this paper, we consider the existence of three solutions and infinitely many solutions for discrete fourth-order boundary value problems with multiple parameters under the different suitable hypotheses, respectively. The approach we use is the critical point theory.

Keywords: Discrete fourth-order boundary value problem, multiple solutions, critical point theory, Lipschitz condition.
2010 MSC: 39A12.

1. Introduction

Consider the following discrete fourth-order boundary value problem

$$\begin{cases} \Delta^4 u(t-2) + \eta\Delta^2 u(t-1) - \xi u(t) \\ = \lambda f(t,u(t)) + \mu g(t,u(t)) + h(u(t)), \quad t \in [a+1,b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b+2) = \Delta^2 u(b+1) = 0, \end{cases} \tag{1.1}$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1)-u(k)$, $\Delta^n u(t) = \Delta(\Delta^{n-1}u(t))$, a, b are two fixed integers, $[a+1, b+1]_{\mathbb{Z}}$ is the discrete interval $\{a+1, a+2, \cdots, b+1\}$. $f, g : [a+1, b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions, $h : \mathbb{R} \to \mathbb{R}$ is a strictly monotone Lipschitz continuous function with Lipschitzian constant $L \geq 0$ and $h(0) = 0$. η, ξ, λ, μ are four real parameters and satisfy

$$\eta < 8\sin^2\frac{\pi}{2(b-a+2)}, \quad \eta^2 + 4\xi \geq 0, \quad \xi + 4\eta\sin^2\frac{\pi}{2(b-a+2)} < 16\sin^4\frac{\pi}{2(b-a+2)}, \quad \lambda > 0, \quad \mu \geq 0,$$

*Corresponding author
Email addresses: wangyanxia8228@163.com (Yanxia Wang), gaokuguo@163.com (Chenghua Gao), 891459322@qq.com (Tianmei Geng)

Received 2015-11-14

where η, ξ are given in [8].

In recent years, much attention has been paid to fourth-order difference equations which are derived from various discrete elastic beam problems. A great deal of work has been done in the research of the existence and multiplicity of solutions for discrete fourth-order boundary value problems by using classical methods, such as the fixed point theory and fixed point index theory [1, 2, 3, 9, 10, 15, 19], the critical point theory [7, 8, 11, 16], Krein-Rutman Theorem and bifurcation theory [12, 13, 14, 18] and references therein for details.

In 2009, by using the Guo-Krasnosel'skii's fixed point theorem and Leggett-Williams Theorem, Anderson, et al. [1] considered the existence, multiplicity and nonexistence of nontrivial solutions to the following problem

$$\begin{cases} \Delta^4 u(t-2) - \eta\Delta^2 u(t-1) = \lambda f(t, u(t)), \quad t \in [a+1, b-1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b) = \Delta^2 u(b-1) = 0. \end{cases} \tag{1.2}$$

Moreover, the clever use of a symmetric Green's function relaxes the nonnegative assumption on the nonlinear term f.

Later, on this basis, depending on the critical point theory and monotone operator theory, He and Su [8] investigated the following problem

$$\begin{cases} \Delta^4 u(t-2) + \eta\Delta^2 u(t-1) - \xi u(t) = \lambda f(t, u(t)), \quad t \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b+2) = \Delta^2 u(b+1) = 0, \end{cases} \tag{1.3}$$

which is the special case of (1.1), i.e., when $\mu = 0$ and $h(u(t)) = 0$. They gave the sufficient conditions for the existence and nonexistence of nontrivial solutions when λ lies in some suitable intervals and η, ξ satisfy certain conditions, respectively.

Motivated by the above results, we will discuss the existence of three and infinitely many solutions of the fourth-order discrete with multiple parameters boundary value problems (1.1) by choosing the suitable η, ξ, λ and μ. The main tools that we use are two critical point theorems due to Bonanno and Marano [6] and Bonanno and Bisci [4], which are two more precise versions of Theorem 3.2 in [5] and Theorem 2.5 in [17]. In details, using the critical point theory, Theorem 2.6 in [6], we obtain the existence of three nontrivial solutions of (1.1) in Theorem 3.1 by establishing precise interval for λ and μ and this theorem extends the Theorem 4.7 in [8]. In Theorem 3.1, we require on the primitive of the function f both a growth more than quadratic in a suitable interval and a growth less than quadratic at infinity and f is nonnegative on a interval, moreover on g an asymptotic condition is requested. Furthermore, using Bonanno and Bisci's theorem, Theorem 2.1 in [4], requiring that the nonlinear term f has a suitable oscillating behavior at infinity, in Theorem 4.1, we establish the existence of a precise interval Λ such that for every λ and every continuous function g which satisfies a certain growth at infinity, and choosing μ sufficiently small, the problem (1.1) admits an unbounded sequence of weak solutions.

The rest of this paper is arranged as follows. In Section 2, we will construct the suitable Banach space and appropriate functionals corresponding to (1.1) and give our basic tools. Moreover, we will show some other preliminaries. In Section 3, under suitable hypotheses, we prove that problem (1.1) admits three nontrivial solutions. In Section 4, we show the conclusion of infinitely many solutions for problem (1.1). In Section 5, an example will be given to demonstrate our main results in Section 3 and Section 4 under the corresponding conditions, respectively.

2. Variational Framework and Main Tools

First, we are going to give the Green's function and the corresponding variation framework associated with (1.1). We introduce some basic notations. Let

$$E := \left\{ u = \{u(t)\}_{t=a+1}^{b+1} : u(t) \in \mathbb{R} \right\}.$$

Then E is a $b-a+1$-dimensional Hilbert space under the following inner product and norm

$$(u,v)=\sum_{t=a+1}^{b+1}u(t)v(t),\quad \|u\|=\left(\sum_{t=a+1}^{b+1}|u(t)|^2\right)^{\frac{1}{2}},\quad u,v\in E.$$

We can obtain the following inequality

$$\max_{t\in[a+1,b+1]_{\mathbb{Z}}}|u(t)|\le\|u\|. \tag{2.1}$$

From [8], we have

$$\Delta^4(t-2)+\eta\Delta^2u(t-1)-\xi u(t)=(-\Delta^2L+r_1)(-\Delta^2L+r_2)u(t)=(-\Delta^2L+r_2)(-\Delta^2L+r_1)u(t),$$

where $u=\{u(t)\}_{t=a-1}^{b+3}$, $Lu(t)=u(t-1)$, $t\in[a+1,b+1]_{\mathbb{Z}}$ and r_1,r_2 are roots of the polynomial $P(r)=r^2+\eta r-\xi$. And by the assumptions on η,ξ, we see that $r_1\ge r_2>-4\sin^2\frac{\pi}{2(b-a+2)}$.

