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Improved version of perturbed Ostrowski type inequalities for n-times differentiable mappings with three-step kernel and its application

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Abstract

New integral inequalities of Ostrowski type are developed for n-times differentiable mappings by using a 3-step kernel when either $f^{(n)} \in L^1[a,b]$ or $f \in L^2[a,b]$. Some new inequalities are derived as special cases of the obtained inequalities. New efficient quadrature rules are also derived with the help of obtained inequalities. The efficiency of the new quadrature rules is demonstrated with the help of specific examples. Finally, applications for cumulative distribution functions is also provided. ©2016 All rights reserved.

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1. Introduction

In 1938, Ostrowski [16] proved a very important inequality, which states that if $f : [a,b] \to \mathbb{R}$ is a differentiable mapping on (a,b) with $\left|f'(x)\right| \leq M$ for all $x \in (a,b)$, then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,M, \tag{1.1}$$

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for all $x \in [a, b]$.

The following result is an extension of the result (1.1) given by Dragomir and Wang [4, 5] for absolutely continuous functions such that f' belongs to $L_p[a, b]$, $1 \le p < \infty$.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$, where p, q are real numbers greater than 1, we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \begin{cases} \frac{1}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right) (b-a)^{\frac{1}{q}} \left\| f' \right\|_p , \\ \frac{1}{b-a} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1 . \end{cases}$$

In [7], Guessab et al. proved some companion of Ostrowski-type inequalities using Lipschitz conditions. Alomari [1], studied and generalized a companion of Ostrowski inequalities, which were proved in [7]. In [13], Liu defined 3-step quadratic kernel and established some Ostrowski type integral inequalities for the functions whose first derivatives are absolutely continuous and second derivatives belong to L^p $(1 \le p \le \infty)$ spaces. The Ostrowski inequality (1.1) has been generalized, extended and refined in different ways; the reader may refer to [6]–[24].

A generalization of Ostrowski's result for n-times differentiable mappings was given by Milovanović and Pečarić in [14, p. 468]. Cerone and Dragomir [3] provided another generalization of the Ostrowski inequality for n-times differentiable functions for two sections of kernel by using the following lemma.

Lemma 1.2 ([2])**.** *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ the following identity holds,*

$$\int_a^b K_n(x,t) f^{(n)}(t)dt = (-1)^{n+1} \left(\sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) - \int_a^b f(t)dt \right),$$

where the kernel $K_n(x, \cdot) : [a, b] \to \mathbb{R}$ is given by

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!}, & \text{if} \quad a \le t \le x, \\ \frac{(t-b)^n}{n!}, & \text{if} \quad x < t \le b, \end{cases}$$

where $n \ge 1$ is a natural number.

The main aim of this paper is to generalize Ostrowski's inequality for n-times differentiable mappings by using a more general three-step kernel.

The rest of the paper is organized as follows: In Section 2, we introduce a new analogue of the Ostrowski inequalities for n-times differentiable functions, which not only improve the results involving Lebesgue norms of the n-th derivative for three-step kernels, but also contain the results from [13] for $n = 2$ as special cases. In Section 3, we use the obtained inequalities of Section 2 to derive new quadrature rules. Their efficiency is demonstrated by using specific examples as well as by deriving their respective error bounds. Composite quadrature rules are derived in Section 4. Section 5 is devoted to applications of the obtained results to probability density functions.

2. Main Results

Consider the Peano kernel $K_n(x, .)$: $[a, b] \to \mathbb{R}$ defined by

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!}, a \le t \le x, \\ \frac{(t-M)^n}{n!}, x < t \le 2M - x, \text{ where } x \in (a, M), \\ \frac{(t-b)^n}{n!}, 2M - x < t \le b, \text{ where } x \in (a, M), \end{cases}$$

where $M = \frac{a+b}{2}$, n is a natural number and a, b are nonnegative real numbers and not both zero simultaneously.

Now we state and prove the following identity which will be the main tool to prove our main results.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be an n-times differentiable function such that $f^{(n-1)}(x)$ for $n \in \mathbb{N}$ is absolutely continuous on $[a,b]$; then*

$$\begin{aligned} &\lambda_n(x;a,b) \\ &:= \frac{1}{b-a}\int_a^b K_n(x,t) f^{(n)}(t)dt = \frac{(-1)^{n+1}}{b-a}\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\Big[(-1)^k (x-a)^{k+1} f^{(k)}(x) \\ &\quad + \Big((-1)^k f^{(k)}(2M-x) + f^{(k)}(x)\Big)(M-x)^{k+1} + (x-a)^{k+1} f^{(k)}(2M-x)\Big] + \frac{(-1)^n}{b-a}\int_a^b f(t)dt, \end{aligned} \tag{2.1}$$

where $M = \frac{a+b}{2}$.

