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Algorithms for common solutions of generalized mixed equilibrium problems and system of variational inclusion problems

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Abstract

In this paper, we introduce a multi-step iterative algorithm for finding a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of solutions of a finite family of variational inclusions for maximal monotone and inverse strong monotone mappings, the set of solutions of general system of variational inequalities and the set of fixed points of a countable family of nonexpansive mappings in a real Hilbert space. This multi-step iterative algorithm is based on Korpelevich's extragradient method, viscosity approximation method, projection method, and strongly positive bounded linear operator and W-mapping approaches. We establish the strong convergence of the sequences generated by the proposed algorithm to a common element of above mentioned problems under appropriate assumptions, which also solves some optimization problem. The result presented in this paper improves and extends some corresponding ones in the earlier and recent literature.

Keywords: Multi-step iterative algorithm, mixed equilibrium problem, Variational inclusion, nonexpansive mapping, maximal monotone mapping, strong convergence.
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1. Introduction

There are many fundamental problems in nonlinear analysis, but here we consider only few, namely,

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generalized mixed equilibrium problems, variational inclusions, system of variational inequalities and fixed point problems. These problems have been focus of many researchers because of their applications in different branches of science, engineering, management and social sciences. We describe these problems as follows:

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and $S : C \to C$ be a mapping. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbb{R}$ the set of all real numbers. Through out the paper, we assume that M and N are integers.

System of Generalized Mixed Equilibrium Problem.

For each $k \in \{1, 2, ..., M\}$, let $\Theta_k : C \times C \to \mathbb{R}$ be a bifunction, $A_k : H \to H$ be a nonlinear operator, and $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. The system of generalized mixed equilibrium problem (SGMEP) is defined as follows:

$$\begin{cases} \text{Find } x \in C \text{ such that for each } k = 1, ..., M, \\ \Theta_k(x, y) + \varphi_k(y) - \varphi_k(x) + \langle A_k x, y - x \rangle \geq 0, \quad \text{for all } y \in C. \end{cases} \tag{1.1}$$

If $k = 1$, then SGMEP reduces to the generalized mixed equilibrium problem considered in [17]. More precisely, let $\varphi : C \to \mathbb{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbb{R}$ be a bifunction. The generalized mixed equilibrium problem (GMEP) is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0, \quad \text{for all } y \in C. \tag{1.2}$$

We denote the set of solutions of GMEP (1.2) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The solution set of SGMEP is equal to $\bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$ which is the set of common solutions of M generalized mixed equilibrium problems. It is worth to mention that the GMEP (1.2) includes several problems, namely, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games as special cases.

When $\varphi \equiv 0$, GMEP (1.2) becomes to the generalized equilibrium problem (GEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \text{for all } y \in C.$$

It was considered by Ansari et al. [5] and further studied by Takahashi and Takahashi [21]. The set of solutions of GEP is denoted by $\mathrm{GEP}(\Theta, A)$.

When $\Theta \equiv 0$, GEP collapses to the classical variational inequality problem (VIP): Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \text{for all } y \in C. \tag{1.3}$$

The theory of variational inequalities is well-known and well-established. There are several books and monographs on different aspects of variational inequalities, but we refer here [4] and the references therein.

When $A \equiv 0$, GMEP (1.2) reduces to the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \text{for all } y \in C.$$

It is considered and studied in [10, 23]. The set of solutions of MEP is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

When $\varphi \equiv 0$, $A \equiv 0$, GMEP (1.2) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C.$$

It is considered and studied in [1–3, 8, 11, 18]. The set of solutions of EP is denoted by $\mathrm{EP}(\Theta)$. It is worth to mention that the EP is an unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, etc.

System of Variational Inequalities.

Let $F_1, F_2 : C \to H$ be two mappings. The general system of variational inequalities (GSVI) problem is to find $(x^*, y^*) \in C \times C$ such that for all $x \in C$

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \end{cases} \tag{1.4}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It is considered and studied in [9]. In particular, if $F_1 = F_2 = A$, then the GSVI (1.4) is considered in [22]. Further, if $x^* = y^*$ additionally, then the GSVI reduces to the classical VIP (1.3). Ceng et al. [9] transformed the GSVI (1.4) into the fixed point problem of the mapping $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$, that is, $Gx^* = x^*$. The set of fixed points of the mapping $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$ is denoted by Γ.