Lemma 2.1 ([8], Lemma 2.1)**.** *Let $v\in E$ and $i\in\{1,2\}$ be fixed. Then the problem*

$$\begin{cases}-\Delta^2u(t-1)+r_iu(t)=v(t), & t\in[a+1,b+1]_{\mathbb{Z}},\\ u(a)=0,\quad u(b+2)=0\end{cases}$$

has a unique solution

$$u(t)=\sum_{k=a+1}^{b+1}G_i(t,k)v(k),\quad t\in[a,b+2]_{\mathbb{Z}},$$

where $G_i(t,k)$ is given by

$$G_i(t,k)=\frac{1}{\rho(1,0)\rho(b+2,a)}\begin{cases}\rho(t,a)\rho(b+2,k), & a\le t\le k\le b+1,\\ \rho(k,a)\rho(b+2,t), & a+1\le k\le t\le b+2,\end{cases}$$

with

(i) $\rho(t,k)=\sin\varphi(t-k)$, $\varphi:=\arctan\frac{\sqrt{-r_i(r_i+4)}}{2+r_i}$, *when* $-4\sin^2\frac{\pi}{2(b-a+2)}<r_i<0$*;*
(ii) $\rho(t,k)=t-k$, *when* $r_i=0$*;*
(iii) $\rho(t,k)=\iota^{t-k}-\iota^{k-t}$, $\iota:=\frac{r_i+2+\sqrt{r_i(r_i+4)}}{2}$, *when* $r_i>0$.

Lemma 2.2 ([8], Lemma 2.2)**.** *Let $\omega\in E$ be fixed. Then the linear discrete fourth-order boundary value problem*

$$\begin{cases}\Delta^4u(t-2)+\eta\Delta^2u(t-1)-\xi u(t)=\omega(t), & t\in[a+1,b+1]_{\mathbb{Z}},\\ u(a)=\Delta^2u(a-1)=0,\quad u(b+2)=\Delta^2u(b+1)=0\end{cases}$$

has a unique solution $u=\{u(t)\}_{t=a-1}^{b+3}$ with

$$\begin{aligned}u(t)&=\sum_{k=a+1}^{b+1}\left(\sum_{s=a+1}^{b+1}G_1(t,s)G_2(s,k)\right)\omega(k)\\&=\sum_{k=a+1}^{b+1}\left(\sum_{s=a+1}^{b+1}G_2(t,s)G_1(s,k)\right)\omega(k),\quad t\in[a+1,b+1]_{\mathbb{Z}},\end{aligned}$$

and

$$u(a-1)=-u(a+1),\quad u(a)=0,\quad u(b+2)=0,\quad u(b+3)=-u(b+1).$$

Lemma 2.3 ([8], Lemma 2.3)**.** *$K : E \to E$ is a linear continuous operator, furthermore K is symmetric, i.e., $(Ku, v) = (u, Kv)$ for all $u, v \in E$.*

Lemma 2.4 ([8], Lemma 2.4)**.** *The eigenvalues of K are $\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}, \cdots, \frac{1}{\varepsilon_{b-a+1}}$, which have the corresponding normal orthonormal eigenfunctions $e_1, e_2, \cdots, e_{b-a+1}$, where $e_k = \{e_k(t)\}_{t=a+1}^{b+1}$, $e_k(t) = \sqrt{\frac{2}{b-a+2}} \sin \frac{t-a}{b-a+2} k\pi$, $t \in [a+1, b+1]_{\mathbb{Z}}$, $k = 1, 2, \cdots, b-a+1$. In addition, the algebraic multiplicity of each eigenvalue $\frac{1}{\varepsilon_k}$ of the linear operator K is equal to 1.*

By the above Lemma 2.1 and Lemma 2.2, we have $G_i(t, k) > 0$ and $G_i(t, k) = G_i(k, t)$ for all $t, k \in [a+1, b+1]_{\mathbb{Z}}$, $i = 1, 2$. Let

$$G(t, k) = \sum_{s=a+1}^{b+1} G_1(t, s) G_2(s, k), \quad t, k \in [a+1, b+1]_{\mathbb{Z}}$$

and $G(t, k) = G(k, t)$ for all $t, k \in [a+1, b+1]_{\mathbb{Z}}$.

Define operators K, T, $A_\lambda : E \to E$, respectively, by

$$\begin{aligned} Ku(t) &= \sum_{k=a+1}^{b+1} G(t, k) u(k), \quad Tu(t) = f(t, u(t)) + \frac{\mu}{\lambda} g(t, u(t)) + \frac{1}{\lambda} h(u(t)), \quad u \in E,\ t \in [a+1, b+1]_{\mathbb{Z}}; \\ A_\lambda &= \lambda K T. \end{aligned}$$

Since the continuity of f, g, h and that E is a $b-a+1$-dimensional Hilbert space imply that A_λ: $E \to E$ is completely continuous.

Remark 2.5. From Lemma 2.2, we easily know that the fixed point $u = \{u(t)\}_{t=a+1}^{b+1} \in E$ of the A_λ exactly is the solution $u = \{u(t)\}_{t=a-1}^{b+3}$ of (1.1), where $u(a-1) = -u(a+1)$, $u(a) = 0$, $u(b+2) = 0$, $u(b+3) = -u(b+1)$.

Again from Lemma 2.3 and Lemma 2.4, we obtain that $Ku(t) = \frac{1}{\varepsilon} u(t)$, $u \in E$, $t \in [a+1, b+1]_{\mathbb{Z}}$, where $\frac{1}{\varepsilon} = \frac{1}{\varepsilon_k}$, and

$$\varepsilon_k = 2 \cos \frac{2k\pi}{b-a+2} + (2\eta - 8) \cos \frac{k\pi}{b-a+2} + 6 - 2\eta - \xi, \quad k \in [1, b-a+1]_{\mathbb{Z}}$$

is given in [8] and

$$0 < \varepsilon_1 < \cdots < \varepsilon_{b-a+1}. \tag{2.2}$$

Moreover, the operator K has unique inverse operator by $K^{-1} u = \varepsilon u$, $u \in E$, where $\varepsilon = \varepsilon_k$, $k \in [1, b-a+1]_{\mathbb{Z}}$.