Proof. The proof of (2.1) will be established by using mathematical induction.

Take $n = 1$; then

$$\lambda_1(x;a,b) = \frac{1}{b-a}\int_a^b K_1(x,t) f'(t)dt,$$

where

$$K_1(x,t) = \begin{cases} t-a, \ a \le t \le x, \\ t-M, \ x < t \le 2M-x, \ \text{where } x \in (a,M), \\ t-b, \ 2M-x < t \le b, \ \text{where } x \in (a,M). \end{cases}$$

After integrating by parts, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b K_1(x,t) f'(t)dt &= \frac{1}{b-a}\left[\{(x-a)f(x) + (f(2M-x)+f(x))(M-x)\}\right. \\ &\quad \left. + (x-a) f(2M-x)\right] - \frac{1}{b-a}\int_a^b f(t)dt. \end{aligned} \tag{2.2}$$

Equation (2.2) is (2.1) for $n = 1$. Assume that (2.1) is true for $n > 1$.

As part of the induction process, we will attempt to prove the equality

$$\begin{aligned} &\frac{1}{b-a}\int_a^b K_{n+1}(x,t) f^{(n+1)}(t)dt \\ &= \frac{1}{b-a}\sum_{k=0}^{n}\frac{(-1)^{n+2}}{(k+1)!}\Big[\Big\{(-1)^k (x-a)^{k+1} f^{(k)}(x) \\ &\quad + \Big((-1)^k f^{(k)}(2M-x) + f^{(k)}(x)\Big)(M-x)^{k+1}\Big\} + (x-a)^{k+1} f^{(k)}(2M-x)\Big] \\ &\quad + \frac{(-1)^{n+1}}{b-a}\int_a^b f(t)dt, \end{aligned}$$

where

$$K_{n+1}(x,t)=\begin{cases}\frac{(t-a)^{n+1}}{(n+1)!}, a\le t\le x,\\ \frac{(t-M)^{n+1}}{(n+1)!},\ x<t\le 2M-x,\\ \frac{(t-b)^{n+1}}{(n+1)!}, 2M-x<t\le b.\end{cases}$$

Consider

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K_{n+1}(x,t)f^{(n+1)}(t)dt\\
&=\frac{1}{(b-a)(n+1)!}\left[\int_a^x (t-a)^{n+1}f^{(n+1)}(t)dt\right.\\
&\quad\left.+\int_x^{2M-x}(t-M)^{n+1}f^{(n+1)}(t)dt+\int_{2M-x}^b (t-b)^{n+1}f^{(n+1)}(t)dt\right]\\
&=\frac{(x-a)^{n+1}f^{(n)}(x)}{(b-a)(n+1)!}-\frac{(n+1)}{(b-a)(n+1)!}\int_a^x (t-a)^n f^{(n)}(t)dt\\
&\quad+\frac{\left((M-x)^{n+1}f^{(n)}(2M-x)-(x-M)^{n+1}f^{(n)}(x)\right)}{(b-a)(n+1)!}\\
&\quad-\frac{(n+1)}{(b-a)(n+1)!}\int_x^{2M-x}(t-M)^n f^{(n)}(t)dt-\frac{(a-x)^{n+1}f^{(n)}(2M-x)}{(b-a)(n+1)!}\\
&\quad-\frac{(n+1)}{(b-a)(n+1)!}\int_{2M-x}^b (t-b)^n f^{(n)}(t)dt\\
&=\frac{1}{(n+1)!(b-a)}\left[(x-a)^{n+1}f^{(n)}(x)+\left((M-x)^{n+1}f^{(n)}(2M-x)\right.\right.\\
&\quad\left.\left.-(x-M)^{n+1}f^{(n)}(x)\right)-(a-x)^{n+1}f^{(n)}(2M-x)\right]\\
&\quad-\frac{1}{(b-a)n!}\left[\int_a^x (t-a)^n f^{(n)}(t)dt+\int_x^{2M-x}(t-M)^n f^{(n)}(t)dt+\int_{2M-x}^b (t-b)^n f^{(n)}(t)dt\right]\\
&=\frac{1}{(b-a)(n+1)!}\left[(x-a)^{n+1}f^{(n)}(x)+\left\{f^{(n)}(2M-x)\right.\right.\\
&\quad\left.\left.+(-1)^{n+2}f^{(n)}(x)\right\}(M-x)^{n+1}+(-1)^{n+2}(x-a)^{n+1}f^{(n)}(2M-x)\right]\\
&\quad-\frac{1}{n!(b-a)}\left[\int_a^x (t-a)^n f^{(n)}(t)dt+\int_x^{2M-x}(t-M)^n f^{(n)}(t)dt+\int_{2M-x}^b (t-b)^n f^{(n)}(t)dt\right]\\
&=\frac{1}{(n+1)!(b-a)}\left[(x-a)^{n+1}f^{(n)}(x)+\left\{f^{(n)}(2M-x)\right.\right.\\
&\quad\left.\left.+(-1)^{n+2}f^{(n)}(x)\right\}(M-x)^{n+1}+(-1)^{n+2}(x-a)^{n+1}f^{(n)}(2M-x)\right]\\
&\quad-\frac{1}{b-a}\int_a^b K_n(x,t)f^{(n)}(t)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K_{n+1}(x,t)f^{(n+1)}(t)dt \\
&= \frac{1}{(n+1)!(b-a)}\Big[(x-a)^{n+1} f^{(n)}(x) + \Big\{f^{(n)}(2M-x) \\
&\quad +(-1)^{n+2} f^{(n)}(x)\Big\}(M-x)^{n+1} + (-1)^{n+2}(x-a)^{n+1} f^{(n)}(2M-x)\Big] \\
&\quad - \frac{1}{b-a}\sum_{k=0}^{n-1}\frac{(-1)^{n+1}}{(k+1)!}\Big[(-1)^k (x-a)^{k+1} f^{(k)}(x) \\
&\quad + \Big((-1)^k f^{(k)}(2M-x) + f^{(k)}(x)\Big)(M-x)^{k+1} \\
&\quad + (x-a)^{k+1} f^{(k)}(2M-x)\Big] + \frac{(-1)^{n+1}}{b-a}\int_a^b f(t)dt \\
&= \frac{1}{b-a}\sum_{k=0}^{n}\frac{(-1)^{n+2}}{(k+1)!}\Big[(-1)^k (x-a)^{k+1} f^{(k)}(x) + \Big((-1)^k f^{(k)}(2M-x) + f^{(k)}(x)\Big) \\
&\quad \times (M-x)^{k+1} + (x-a)^{k+1} f^{(k)}(2M-x)\Big] + \frac{(-1)^{n+1}}{b-a}\int_a^b f(t)dt.
\end{aligned}$$