System of Variational Inclusion Problems.

For each $i \in \{1, 2, ..., N\}$, let $R_i : C \to 2^H$ be a set-valued mapping with nonempty values, and $B_i : C \to H$ be single-valued mapping. The system of variational inclusions (SVI) is to find $x \in C$ such that for each $i \in \{1, 2, ..., N\}$

$$0 \in B_i x + R_i x. \tag{1.5}$$

If $i = 1$, then SVI is called variational inclusion. More precisely, let $B : C \to H$ be a single-valued mapping and $R : C \to 2^H$ be a multivalued mapping with nonempty values, where domain of R, $D(R) = C$. The variational inclusion (VI) problem is to find $x \in C$ such that

$$0 \in Bx + Rx. \tag{1.6}$$

We denote by $\mathrm{I}(B, R)$ the solution set of the variational inclusion (1.6). Then the solution set of SVI is equal to $\bigcap_{i=1}^N \mathrm{I}(B_i, R_i)$ which is the set of common solutions of N variational inclusions. For further details on variational inclusions, we refer to [12–15, 27] and the references therein.

If $B \equiv R \equiv 0$, then $\mathrm{I}(B, R) = C$. If $B \equiv 0$, then problem (1.6) becomes the inclusion problem introduced by Rockafellar [19]. It is known that problem (1.6) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, etc. Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \text{for all } x \in H,$$

where λ is a positive number.

Common Fixed Point Problem.

Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C. The common fixed point problem (CFPP) is to find $x \in C$ such that $x \in T_i(x)$ for each $i \in \mathbb{N}$. The set of common fixed point of $\{T_n\}_{n=1}^\infty$ is denoted by $\bigcap_{n=1}^\infty \mathrm{Fix}(T_n)$.

If $n = 1$, then CFPP reduces to the fixed point problem. More precisely, let C be a nonempty subset of an H and $T : C \to C$ be a mapping. The fixed point problem (FPP) is to find an element $x \in C$ such that $T(x) = x$.

It is well-known problem and has tremendous applications in different branches of science, engineering, social sciences and management.

Problem to be considered.

For each $k \in \{1, 2, ..., M\} \subset \mathbb{N}$, let $\Theta_k : C \times C \to \mathbb{R}$ be a bifunction, $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function and $A_k : H \to H$ be a mapping. For each $i \in \{1, 2, ..., N\} \subset \mathbb{N}$, let $R_i : C \to 2^H$ be a set-valued mapping and $B_i : C \to H$ be a single-valued mapping. For $j = 1, 2$, let the mapping $F_j : C \to H$ be a mapping. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a bounded linear operator on H and $f : H \to H$ be a mapping. Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i) \bigcap \Gamma \neq \emptyset$.

We consider the following problem:

Problem 1.1. Find $x^* \in \Omega$, where $x^* = P_\Omega(I - (V - \gamma f))x^*$ is a unique solution of the following variational inequality problem (VIP):

$$\langle (V - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \text{for all } x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Vx, x\rangle - h(x),$$

where h is a potential function for γf.

Remark 1.2. The Problem 1.1 is very different from the problem of finding a point

$$x^* \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \bigcap \mathrm{GMEP}(\Theta, \varphi, A) \bigcap \mathrm{I}(B, R),$$

considered in [25, Theorem 3.2]. There is no doubt that the Problem 1.1 is more general and more subtle than the problem considered in [25, Theorem 3.2].

During the last decade, many authors proposed different kinds of algorithms to find the common solutions of some problems mentioned above; See, for example, [13, 15, 23, 25, 26] and the references therein.