Then, for $u \in E$, we define the functionals $\Phi, \Psi : E \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}(K^{-1}u, u) - \sum_{t=a+1}^{b+1} H(u(t)), \quad \Psi(u) = \sum_{t=a+1}^{b+1} \left[F(t, u(t)) + \frac{\mu}{\lambda} G(t, u(t)) \right],$$

where

$$F(t, x) = \int_0^x f(t, s) ds, \quad G(t, x) = \int_0^x g(t, s) ds, \quad H(x) = \int_0^x h(s) ds$$

for every $t \in [a+1, b+1]_{\mathbb{Z}}$ and $x \in \mathbb{R}$.

Put $I_\lambda(u) = \Phi(u) - \lambda \Psi(u)$. Obviously, for every $\lambda > 0$, I_λ is continuously Frêchet differentiable function, whose differential at the point $u \in E$ is

$$\begin{aligned} I'_\lambda(u) = \Phi'(u) - \lambda \Psi'(u) &= K^{-1}u - \sum_{t=a+1}^{b+1} h(u(t)) - \lambda \sum_{t=a+1}^{b+1} \left[f(t, u(t)) + \frac{\mu}{\lambda} g(t, u(t)) \right] \\ &= K^{-1}u - \lambda T u. \end{aligned}$$

Remark 2.6. Since the operator equation $u = A_\lambda u$ is equivalent to the operator equation $K^{-1}u = \lambda Tu$. Then, we know that every critical point of the functional $I_\lambda(u)$ in E is a solution of (1.1). Therefore, it will be enough only to find the critical point of the functional I_λ in E.

Now, let us give the main tools we will use.

Theorem 2.7 ([6], Theorem 2.6)**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and satisfies $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that*

(i) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(ii) for each $\lambda \in \Lambda_r := \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right)$, *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.8 ([4], Theorem 2.1)**.** *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semi-continuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{\nu\in\Phi^{-1}((-\infty,r))}\Psi(\nu) - \Psi(u)}{r - \Phi(u)},$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, we have

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, (1/\varphi(r)))$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$ then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minimum) of I_λ such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$ then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(c_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(c_2) there is a sequence of pairwise distinct critical points (local minimum) of I_λ which converges to a global minimum of Φ.

3. Existence of three solutions of (1.1)

($A1$) Suppose that $f, g : [a+1, b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions;

($A2$) $h : \mathbb{R} \to \mathbb{R}$ is a strictly monotone Lipschitz continuous function with Lipschitzian constant L satisfying $0 \le L < \frac{\varepsilon_1}{2}$, i.e.,

$$|h(t_1) - h(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$ and $h(0) = 0$.

Set

$$
\begin{aligned}
F^c &= \sum_{t=a+1}^{b+1} \max_{|x|\le c} F(t,x), \quad G^c = \sum_{t=a+1}^{b+1} \max_{|x|\le c} G(t,x) \ \ \forall c>0,\\
F_d &= \sum_{t=a+1}^{b+1} F(t,d), \quad G_d = \sum_{t=a+1}^{b+1} G(t,d) \ \ \forall d>0,\\
\sigma_1 &= \left|\frac{c^2(\varepsilon_1 - L) - 2\lambda F^c}{2G^c}\right|,\\
\sigma_2 &= \left|\frac{(\varepsilon_{b-a+1} + L)(b-a+1)d^2 - 2\lambda F_d}{2\min\{0, G_d\}}\right|,\\
\sigma_3 &= \frac{\varepsilon_1}{\max\left\{0, 4\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} G(t,x)}{|x|^2}\right\}},\\
\sigma &= \min\{\sigma_1, \sigma_2, \sigma_3\}.
\end{aligned}
\tag{3.1}
$$

Suppose $(A1)$, $(A2)$ hold, we have the following theorem.

Theorem 3.1. *Assume that there exist two positive constants c, d with $c^2 < (b-a+1)d^2$, such that*

(H1) $f(t,x) \ge 0$ for each $(t,x) \in [a+1, b+1]_{\mathbb{Z}} \times [-S, S]$ and $f(t,x) \not\equiv 0$ on $[a+1,b+1]_{\mathbb{Z}} \times [0,S]$, where $S = \max\{c,d\}$;

(H2) $\frac{F^c}{c^2} < \frac{(\varepsilon_1 - L)F_d}{(\varepsilon_{b-a+1}+L)(b-a+1)d^2}$;

(H3) $\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} F(t,x)}{|x|^2} < \frac{F^c(\varepsilon_1 - 2L)}{2c^2(\varepsilon_1 - L)}$;

(H4) $\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} G(t,x)}{|x|^2} < +\infty$.

Then, for every $\lambda \in \Lambda := \left(\frac{(\varepsilon_{b-a+1}+L)(b-a+1)d^2}{2F_d}, \frac{(\varepsilon_1 - L)c^2}{2F^c}\right)$ and each $\mu \in [0, \sigma)$, the problem (1.1) has at least three solutions.

Proof. First, since condition $(H2)$, we can affirm that Λ is nonempty. Then, let us prove that functionals Φ and Ψ satisfy the conditions of Theorem 2.7. According to the definitions of the Φ, we easily know that Φ and Ψ are both continuously Gâteaux differentiable functional. Moreover, since K^{-1} has a inverse and h is strictly monotone, we get that the Gâteaux derivative of Φ admits a continuous inverse on E^*. Now, we claim that the Gâteaux derivative of Ψ is compact. In fact, let Ω be a bounded subset of E, i.e., there is a constant $M_1 > 0$ such that $\|u\| \le M_1$, for all $u \in \Omega$. Combining with the continuity of f, g for $u \in \Omega$, we have

$$
\begin{aligned}
|\Psi'(u)| &\le \sum_{t=a+1}^{b+1} \left| f(t,u(t)) + \frac{\mu}{\lambda} g(t,u(t)) \right|\\
&\le (b-a+1)\left(\max_{t\in[a+1,b+1]_{\mathbb{Z}}, -M_1 \le u \le M_1} f(t,u) + \frac{\mu}{\lambda} \max_{t\in[a+1,b+1]_{\mathbb{Z}}, -M_1\le u\le M_1} g(t,u) \right),
\end{aligned}
$$

which implies that $\Psi'(u)$ is uniformly bounded.