Hence the lemma is proved. □

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. If $f^{(n)}\in L^1[a,b]$ and $\gamma\le f^{(n)}(t)\le\Gamma$, for all $t\in[a,b]$, then the following inequalities hold for all $x\in\left(a,\frac{a+b}{2}\right)$,*

$$|\lambda_n(x;a,b)| \le \begin{cases} (S-\gamma)\,\eta_n(x;a,b) \\ \\ (\Gamma-S)\,\eta_n(x;a,b) \end{cases}, n \text{ is an odd integer,} \tag{2.3}$$

$$\left|\lambda_n(x;a,b) - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\nu_n(x;a,b)\right| \le \begin{cases} (S-\gamma)\,\mu_n(x;a,b) \\ (\Gamma-S)\,\mu_n(x;a,b) \end{cases}, n \text{ is an even integer} \tag{2.4}$$

where $\lambda_n(x;a,b)$ is defined in Lemma 2.1, $S=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$,

$$\eta_n(x;a,b) := \max\left\{\frac{(x-a)^n}{n!}, \frac{|x-M|^n}{n!}\right\},$$

$$\nu_n(x;a,b) := \frac{2\left\{(x-a)^{n+1}+(M-x)^{n+1}\right\}}{(n+1)!\,(b-a)},$$

and

$$\mu_n(x;a,b) := \max\left\{\left|\frac{1}{n!}(x-a)^n - \nu_n(x;a,b)\right|, \left|\frac{1}{n!}(M-x)^n - \nu_n(x;a,b)\right|, |\nu_n(x;a,b)|\right\}.$$

Proof. Let

$$R_n(x) = \frac{1}{b-a}\int_a^b K_n(x,t)f^{(n)}(t)dt - \frac{1}{(b-a)^2}\int_a^b K_n(x,t)dt\int_a^b f^{(n)}(t)dt$$

$$
\begin{aligned}
&= \lambda_n(x;a,b) - \left(\frac{f^{n-1}(b)-f^{n-1}(a)}{b-a}\right)\frac{[1+(-1)^n]}{(b-a)(n+1)!}\left[(x-a)^{n+1}+(M-x)^{n+1}\right] \\
&= \lambda_n(x;a,b) - S\frac{[1+(-1)^n]}{(b-a)(n+1)!}\left[(x-a)^{n+1}+(M-x)^{n+1}\right].
\end{aligned}
$$

For an arbitrary constant $C \in \mathbb{R}$, $R_n(x)$ can be written as

$$
R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t)-C\right)\left[K_n(x,t)-\frac{1}{b-a}\int_a^b K_n(x,s)ds\right]dt.
$$