Inspired by the research going on in this area, in this paper, we introduce the multi-step iterative algorithm for finding a solution of Problem 1.1. This multi-step iterative algorithm is based on Korpelevich's extragradient method, viscosity approximation method, projection method, and strongly positive bounded linear operator and W-mapping approaches. We prove the strong convergence of the sequences generated by the proposed algorithm to a solution of Problem 1.1. Our result improves and extends the corresponding results in [18, Theorem 3.1], [25, Theorem 3.2] and [23, Theorem 3.1].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot, \cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \to x$ (respectively, $x_n \rightharpoonup x$) to indicate that the sequence $\{x_n\}$ converges (respectively, weakly) to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Lemma 2.1 ([13])**.** *Let X be a real inner product space. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2 ([13])**.** *Let H be a real Hilbert space. Then, the following assertions hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, $\forall x, y \in H$*;*
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$, $\forall x, y \in H$ *and* $\lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$*;*

(c) If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

A mapping V is called strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Vx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Lemma 2.3 ([16])**.** *Let V be a $\bar{\gamma}$-strongly positive bounded linear operator on H and assume $0 < \rho \leq \|V\|^{-1}$. Then $\|I - \rho V\| \leq 1 - \rho\bar{\gamma}$.*

Definition 2.4. A mapping $T : H \to H$ is said to be

(a) L-Lipsctiz continuous if there exists a constant $L \geq 0$ such that if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H;$$

In particular, if $L = 1$ then T is called a nonexpansive mapping; if $L \in [0, 1)$ then T is called a contraction.

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.5. Let T be a nonlinear operator with the domain $D(T) \subset H$ and the range $R(T) \subset H$. Then T is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in D(T);$$

(b) β-strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in D(T);$$

(c) ν-inverse-strongly monotone if there exists a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

On the other hand, it is obvious that if A is ζ-inverse-strongly monotone, then A is monotone and $\frac{1}{\zeta}$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle Au - Av, u - v\rangle + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\zeta)\|Au - Av\|^2. \end{aligned} \tag{2.1}$$

So, if $\lambda \leq 2\zeta$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n on C as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{2.2}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\lambda_n, \lambda_{n-1}, ..., \lambda_1$.

Lemma 2.6 ([20])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, $\lim_{n\to\infty} U_{n,k}x$ exists, where $U_{n,k}$ is defined by* (2.2).

Remark 2.7 ([26], Remark 3.1). It can be known from Lemma 2.6 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$, there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon.$$

Remark 2.8 ([26], Remark 3.2). Utilizing Lemma 2.6, we define a mapping $W : C \to C$ by

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

This mapping W is called the W-mapping generated by $T_1, T_2, ...$ and $\lambda_1, \lambda_2, ...$ Since W_n is nonexpansive, so $W : C \to C$ is too.

Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 1\}$. Hence, it is clear from Remark 2.7 that for an arbitrary $\epsilon > 0$, there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.9 ([20])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$, and let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then,* $\mathrm{Fix}(W) = \cap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$, the unique point $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

The following properties of a projection are useful and pertinent to our purpose.

Proposition 2.10 ([13]). *Given any $x \in H$ and $z \in C$, the following assertions hold:*

(a) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \le 0,\ \forall y \in C;$
(b) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C;$
(c) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2$, *for all $y \in H$, which hence implies that P_C is nonexpansive and monotone.*

The following lemma provides the characterization of a solution of the variational inequality problem in terms of projection operator.

Lemma 2.11 ([9]). *Let $A : C \to H$ be a monotone mapping. Then,*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \textit{for all } \lambda > 0.$$

Ceng et. al. [9] transformed problem (1.4) into a fixed point problem in the following way:

Proposition 2.12 ([9]). *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of the GSVI (1.4) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x, \quad \textit{for all } x \in C,$$

where $\bar{y} = P_C(I - \nu_2 F_2)\bar{x}$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$.

Throughout this paper, it is assumed that $\Theta : C \times C \to \mathbb{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbb{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$, for all $x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \le 0$, for all $x, y \in C$;
(A3) Θ is upper-hemicontinuous, i.e., for all $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1 - t)x, y) \le \Theta(x, y);$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;
(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \left\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \ge 0, \text{for all } z \in C \right\}.$$

Next we list some elementary conclusions for the MEP.

Proposition 2.13 ([10]). *Assume that $\Theta : C \times C \to \mathbb{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ by*

$$T_r^{(\Theta,\varphi)}(x) = \left\{ z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}, \quad \textit{for all } x \in H.$$

Then, the following assertions hold:

(a) For each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ *is nonempty and single-valued;*

(b) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\left\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle;$$

(c) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta,\varphi)$*;*

(d) $\mathrm{MEP}(\Theta,\varphi)$ *is closed and convex;*

(e) $\left\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\right\|^2 \leq \frac{s-t}{s}\left\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\right\rangle$*, for all* $s, t > 0$ *and* $x \in H$.