From the continuity of f, g, then, f, g are uniformly continuous for $(t,u) \in [a+1,b+1]_{\mathbb{Z}} \times [-M_1, M_1]$. Thus, for $\forall \epsilon > 0$, there exists $\delta > 0$ for $(t_1,u), (t_2,u) \in [a+1,b+1]_{\mathbb{Z}} \times [-M_1, M_1]$ and $|t_1 - t_2| < \delta$, we have

$$
|f(t_1,u) - f(t_2,u)| < \epsilon, \quad |g(t_1,u) - g(t_2,u)| < \epsilon.
$$

Then, for $\forall u \in E$, we obtain

$$\begin{aligned}\left|\Psi'(u(t_1)) - \Psi'(u(t_2))\right| &\leq \sum_{t=a+1}^{b+1} \left(|f(t_1,u) - f(t_2,u)| + \frac{\mu}{\lambda}|g(t_1,u) - g(t_2,u)|\right)\\ &\leq (b-a+1)(1+\frac{\mu}{\lambda})\epsilon.\end{aligned}$$

Namely, $\Psi'(u)$ is equicontinuous. According to the Arzela-Ascoli Theorem, we easily show that the Gâteaux derivative of Ψ is compact. And Φ, Ψ satisfy $\Phi(0) = \Psi(0) = 0$. Thus, we have $\Phi - \lambda\Psi \in \mathcal{C}^1(E,\mathbb{R})$ and the critical points of $\Phi - \lambda\Psi$ are exactly the solutions of problem (1.1) for $\lambda \in \Lambda$. Moreover, we can see Ψ is continuous, so Ψ is sequentially weakly upper semi-continuous. And We can assert that Φ is sequentially weakly lower semi-continuous. As a matter of fact, owing to $\sum_{t=a+1}^{b+1} H(u(t))$ is continuous, we putting $M(u) = \sum_{t=a+1}^{b+1} H(u(t))$ for $u \in E$. For any $u_n \in E$ with $u_n \to u$ weakly in E. Since the inner product is sequentially weakly lower semi-continuous in Banach Space. Then, we have

$$\liminf_{n\to\infty} \Phi(u_n) = \liminf_{n\to\infty} \frac{(K^{-1}u_n, u_n)}{2} - \lim_{n\to\infty} M(u_n) \geq \frac{(K^{-1}u,u)}{2} - M(u) = \Phi(u).$$

Next, we show that Φ is coercive. Consider that $(A2)$ and combining (2.2), we have

$$\begin{aligned}\Phi(u) = \frac{(K^{-1}u,u)}{2} - \sum_{t=a+1}^{b+1} H(u(t)) &\geq \frac{\varepsilon_1}{2}\|u\|^2 - \sum_{t=a+1}^{b+1} |H(u(t))|\\ &\geq \frac{\varepsilon_1}{2}\|u\|^2 - \sum_{t=a+1}^{b+1}\left(\int_0^{u(t)} |h(x) - h(0)|dx\right) \geq \frac{\varepsilon_1}{2}\|u\|^2 - \sum_{t=a+1}^{b+1} L\int_0^{u(t)} |x|dx \\ &\geq \frac{\varepsilon_1}{2}\|u\|^2 - \frac{L}{2}\sum_{t=a+1}^{b+1} |u(t)|^2 = \frac{\varepsilon_1 - L}{2}\|u\|^2,\end{aligned} \tag{3.2}$$

which owing to (A2) with the fact that $L < \frac{\varepsilon_1}{2}$, we clearly observe that $L < \varepsilon_1$. Therefore, the functionals Φ and Ψ, as required in Theorem 2.7, are verified.

In addition, using the same method, we obtain

$$\Phi(u) = \frac{(K^{-1}u,u)}{2} - \sum_{t=a+1}^{b+1} H(u(t)) \leq \frac{\varepsilon_{b-a+1}}{2}\|u\|^2 + \sum_{t=a+1}^{b+1} |H(u(t))| \leq \frac{\varepsilon_{b-a+1} + L}{2}\|u\|^2.$$

Set

$$r = \frac{\varepsilon_1 - L}{2}c^2.$$

Then, from (2.1), (3.2), we have

$$\begin{aligned}\sup_{\Phi(u)\leq r} \Psi(u) &= \sup_{\Phi(u)\leq r} \sum_{t=a+1}^{b+1} \left[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\right]\\ &\leq \sum_{t=a+1}^{b+1} \max_{|x|\leq c} F(t,x) + \frac{\mu}{\lambda}\sum_{t=a+1}^{b+1} \max_{|x|\leq c} G(t,x)\\ &= F^c + \frac{\mu}{\lambda}G^c,\end{aligned}$$

which implies

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq \frac{2}{c^2(\varepsilon_1 - L)}\left(F^c + \frac{\mu}{\lambda}G^c\right).$$

If $G^c \le 0$, by the fact that

$$\lambda < \frac{(\varepsilon_1 - L)c^2}{2F^c},$$

we obtain

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} \le \frac{2F^c}{(\varepsilon_1 - L)c^2} < \frac{1}{\lambda}. \tag{3.3}$$

If $G^c > 0$, by (3.1) and the facts $\mu < \sigma$ and $\lambda \in \Lambda$, we get

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{2F^c}{(\varepsilon_1 - L)c^2} + \frac{2G^c}{(\varepsilon_1 - L)c^2} \frac{\left|\frac{(\varepsilon_1 - L)c^2 - 2\lambda F^c}{2G^c}\right|}{\lambda} = \frac{1}{\lambda}. \tag{3.4}$$