Hence

$$
\begin{aligned}
|R_n(x)| &= \left|\lambda_n(x;a,b) - S\frac{[1+(-1)^n]}{(b-a)(n+1)!}\left[(x-a)^{n+1}+(M-x)^{n+1}\right]\right| \\
&\leq \frac{1}{b-a}\max_{t\in[a,b]}\left|K_n(x,t)-\frac{1}{b-a}\int_a^b K_n(x,s)ds\right|\int_a^b\left|f^{(n)}(t)-C\right|dt.
\end{aligned} \tag{2.5}
$$

It can be easily seen that

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b K_n(x,s)ds &= \frac{[1+(-1)^n]}{(b-a)(n+1)!}\left[(x-a)^{n+1}+(M-x)^{n+1}\right] \\
&= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2\left[(x-a)^{n+1}+(M-x)^{n+1}\right]}{(b-a)(n+1)!}, & \text{if } n \text{ is even.} \end{cases}
\end{aligned} \tag{2.6}
$$

By choosing $C=\gamma$ and $C=\Gamma$, we also have

$$
\int_a^b\left|f^{(n)}(t)-C\right|dt = \int_a^b\left|f^{(n)}(t)-\gamma\right| = (S-\gamma)(b-a), \tag{2.7}
$$

and

$$
\int_a^b\left|f^{(n)}(t)-C\right|dt = \int_a^b\left|f^{(n)}(t)-\Gamma\right| = (\Gamma-S)(b-a). \tag{2.8}
$$

The inequalities (2.3) and (2.4) can be obtained by using (2.6)–(2.8) in (2.5). □

Corollary 2.3. *Some special cases of Theorem* 2.2 *can be obtained as follows:*

1. *When $x=a$, we have*

$$
\begin{aligned}
\left|\sum_{k=0}^{2n-2}A_k^{(1)} - \int_a^b f(t)dt\right| &\leq \frac{(S-\gamma)(b-a)^{2n}}{2^{2n-1}(2n-1)!}, \\
\left|\sum_{k=0}^{2n-2}A_k^{(1)} - \int_a^b f(t)dt\right| &\leq \frac{(\Gamma-S)(b-a)^{2n}}{2^{2n-1}(2n-1)!},
\end{aligned}
$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(1)} + C_n^{(1)} - \int_a^b f(t)dt\right| \leq \frac{n(S-\gamma)(b-a)^{2n+1}}{2^{2n-1}(2n+1)!},$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(1)} + C_n^{(1)} - \int_a^b f(t)dt\right| \leq \frac{n(\Gamma-S)(b-a)^{2n+1}}{2^{2n-1}(2n+1)!}.$$

2. *if* $x = \frac{3a+b}{4}$, *then*

$$\left|\sum_{k=0}^{2n-2} A_k^{(3)} - \int_a^b f(t)dt\right| \leq \frac{(S-\gamma)(b-a)^{2n}}{2^{4n-2}(2n-1)!},$$

$$\left|\sum_{k=0}^{2n-2} A_k^{(3)} - \int_a^b f(t)dt\right| \leq \frac{(\Gamma-S)(b-a)^{2n}}{2^{4n-2}(2n-1)!},$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(3)} + \frac{C_n^{(1)}}{2^{2n}} - \int_a^b f(t)dt\right| \leq \frac{n(S-\gamma)(b-a)^{2n+1}}{2^{4n-1}(2n+1)!}, \tag{2.9}$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(3)} + \frac{C_n^{(1)}}{2^{2n}} - \int_a^b f(t)dt\right| \leq \frac{n(\Gamma-S)(b-a)^{2n+1}}{2^{4n-1}(2n+1)!}, \tag{2.10}$$

where

$$\begin{aligned}
A_k^{(1)} &= \frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[(-1)^k f^{(k)}(b) + f^{(k)}(a)\right],\\
A_k^{(3)} &= \frac{\left[(-1)^k + 1\right]}{(k+1)!}\left(\frac{b-a}{4}\right)^{k+1}\left[f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)\right],\\
C_n^{(1)} &= \frac{\left[f^{(2n-1)}(b) - f^{(2n-1)}(a)\right](b-a)^{2n+1}}{2^{2n}(2n+1)!},
\end{aligned}$$

and $S = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$.

Theorem 2.4. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$ *and* $f^{(n)} \in L^2[a,b]$*; then for all* $x \in \left(a, \frac{a+b}{2}\right)$ *we have*

$$|\lambda_n(x;a,b)| \leq \frac{\left\|f^{(n+1)}\right\|_2}{\pi}\sqrt{\frac{2\left[(x-a)^{2n+1} + (M-x)^{2n+1}\right]}{(n!)^2(2n+1)}}, \text{ if } n \text{ is odd}$$

and

$$\left|\lambda_n(x;a,b) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\nu_n(x;a,b)\right|$$
$$\leq \frac{1}{\pi}\left\{\frac{2\left[(x-a)^{2n+1} + (M-x)^{2n+1}\right]}{(n!)^2(2n+1)}\right.$$

$$-\frac{4}{b-a}\left[\frac{(x-a)^{n+1}+(M-x)^{n+1}}{(n+1)!}\right]^2\Bigg\}^{\frac{1}{2}}\left\|f^{(n+1)}\right\|_2, \text{ if } n \text{ is even,}$$

where $\nu_n(x;a,b) := \frac{2\{(x-a)^{n+1}+(M-x)^{n+1}\}}{(n+1)!(b-a)}$.

Proof. From Theorem 2.2, we have

$$R_n(x) = \frac{1}{b-a}\int_a^b K_n(x,t)f^{(n)}(t)dt - \frac{1}{(b-a)^2}\int_a^b K_n(x,t)dt\int_a^b f^{(n)}(t)dt,$$

which can be rewritten as

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t)-C\right)\left[K_n(x,t)-\frac{1}{b-a}\int_a^b K_n(x,s)ds\right]dt, \tag{2.11}$$

for any arbitrary constant $C \in \mathbb{R}$. Consider $C = f^{(n)}\left(\frac{a+b}{2}\right)$ in (2.11) and use the Cauchy inequality to get

$$\begin{aligned}|R_n(x)| &\leq \frac{1}{b-a}\int_a^b \left|f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right|\left|K_n(x,t)-\frac{1}{b-a}\int_a^b K_n(x,s)ds\right|dt \\ &\leq \frac{1}{b-a}\left[\int_a^b\left(f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right)^2dt\right]^{\frac{1}{2}}\left[\int_a^b\left(K_n(x,t)-\frac{1}{b-a}\int_a^b K_n(x,s)ds\right)^2dt\right]^{\frac{1}{2}}.\end{aligned} \tag{2.12}$$

By using the Diaz–Metcalf inequality, we obtain

$$\int_a^b\left(f^{(n)}(t)-f^{(n)}\left(\frac{a+b}{2}\right)\right)^2dt \leq \frac{(b-a)^2}{\pi^2}\left\|f^{(n+1)}\right\|_2^2. \tag{2.13}$$

Also we have

$$\begin{aligned}&\int_a^b\left(K_n(x,t)-\frac{1}{b-a}\int_a^b K_n(x,s)ds\right)^2dt \\ &= \int_a^b (K_n(x,t))^2dt-(b-a)\left(\frac{1}{b-a}\int_a^b K_n(x,t)dt\right)^2 \\ &= \begin{cases}\frac{2\left[(x-a)^{2n+1}+(M-x)^{2n+1}\right]}{(n!)^2(2n+1)}, & \text{if } n \text{ is odd,}\\ \frac{2\left[(x-a)^{2n+1}+(M-x)^{2n+1}\right]}{(n!)^2(2n+1)}-\frac{4}{b-a}\left[\frac{(x-a)^{n+1}+(M-x)^{n+1}}{(n+1)!}\right]^2, & \text{if } n \text{ is even.}\end{cases}\end{aligned} \tag{2.14}$$

Using (2.13) and (2.14) in (2.12), we get the desired result. □

Corollary 2.5. *Suppose the assumptions of Theorem* 2.4 *are satisfied and* n *is a natural number. Then*

1. *for $x = a$, we have*

$$\left|\sum_{k=0}^{2n-2} A_k^{(1)} - \int_a^b f(t)dt\right| \leq \frac{(b-a)^{2n+1/2}\left\|f^{(2n)}\right\|_2}{\pi 2^{2n-1}(2n-1)!\sqrt{4n-1}},$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(1)} + C_n^{(1)} - \int_a^b f(t)dt\right| \leq \frac{n(b-a)^{2n+3/2}\left\|f^{(2n+1)}\right\|_2}{\pi(2n+1)!2^{2n+1}\sqrt{4n+1}}.$$

2. *for $x = \frac{3a+b}{4}$, we get*

$$\left|\sum_{k=0}^{2n-2} A_k^{(3)} - \int_a^b f(t)dt\right| \leq \frac{(b-a)^{2n+1/2}\left\|f^{(2n)}\right\|_2}{2^{4n-2}(2n-1)!\pi\sqrt{4n-1}},$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(3)} + \frac{C_n^{(1)}}{2^{2n}} - \int_a^b f(t)dt\right| \leq \frac{n(b-a)^{2n+3/2}\left\|f^{(2n+1)}\right\|_2}{2^{4n-1}(2n+1)!\pi\sqrt{4n+1}}.$$