Choosing $\bar{u} = d$, we assert that $\Phi(\bar{u}) > r$. In fact, we consider the condition $(A2)$, combining $c^2 < (b-a+1)d^2$, we obtain

$$\begin{aligned}
\Phi(\bar{u}) = \frac{(K^{-1}\bar{u}, \bar{u})}{2} - \sum_{t=a+1}^{b+1} H(\bar{u}(t)) &\ge \frac{\varepsilon_1}{2}\|\bar{u}\|^2 - \sum_{t=a+1}^{b+1} |H(\bar{u}(t))| \\
&\ge \frac{\varepsilon_1}{2}(b-a+1)d^2 - \sum_{t=a+1}^{b+1} \left(\int_0^d |h(x) - h(0)| dx\right) \\
&\ge \frac{\varepsilon_1}{2}(b-a+1)d^2 - \sum_{t=a+1}^{b+1} L \int_0^d |x| dx \\
&= \frac{\varepsilon_1 - L}{2}(b-a+1)d^2 > \frac{\varepsilon_1 - L}{2}c^2 = r.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\frac{\Psi(\bar{u})}{\Phi(\bar{u})} &= \frac{\sum_{t=a+1}^{b+1}[F(t,d) + (\mu/\lambda)G(t,d)]}{\frac{(K^{-1}\bar{u},\bar{u})}{2} - \sum_{t=a+1}^{b+1} H(d)} \\
&\ge \frac{\sum_{t=a+1}^{b+1}[F(t,d) + (\mu/\lambda)G(t,d)]}{\frac{\varepsilon_{b-a+1}+L}{2}(b-a+1)d^2} \\
&= \frac{2F_d}{(\varepsilon_{b-a+1}+L)(b-a+1)d^2} + \frac{2\mu G_d}{\lambda(\varepsilon_{b-a+1}+L)(b-a+1)d^2}.
\end{aligned}$$

If $G_d \ge 0$, it follows that

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \ge \frac{2F_d}{(\varepsilon_{b-a+1}+L)(b-a+1)d^2} > \frac{1}{\lambda}. \tag{3.5}$$

If $G_d < 0$, owing to $\mu < \sigma$ and $\lambda \in \Lambda$, we also get

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} > \frac{2F_d}{(\varepsilon_{b-a+1}+L)(b-a+1)d^2} + \frac{\left|\frac{(\varepsilon_{b-a+1}+L)(b-a+1)d^2 - 2\lambda F_d}{2G_d}\right|}{\lambda(\varepsilon_{b-a+1}+L)(b-a+1)d^2} 2G_d = \frac{1}{\lambda}. \tag{3.6}$$

Therefore, from (3.3), (3.4), (3.5) and (3.6), the condition (i) of Theorem 2.7 holds.

Now, we prove that the functional $\Phi - \lambda\Psi$ is coercive. The proof we will divide into two cases.

Case 1. When

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} F(t,x)}{|x|^2} > 0.$$

From $(H3)$, there exists $\alpha > 0$ such that

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} F(t,x)}{|x|^2} < \alpha < \frac{F^c(\varepsilon_1 - 2L)}{2c^2(\varepsilon_1 - L)}.$$

Then, for each $(t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}$, there exists a positive constant β_α such that

$$\sum_{t=a+1}^{b+1} F(t,x) \le \alpha |x|^2 + \beta_\alpha.$$

Combining (2.1) with the fact that $\lambda < \frac{(\varepsilon_1 - L)c^2}{2F^c}$, we get, for each $u \in E$

$$\lambda \sum_{t=a+1}^{b+1} F(t,u(t)) \le \lambda\alpha |u(t)|^2 + \lambda\beta_\alpha \le \frac{(\varepsilon_1 - L)c^2\alpha}{2F^c}\|u\|^2 + \frac{(\varepsilon_1 - L)c^2}{2F^c}\beta_\alpha. \tag{3.7}$$

Moreover, due to $\mu < \sigma$, we have

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} G(t,x)}{|x|^2} < \frac{\varepsilon_1}{4\mu}.$$

Furthermore, for every $(t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}$, there exists a positive constant δ_μ such that

$$\sum_{t=a+1}^{b+1} G(t,x) \le \frac{\varepsilon_1}{4\mu}|x|^2 + \delta_\mu.$$

By (2.1), for every $u \in E$, it follows that

$$\sum_{t=a+1}^{b+1} G(t,u(t)) \le \frac{\varepsilon_1}{4\mu}|u(t)|^2 + \delta_\mu \le \frac{\varepsilon_1}{4\mu}\|u\|^2 + \delta_\mu. \tag{3.8}$$

Then, by (3.7) and (3.8), we obtain

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}(K^{-1}u,u) - \sum_{t=a+1}^{b+1} H(u(t)) - \lambda \sum_{t=a+1}^{b+1}\left[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\right] \\
&\ge \frac{\varepsilon_1 - L}{2}\|u\|^2 - \frac{(\varepsilon_1 - L)c^2\alpha}{2F^c}\|u\|^2 - \frac{(\varepsilon_1 - L)c^2\beta_\alpha}{2F^c} - \frac{\varepsilon_1}{4}\|u\|^2 - \mu\delta_\mu \\
&= \frac{F^c(\varepsilon_1 - 2L) - 2(\varepsilon_1 - L)c^2\alpha}{4F^c}\|u\|^2 - \frac{(\varepsilon_1 - L)c^2\beta_\alpha}{2F^c} - \mu\delta_\mu.
\end{aligned}$$

Case 2. When

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} F(t,x)}{|x|^2} \le 0.$$

Then, there is a positive constant ϑ such that $\sum_{t=a+1}^{b+1} F(t,x) \le \vartheta$ for $(t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}$, then

$$\lambda \sum_{t=a+1}^{b+1} F(t,x) \le \frac{(\varepsilon_1 - L)c^2\vartheta}{2F^c}.$$

We also get

$$\Phi(u) - \lambda\Psi(u) \ge \frac{\varepsilon_1 - 2L}{4}\|u\|^2 - \frac{(\varepsilon_1 - L)c^2\vartheta}{2F^c} - \mu\delta_\mu.$$

Therefore, by the fact that $L < \frac{\varepsilon_1}{2}$, $\Phi - \lambda\Psi$ is coercive. The condition (ii) of Theorem 2.7 holds. Moreover, we have

$$\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right).$$

Hence, the functional $\Phi - \lambda\Psi$ has at least three critical points. $\square$

4. Existence of infinitely many solutions of (1.1)

$(A3)$ Suppose that f is continuous function and for every continuous function g, whose potential $G(t,x) = \int_0^x g(t,s)ds$ is non-negative function for each $(t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}$.

$(A4)$ $h : \mathbb{R} \to \mathbb{R}$ satisfies the assumption $(A2)$ in Section 3. However, in this section, the Lipschitzian constant L only needs to satisfy $0 \le L < \varepsilon_1$.