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L^2[a,b]$; then for all $x \in \left(a, \frac{a+b}{2}\right)$ we have*

$$|\lambda_n(x;a,b)| \leq \frac{\sqrt{\sigma\left(f^{(n-1)}\right)}}{b-a}\sqrt{\frac{2\left[(x-a)^{2n+1} + (M-x)^{2n+1}\right]}{(n!)^2(2n+1)}}, \text{ if } n \text{ is odd}$$

and

$$\begin{aligned}&\left|\lambda_n(x;a,b) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\nu_n(x;a,b)\right| \\ &\leq \frac{\sqrt{\sigma\left(f^{(n-1)}\right)}}{b-a}\left\{\frac{2\left[(x-a)^{2n+1} + (M-x)^{2n+1}\right]}{(n!)^2(2n+1)}\right. \\ &\left.-\frac{4}{b-a}\left[\frac{(x-a)^{n+1} + (M-x)^{n+1}}{(n+1)!}\right]^2\right\}^{\frac{1}{2}}, \text{ if } n \text{ is even,}\end{aligned}$$

where

$$\nu_n(x;a,b) := \frac{2\left\{(x-a)^{n+1} + (M-x)^{n+1}\right\}}{(n+1)!(b-a)},$$

$$\sigma\left(f^{(n)}\right) = \left\|f^{(n)}\right\|_2^2 - S^2(b-a),$$

and

$$S = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

Proof. It has been observed that

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t) - C\right)\left[K_n(x,t) - \frac{1}{b-a}\int_a^b K_n(x,s)ds\right]dt, \tag{2.15}$$

for any constant $C \in \mathbb{R}$.

Let us choose

$$C = \frac{1}{b-a}\int_a^b f^{(n)}(s)ds,$$

in (2.15), and use the Cauchy inequality to get

$$\begin{aligned}
&|R_n(x)| \\
&\leq \frac{1}{b-a}\left[\int_a^b \left(f^{(n)}(t) - \frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)^2 dt\right]^{\frac{1}{2}} \left[\int_a^b \left(K_n(x,t) - \frac{1}{b-a}\int_a^b K_n(x,s)ds\right)^2 dt\right]^{\frac{1}{2}} \\
&\leq \begin{cases} \left\{ \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\sqrt{\frac{2\left[(x-a)^{2n+1}+(M-x)^{2n+1}\right]}{(n!)^2(2n+1)}}, \right. & \text{if } n \text{ is odd,} \\ \left\{ \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a} \begin{array}{l} \frac{2\left[(x-a)^{2n+1}+(M-x)^{2n+1}\right]}{(n!)^2(2n+1)} \\ -\frac{4}{b-a}\left[\frac{(x-a)^{n+1}+(M-x)^{n+1}}{(n+1)!}\right]^2, \end{array} \right. & \text{if } n \text{ is even.} \end{cases}
\end{aligned} \tag{2.16}$$

The result follows immediately from (2.16). □

Corollary 2.7. *Suppose the assumptions of Theorem* 2.6 *are satisfied and n is a natural number; then*

1. *for $x = a$, we have*

$$\left|\sum_{k=0}^{2n-2} A_k^{(1)} - \int_a^b f(t)dt\right| \leq \frac{(b-a)^{2n-1/2}}{2^{2n-1}(2n-1)!}\sqrt{\frac{\sigma\left(f^{(2n-1)}\right)}{(4n-1)}},$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(1)} + C_n^{(1)} - \int_a^b f(t)dt\right| \leq \frac{n(b-a)^{2n+1/2}}{(4n+1)!2^{2n-1}}\sqrt{\frac{\sigma\left(f^{(2n)}\right)}{(4n+1)}},$$

2. *for $x = \frac{3a+b}{4}$, we obtain*

$$\left|\sum_{k=0}^{2n-2} A_k^{(3)} - \int_a^b f(t)dt\right| \leq \frac{(b-a)^{n-1/2}}{2^{4n-2}(2n-1)!}\sqrt{\frac{\sigma\left(f^{(2n-1)}\right)}{(4n-1)}},$$

$$\left|\sum_{k=0}^{2n-1} A_k^{(3)} + \frac{C_n^{(1)}}{2^{2n}} - \int_a^b f(t)dt\right| \leq \frac{n(b-a)^{n+1/2}}{(2n+1)!2^{4n-1}}\sqrt{\frac{\sigma\left(f^{(2n)}\right)}{(4n+1)}}.$$

3. Derivation and Applications of Quadrature Rules

We propose some new quadrature rules involving higher order derivatives of the function f. The following new quadrature rules can be obtained by investigating error bounds using Theorem 2.2.