Set

$$G_\infty := \liminf_{s\to+\infty} \frac{\sum_{t=a+1}^{b+1} \sup_{|x|\le s} G(t,x)}{s^2}, \quad F_\infty := \liminf_{s\to+\infty} \frac{\sum_{t=a+1}^{b+1} \sup_{|x|\le s} F(t,x)}{s^2},$$

$$F^\infty := \limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} F(t,x)}{|x|^2}, \quad \kappa := \frac{\varepsilon_1 - L}{\varepsilon_1 + L},$$

$$\lambda_1 := \frac{\varepsilon_1 + L}{2F^\infty}, \quad \lambda_2 := \frac{\varepsilon_1 - L}{2F_\infty + L}, \quad \mu_{g,\lambda} := \lambda \frac{L}{2G_\infty}.$$

Theorem 4.1. *Assume that (A3) and (A4) hold and*

(H5) $G_\infty < +\infty$*;*

(H6) $F_\infty + \frac{L}{2} < \kappa F^\infty$ *and* $F_\infty \ge -\frac{L}{2}$.

Then, for any $\lambda \in \Lambda := (\lambda_1, \lambda_2)$ *and each* $\mu \in [0, \mu_{g,\lambda})$*, problem* (1.1) *admits an unbounded sequence of solutions.*

Proof. We will apply Theorem 2.8 (b) to our problem. First, since condition $(H6)$, we clearly observe that the interval (λ_1, λ_2) is non-empty. So we can fix $\bar\lambda$ in (λ_1, λ_2) and let g be a continuous function satisfying condition $(H5)$. Due to $\bar\lambda > \lambda_1 > 0$, we have $\mu_{g,\bar\lambda} \ge 0$. Then, we fix again $\bar\mu \in [0, \mu_{g,\bar\lambda})$ as in the conclusion.

We will divide the proof into three steps.

Step 1. Let us verify that the functionals Φ and Ψ satisfy the conditions of Theorem 2.8. From Section 3, the functionals Φ and Ψ, as required in Theorem 2.8, are easily verified. Thus, we have $I_{\bar\lambda} = \Phi - \bar\lambda\Psi \in \mathcal{C}^1(E,\mathbb{R})$ and the critical points of $I_{\bar\lambda}$ are exactly the solutions of problem (1.1) for the fixed $\bar\lambda \in (\lambda_1, \lambda_2)$.

Step 2. We prove $\gamma < +\infty$ and $(\lambda_1, \lambda_2) \subseteq (0, 1/\gamma)$. Let $\{s_n\}$ be a real sequence for all $n \in \mathbb{N}$ and $s_n \to +\infty$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \frac{\sum_{t=a+1}^{b+1} \sup_{|x|\le s_n} \left[F(t,x) + (\bar\mu/\bar\lambda)G(t,x)\right]}{s_n^2} = \liminf_{s\to+\infty} \frac{\sum_{t=a+1}^{b+1} \sup_{|x|\le s} \left[F(t,x) + (\bar\mu/\bar\lambda)G(t,x)\right]}{s^2}. \quad (4.1)$$

Set

$$r_n = \frac{\varepsilon_1 - L}{2} s_n^2$$

for all $n \in \mathbb{N}$. Moreover, by (2.1), we get $\max_{t\in[a+1,b+1]_{\mathbb{Z}}} |u(t)| \le s_n$, if $\Phi(u) < r_n$. Then, by the fact $\Phi(0) = \Psi(0) = 0$, we have

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\sup_{\nu\in\Phi^{-1}((-\infty,r_n))} \Psi(\nu) - \Psi(u)}{\frac{\varepsilon_1-L}{2}s_n^2 - \Phi(u)} \\
&\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_n))} \Psi(u) - \Psi(0)}{\frac{\varepsilon_1-L}{2}s_n^2 - \Phi(0)} \\
&= \frac{\sup_{u\in\Phi^{-1}((-\infty,r_n))} \sum_{t=a+1}^{b+1} \left[F(t,u(t)) + (\bar\mu/\bar\lambda)G(t,u(t))\right]}{\frac{\varepsilon_1-L}{2}s_n^2} \\
&\le \frac{2}{\varepsilon_1 - L}\left[\frac{\sum_{t=a+1}^{b+1} \sup_{|x|\le s_n} F(t,x)}{s_n^2} + \frac{\bar\mu}{\bar\lambda}\frac{\sum_{t=a+1}^{b+1} \sup_{|x|\le s_n} G(t,x)}{s_n^2}\right].
\end{aligned}$$

Therefore, from the assumptions $(H5)$ and $(H6)$, and by the definition of γ, as well as combining (4.1), we have

$$\gamma = \liminf_{n\to+\infty} \varphi(r_n) \le \lim_{n\to\infty} \varphi(r_n) \le \frac{2}{\varepsilon_1 - L}\left(F_\infty + (\bar{\mu}/\bar{\lambda})G_\infty\right) < \infty. \tag{4.2}$$

Hence, by Inequality (4.2), and since $\bar{\mu} \in [0, \mu_{g,\bar{\lambda}})$, we obviously get

$$\frac{1}{\gamma} \ge \frac{\varepsilon_1 - L}{2\left(F_\infty + (\bar{\mu}/\bar{\lambda})G_\infty\right)} \ge \frac{\varepsilon_1 - L}{2\left(F_\infty + (\mu_{g,\bar{\lambda}}/\bar{\lambda})G_\infty\right)} = \lambda_2. \tag{4.3}$$

Since $G(t,x)$ is non-negative and $\bar{\lambda} > 0$, $\bar{\mu} \ge 0$, we obtain

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1}\left[F(t,x) + (\bar{\mu}/\bar{\lambda})G(t,x)\right]}{|x|^2} \ge \limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1} F(t,x)}{|x|^2}. \tag{4.4}$$

Therefore, from (4.4), we have

$$\lambda_1 = \frac{\varepsilon_1 + L}{2F^\infty} \ge \frac{\varepsilon_1 + L}{2\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1}\left[F(t,x)+(\bar{\mu}/\bar{\lambda})G(t,x)\right]}{|x|^2}}. \tag{4.5}$$

Hence, by (4.3) and (4.5), we have

$$\bar{\lambda} \in (\lambda_1, \lambda_2) \subseteq \left(\frac{\varepsilon_1 + L}{2\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1}\left[F(t,x)+(\bar{\mu}/\bar{\lambda})G(t,x)\right]}{|x|^2}}, \frac{\varepsilon_1 - L}{2\left(F_\infty + (\bar{\mu}/\bar{\lambda})G_\infty\right)} \right) \subseteq \left(0, \frac{1}{\gamma}\right).$$

Step 3. We will prove that the restrict functional $I_{\bar{\lambda}} = \Phi - \bar{\lambda}\Psi$ is unbounded. Since $\bar{\lambda} > \lambda_1$, then

$$\begin{aligned}\frac{1}{\bar{\lambda}} < \frac{1}{\lambda_1} &= \frac{2}{\varepsilon_1 + L}F^\infty \\ &\le \frac{2}{\varepsilon_1 + L}\limsup_{|x|\to+\infty} \frac{\sum_{t=a+1}^{b+1}\left[F(t,x) + (\bar{\mu}/\bar{\lambda})G(t,x)\right]}{|x|^2}.\end{aligned}$$

Now, we can consider a real sequence $\{d_n e_1\}$ and a positive constant θ, such that $\|d_n e_1\| = d_n \to +\infty$ as $n \to \infty$ and

$$\begin{aligned}\frac{1}{\bar{\lambda}} < \theta < \frac{2}{\varepsilon_1 + L} &\frac{\sum_{t=a+1}^{b+1}\left[F(t, d_n e_1(t)) + (\bar{\mu}/\bar{\lambda})G(t, d_n e_1(t))\right]}{\|d_n e_1\|^2} \\ = \frac{2}{\varepsilon_1 + L} &\frac{\sum_{t=a+1}^{b+1}\left[F(t, d_n e_1(t)) + (\bar{\mu}/\bar{\lambda})G(t, d_n e_1(t))\right]}{d_n^2},\end{aligned}$$

where e_1 is given in Lemma 2.4, $e_1(t) > 0$ for each $t \in [a+1, b+1]$ and $\|e_1\| = 1$. Furthermore,

$$\frac{\varepsilon_1 + L}{2}\theta\bar{\lambda} < \bar{\lambda}\frac{\sum_{t=a+1}^{b+1}\left[F(t, d_n e_1(t)) + (\bar{\mu}/\bar{\lambda})G(t, d_n e_1(t))\right]}{d_n^2} \tag{4.6}$$

for all $n \in \mathbb{N}$ large enough. Set

$$w_n(t) := d_n e_1(t), \quad t \in [a+1, b+1]_{\mathbb{Z}}. \tag{4.7}$$

Obviously, $w_n \in E$.

Furthermore, from (A2), as well as $\|e_1\| = 1$, using the same method, we obtain

$$\begin{aligned}\Phi(w_n) &= \frac{(K^{-1}(d_n e_1), d_n e_1)}{2} - \sum_{t=a+1}^{b+1} H(d_n e_1(t)) \\ &\le \frac{\varepsilon_1}{2} d_n^2 + \sum_{t=a+1}^{b+1} |H(d_n e_1(t))| \\ &\le \frac{\varepsilon_1}{2} d_n^2 + \sum_{t=a+1}^{b+1} \int_0^{d_n e_1(t)} |h(x) - h(0)| dx \\ &\le \frac{\varepsilon_1}{2} d_n^2 + \frac{L}{2} d_n^2 \|e_1\|^2 = \frac{\varepsilon_1 + L}{2} d_n^2.\end{aligned} \tag{4.8}$$

Then, from (4.6) and (4.8), we obtain for all $n \in \mathbb{N}$ large enough,

$$\begin{aligned}I_{\bar\lambda}(w_n) &= (\Phi - \bar\lambda\Psi)(w_n) = \Phi(w_n) - \bar\lambda\Psi(w_n) \\ &\le \frac{\varepsilon_1 + L}{2} d_n^2 - \bar\lambda \sum_{t=a+1}^{b+1} F(t, d_n e_1(t)) - \bar\mu \sum_{t=a+1}^{b+1} G(t, d_n e_1(t)) \\ &= \left(\frac{\varepsilon_1 + L}{2} - \bar\lambda \frac{\sum_{t=a+1}^{b+1} \left[F(t, d_n e_1(t)) + (\bar\mu/\bar\lambda) G(t, d_n e_1(t))\right]}{d_n^2} \right) d_n^2 \\ &< \frac{\varepsilon_1 + L}{2} (1 - \theta\bar\lambda) d_n^2.\end{aligned}$$

Thus, $\Phi - \bar\lambda\Psi$ is unbounded from below and it has no global minimum. Hence, using Theorem 2.8, we verify that there is a sequence $\{u_n\} \subset E$ of critical points of $I_{\bar\lambda} = \Phi - \bar\lambda\Psi$ such that $\lim_{n\to\infty} \|u_n\| = +\infty$. Then, since Φ is coercive, we obtain $\lim_{n\to+\infty} \Phi(u_n) = +\infty$. Therefore, we establish the Theorem 4.1 holds. □

Corollary 4.2. *Assume that* (H5) *and* (H6) *hold, and* $F_\infty = 0$, $F^\infty = +\infty$ *and* $G_\infty > 0$. *Then, for every* $\lambda \in (0, \frac{\varepsilon_1 - L}{L})$, *and for each* $\mu \in [0, \frac{\lambda L}{2G_\infty})$, *the problem* (1.1) *possesses infinitely many weak solutions in* E.

In particular, when $F_\infty = -\frac{L}{2}$ and $G_\infty = 0$ in Corollary 4.2. Thus, for each $\lambda > 0$ and $\mu \ge 0$, the result of Theorem 4.1 holds again.

Corollary 4.3. *Assume that* (H5) *and* (H6) *hold, and* $L = 0$. *Then, for each* $\lambda \in (\frac{\varepsilon_1}{2F^\infty}, \frac{\varepsilon_1}{2F_\infty})$ *and* $\mu = 0$, *we have the following problem*

$$\begin{cases} \Delta^4 u(t-2) + \eta\Delta^2(u(t-1)) - \xi u(t) = \lambda f(t, u(t)) + h(u(t)), & t \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \quad u(b+2) = \Delta^2 u(b+1) = 0, \end{cases} \tag{4.9}$$

then, the result of Theorem 4.1 *holds again.*

5. Example

Example 5.1. Let $\eta = \xi = 0$, $a = 0$ and $b = 5$. Consider the following discrete fourth-order problem

$$\begin{cases} \Delta^4 u(t-2) = \lambda f(t, u(t)) + \mu g(t, u(t)) + h(u(t)), & t \in [1, 6]_{\mathbb{Z}}, \\ u(0) = \Delta^2 u(-1), \quad u(7) = \Delta^2 u(6) \end{cases} \tag{5.1}$$

for every $t \in [1, 6]_{\mathbb{Z}}$ and $x \in \mathbb{R}$, where

$$f(t, x) = 2x^2(2 - \frac{x}{2}), \quad g(t, x) = \frac{tx}{21}, \quad h(x) = \arctan 0.002x.$$