$$Q_{n3o} := \int_a^b f(t)dt \approx \sum_{k=0}^{2n-2} A_k^{(3)}$$

$$, Q_{n1e} := \int_a^b f(t)dt \approx \sum_{k=0}^{2n-1} A_k^{(1)} + \frac{S(b-a)^{2n+1}}{2^{2n}(2n+1)!},$$

$$Q_{n3e} := \int_a^b f(t)dt \approx \sum_{k=0}^{2n-1} A_k^{(3)} + \frac{S\,(b-a)^{2n+1}}{2^{4n}\,(2n+1)!}. \tag{3.1}$$

To investigate the efficiency of these quadrature rules, we integrate various types of functions including polynomial, trigonometric, exponential and logarithmic functions. The results of the numerical integrations are given in Table 1 with absolute error and the value of n used to obtain the mentioned accuracy. The default error is chosen of the order of 10^{-5} to make a comparison amongst all the quadrature rules possible. The error type shown in Table 1 is the absolute value of the difference of the exact value of the integral and its numerical value. Naturally, if a quadrature rule requires smaller value of n to achieve the desired accuracy it will be considered more efficient.

The following nine diverse types of functions are used to demonstrate the efficiency of the various quadrature rules namely $Q_{n1e}(f)$ and $Q_{n3e}(f)$.

Table 1: Performance of the proposed quadrature rules

S. No	**Method**	$n:\quad Q_{n1e}(f)$	$n:\quad Q_{n3e}(f)$	**Exact Value**
1	$\int_0^{\pi} f_1(x)dx$	**7**: 26.97381399	**7**: 26.97383822	26.9738436
	Error:	2.4×10^{-5}	5.4×10^{-6}	
2	$\int_0^{\pi/2} f_2(x)dx$	**4**: 2.90523453	**3**: 2.905243499	2.905238690
	Error:	4.1×10^{-6}	4.8×10^{-6}	
3	$\int_0^{1} f_3(x)dx$	**5**: 1.313799439	**3**: 1.313835726	1.313831895
	Error:	3.2×10^{-5}	3.8×10^{-6}	
4	$\int_0^{\pi/2} f_4(x)dx$	**3**: 2.233693811	**3**: 2.233700832	2.233700550
	Error:	6.7×10^{-6}	2.8×10^{-7}	
5	$\int_0^{1} f_5(x)dx$	**4**: 0.6297568214	**2**: 0.6297713872	0.6297685230
	Error:	1.1×10^{-5}	2.2×10^{-5}	
6	$\int_0^{1} f_6(x)dx$	**7**: -1.176912847	**4**: -1.176900063	-1.176887888
	Error	2.4×10^{-5}	1.2×10^{-5}	
7	$\int_0^{1} f_7(x)dx$	**5**: 0.2415670039	**2**: 0.2415547713	0.2415491347
	Error:	1.7×10^{-5}	5.6×10^{-6}	
8	$\int_0^{1} f_8(x)dx$	**4**: 0.2639951553	**2**: 0.2639404958	0.2639435074
		5.1×10^{-5}	3.01×10^{-6}	
9	$\int_0^{1} f_9(x)dx$	**5**: 1.462637863	**3**: 1.46258681	1.462651746
		1.3×10^{-5}	6.9×10^{-6}	

$$f_1(x) = x^4 \sin x, \qquad f_2(x) = e^x \sin x, \qquad f_3(x) = e^x \cos\left(e^x - 2x\right),$$
$$f_4(x) = \cos x - x, \qquad f_5(x) = \log\left(x^2+2\right) \sin\left[\log\left(x^2+2\right)\right],$$

$$f_6(x) = e^{2x}\cos(e^x), \qquad f_7(x) = \frac{1}{x^4+4x^2+3},$$
$$f_8(x) = \log\left[x^2+1\right], \qquad f_9(x) = e^{x^2}.$$

Looking at the table, we discuss the accuracy of the two proposed quadrature rules. At the first sight at the third column, it can be seen that $Q_{n1e}(f)$ gives the desired accuracy of order 10^{-5} for all of the functions except f_4, which is a relatively simple function. Although the error reported by $Q_{n1e}(f)$ is acceptable and is achieved for not very large values of n, it is noticed that $Q_{n3e}(f)$ gives error of the same magnitude for much lower values of n. For example, when f_3 is integrated with $Q_{n1e}(f)$, it gives an error of order 10^{-5} for $n = 5$, but $Q_{n3e}(f)$ gives an error of order 10^{-6} magnitude for $n = 3$. Similar performance can be seen for all the functions except f_1, where $Q_{n1e}(f)$ marginally beat $Q_{n3e}(f)$. Therefore it can be conjectured, based on this observation, that the convergence of $Q_{n3e}(f)$ is faster than $Q_{n1e}(f)$, as well as standard quadrature rules such as the Simpson rule.