Taking into account that $L = 0.002$. By simple computation, we get $\varepsilon_1 = 16\sin^4\frac{\pi}{14}$ and $\varepsilon_6 = 16\sin^4\frac{3\pi}{7}$. Now, we take $c = \frac{1}{2^{12}}$ and $d = 4$. Clearly, $f(t,x) \geq 0$ for $t \in [1,6]_{\mathbb{Z}}$, $x \in [-d,d]$ and $f(t,x) \not\equiv 0$ for $(t,x) \in [1,6]_{\mathbb{Z}} \times [0,d]$. Then

$$\frac{F^c}{c^2} = c(8-\frac{3}{2}c) = \frac{1}{2^9} - \frac{3}{2^{25}}, \qquad \frac{F_d(\varepsilon_1 - L)}{6d^2(\varepsilon_6 + L)} = 0.0034.$$

The condition $(H2)$ of Theorem 3.1 holds. Obviously,

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=1}^{6} F(t,x)}{|x|^2} = \limsup_{|x|\to+\infty} x(8-\frac{3}{2}x) = -\infty < \frac{F^c(\varepsilon_1 - 2L)}{2c^2(\varepsilon_1 - L)}.$$

Thus, the condition $(H3)$ of Theorem 3.1 holds. Moreover,

$$\limsup_{|x|\to+\infty} \frac{\sum_{t=1}^{6} G(t,x)}{|x|^2} = \frac{1}{2} < +\infty$$

satisfies the condition $(H4)$ of Theorem 3.1. Moreover, we get $\sigma_1 = 0.0372 - 0.0039\lambda$ and $\sigma_3 = 0.0196$. We take $\sigma = \min\{\sigma_1, \sigma_3\}$, then for $\lambda \in \Lambda := (5.4213, 9.5306)$ and $\mu \in [0,\sigma)$, we get the problem (5.1) has at least three solutions.

In Example 5.1, if we put

$$f(t,x) = \frac{1}{2}tx + \frac{5}{9}tx\sin(\frac{\pi}{6} - \ln|x|),$$

by direct computation, we obtain $\kappa = 0.903$, and g satisfy the hypothesis $(A3)$, $G_\infty = \frac{1}{2}$, as well as

$$\liminf_{s\to+\infty} \frac{\sum_{t=1}^{6} \sup_{|x|\leq s} F(t,x)}{s^2} + \frac{L}{2} = 5.25 - \frac{7}{3}\sqrt{5} + 0.001,$$
$$\limsup_{|x|\to+\infty} \frac{\sum_{t=1}^{6} F(t,x)}{|x|^2} = 5.25 + \frac{7}{3}\sqrt{5}.$$

Then, the conditions of Theorem 4.1 holds. Hence, for each $\lambda \in (0.002, 0.555)$ and for every $\mu \in [0, 0.002\lambda)$, we obtain that problem (5.1) possesses an unbounded sequence of solutions.

Acknowledgements

The authors are very appreciated to the reviewer for his/her valuable suggestions. The second author is supported by NSFC(11401479,71561024,11301059), Natural Science Foundation of Gansu Province (145RJYA237), China Postdoctoral Science Foundation (2014M562472) and the Postdoctoral Science Foundation of Gansu Province.

References

[1] D. R. Anderson, F. Minhós, *A discrete fourth-order Lidstone problem with parameters*, Appl. Math. Comput., **214** (2009), 523–533. 1

[2] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York, (2000). 1

[3] R. P. Agarwal, P. J. Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic Publishers, Dordrecht, (1997). 1

[4] G. Bonanno, G. M. Bisci, *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl., **2009** (2009), 20 pages. 1, 2.8

[5] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations, **244** (2008), 3031–3059. 1

[6] G. Bonanno, S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal., **89** (2010), 1–10. 1, 2.7

[7] X. Deng, *Nonexistence and existence results for a class of fourth-order difference mixed boundary value problems*, J. Appl. Math. Comput., **45** (2014), 1–14. 1

[8] T. He, Y. Su, *On discrete fourth-order boundary value problems with three parameters*, J. Comput. Appl. Math., **233** (2010), 2506–2520. 1, 1, 1, 2, 2.1, 2.2, 2.3, 2.4, 2

[9] Z. He, J. Yu, *On the existence of positive solutions of fourth-order difference equations*, Appl. Math. Comput., **161** (2005), 139–148. 1

[10] J. Henderson, *Positive solutions for nonlinear difference equations*, Nonlinear Stud., **4** (1997), 29–36. 1

[11] X. Liu, Y. Zhang, H. Shi, X. Deng, *Periodic and subharmonic solutions for fourth-order nonlinear difference equations*, Appl. Math. Comput., **236** (2014), 613–620. 1

[12] R. Ma, C. Gao, *Bifurcation of positive solutions of a nonlinear discrete fourth-order boundary value problem*, Z. Angew. Math. Phys., **64** (2013), 493–506. 1

[13] R. Ma, C. Gao, Y. Chang, *Existence of solutions of a discrete fourth-order boundary value problem*, Discrete Dyn. Nat. Soc., **2010** (2010), 19 Pages. 1

[14] R. Ma, J. Li, C. Gao, *Existence of positive solutions of a discrete elastic beam equation*, Discrete Dyn. Nat. Soc., **2010** (2010), 15 Pages. 1

[15] R. Ma, Y. Xu, *Existence of positive solution for nonlinear fourth-order difference equations*, Comput. Math. Appl., **59** (2010), 3770–3777. 1

[16] M. K. Moghadam, S. Heidarkhani, J. Henderson, *Infinitely many solutions for perturbed difference equations*, J. Difference Equ. Appl., **20** (2014), 1055–1068. 1

[17] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401–410. 1

[18] Y. Xu, C. Gao, R. Ma, *Solvability of a nonlinear fourth-order discrete problem at resonance*, Appl. Math. Comput., **216** (2010), 662–670. 1

[19] B. Zhang, L. Kong, Y. Sun, X. Deng, *Existence of positive solutions for BVPs of fourth-order difference equations*, J. Appl. Math. Comput., **131** (2002), 583–591. 1