4. Composite Quadrature Rule

Let $I_m : a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ be a partition of the interval $[a, b]$ and

$$h_i = x_{i+1} - x_i (i = 0, 1, 2, \cdots, m-1).$$

Consider the following perturbed composite quadrature rule defined in (3.1) for f which is $2n$-times differentiable mapping such that $\gamma \le f^{(2n)}(t) \le \Gamma$ for all $t \in [a, b]$

$$Q_{n3e}(I_m, f) := \sum_{i=0}^{m-1}\left(\left\{\sum_{k=0}^{2n-1}\frac{\left[(-1)^k+1\right]}{(k+1)!}\left(\frac{h_i}{4}\right)^{k+1}\left[f^{(k)}\left(\frac{3x_i+x_{i+1}}{4}\right)\right.\right.\right.$$
$$\left.\left.\left.+f^{(k)}\left(\frac{x_i+3x_{i+1}}{4}\right)\right]\right\} + \frac{\left(f^{(2n-1)}(x_{i+1}) - f^{(2n-1)}(x_i)\right)(h_i)^{2n+1}}{2^{4n}(2n+1)!}\right).$$

Theorem 4.1. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous on $[a, b]$. If $f^{(2n)} \in L^1[a, b]$ and $\gamma \le f^{(2n)}(t) \le \Gamma$ for all $t \in [a, b]$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\int_a^b f(t)dt = Q_{n3e}(I_m, f) + R_{n3e}(I_m, f),$$

where the term $Q_{n3e}(I_m, f)$ is obtained in (3.1) and the remainder $R_{n3o}(I_m, f)$ has the error bound

$$|R_{n3e}(f)| \le \frac{n(S-\gamma)\sum_{i=0}^{m-1}(h_i)^{2n+1}}{2^{4n-1}(2n+1)!},$$

and

$$|R_{n3e}(f)| \le \frac{n(\Gamma-S)\sum_{i=0}^{m-1}(h_i)^{2n+1}}{2^{4n-1}(2n+1)!}.$$

Proof. Applying inequalities (2.9) and (2.10) to the intervals $[x_{i+1}, x_i]$ and summing the resulting inequalities for $i = 0, 1, 2, \ldots, m-1$, we get the required estimates. □

Remark 4.2. We can get the error bounds for other composite quadrature rules in a similar fashion.

5. Application to probability density functions

Let X be a continuous random variable with probability density function $f : [a, b] \to [0, 1]$. Let F be the cumulative distribution function of f i.e.,

$$F(x) = \int_a^x f(t)dt.$$

Now we present applications of our results for continuous random variable X.

Theorem 5.1. *With the assumptions of Theorem 2.2, we have*

$$\begin{aligned}
&\Bigg| \frac{1}{b-a} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \Big[(-1)^k (x-a)^{k+1} f^{(k-1)}(x) \\
&+ \left((-1)^k f^{(k-1)} (2M - x) + f^{(k-1)}(x) \right) (M-x)^{k+1} \\
&+ (x-a)^{k+1} f^{(k)} (2M - x) \Big] - \frac{b - E(X)}{b-a} \Bigg| \\
&\qquad \le \begin{cases} (S - \gamma)\, \eta_n (x; a, b) \\ (\Gamma - S)\, \eta_n (x; a, b) \end{cases}, \ n \text{ is an odd integer.}
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg| \frac{1}{b-a} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \Big[(-1)^k (x-a)^{k+1} f^{(k-1)}(x) \\
&+ \left((-1)^k f^{(k-1)} (2M - x) + f^{(k-1)}(x) \right) (M-x)^{k+1} \\
&+ (x-a)^{k+1} f^{(k)} (2M - x) \Big] \\
&+ \frac{(-1)^n \left[f^{(n-2)}(b) - f^{(n-2)}(a) \right] \nu_n (x; a, b)}{b-a} - \frac{b - E(X)}{b-a} \Bigg| \\
&\qquad \le \begin{cases} (S - \gamma)\, \mu_n (x; a, b) \\ (\Gamma - S)\, \mu_n (x; a, b) \end{cases}, \ n \text{ is an even integer,}
\end{aligned}$$

for all $x \in \left(a, \frac{a+b}{2}\right)$, where

$$\nu_n (x; a, b) := \frac{2 \left\{ (x-a)^{n+1} + (M-x)^{n+1} \right\}}{(n+1)!\,(b-a)},$$

$$\mu_n (x; a, b) := \max \left\{ \left| \frac{1}{n!} (x-a)^n - \nu_n (x; a, b) \right|, \ \left| \frac{1}{n!} (M-x)^n - \nu_n (x; a, b) \right|, |\nu_n (x; a, b)| \right\},$$

and $E(X)$ is the expectation of X.

Proof. By choosing $f = F$ in Theorem 2.2 and using the fact that

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we get the required result. □

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