Now we present the demiclosedness principle which will be used in the sequel.

Lemma 2.14 (Demiclosedness principle)**.** [13] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $S : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(S) \neq \emptyset$. *Then,* $I - S$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I - S)x_n\}$ *strongly converges to some* y*, it follows that* $(I - S)x = y$. *Here* I *is the identity operator of* H.

Recall that a set-valued mapping $\widetilde{T} : D(\widetilde{T}) \subset H \to 2^H$ is called monotone if for all $x, y \in D(\widetilde{T})$, $f \in \widetilde{T}x$ and $g \in \widetilde{T}y$, we have

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping $\widetilde{T} : D(\widetilde{T}) \subset H \to 2^H$ is called maximal monotone if $\widetilde{T}$ is monotone and $(I + \lambda\widetilde{T})D(\widetilde{T}) = H$ for each $\lambda > 0$, where I is the identity mapping of H.

We denote by $G(\widetilde{T})$ the graph of $\widetilde{T}$. It is known that a monotone mapping $\widetilde{T}$ is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(\widetilde{T})$ implies $f \in \widetilde{T}x$.

Let $A : C \to H$ be a monotone and k-Lipschitz-continuous mapping, and $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{u \in H : \langle v - x, u\rangle \geq 0, \ \forall x \in C\}.$$

Define

$$\widetilde{T}v = \begin{cases} Av + N_C v, & \text{if } v \in C,\\ \emptyset, & \text{if } v \notin C.\end{cases}$$

Then, $\widetilde{T}$ is maximal monotone (see [19]) and

$$0 \in \widetilde{T}v \ \Leftrightarrow \ v \in \mathrm{VI}(C, A). \tag{2.3}$$

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 2.15 ([6])**.** *There holds the resolvent identity*

$$J_{R,\lambda}x = J_{R,\mu}\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_{R,\lambda}x\right), \quad \textit{for all } x \in H.$$

Remark 2.16. For $\lambda, \mu > 0$, we have

$$\|J_{R,\lambda}x - J_{R,\mu}y\| \leq \|x - y\| + |\lambda - \mu|\left(\frac{1}{\lambda}\|J_{R,\lambda}x - y\| + \frac{1}{\mu}\|x - J_{R,\mu}y\|\right), \quad \text{for all } x, y \in H. \tag{2.4}$$

Indeed, whenever $\lambda \geq \mu$, utilizing Lemma 2.15 we deduce

$$\begin{aligned}\|J_{R,\lambda}x - J_{R,\mu}y\| &= \|J_{R,\mu}(\tfrac{\mu}{\lambda}x + (1 - \tfrac{\mu}{\lambda})J_{R,\lambda}x) - J_{R,\mu}y\|\\ &\leq \|\tfrac{\mu}{\lambda}x + (1 - \tfrac{\mu}{\lambda})J_{R,\lambda}x - y\|\\ &\leq \tfrac{\mu}{\lambda}\|x - y\| + (1 - \tfrac{\mu}{\lambda})\|J_{R,\lambda}x - y\|\\ &\leq \|x - y\| + \tfrac{|\lambda-\mu|}{\lambda}\|J_{R,\lambda}x - y\|.\end{aligned}$$

Similarly, whenever $\lambda < \mu$, we get

$$\|J_{R,\lambda}x - J_{R,\mu}y\| \leq \|x - y\| + \frac{|\lambda - \mu|}{\mu}\|x - J_{R,\mu}y\|.$$

Combining the above two cases, we conclude that (2.4) holds.

Now, we present some properties for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 2.17 ([6]). *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, i.e.,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \geq \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 2.18 ([7]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (2.2) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu).$$

Lemma 2.19 ([27]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the inclusion $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.20 ([7]). *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

Finally, we present a result related to the convergence of real sequences.

Lemma 2.21 ([24]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \sigma_n\gamma_n, \quad \textit{for all } n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$ and $\{\sigma_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Algorithm and Convergence Result

We propose the following multi-step iterative algorithm for finding a solution of Problem 1.1 such that it is also a solution of an optimization problem.

Algorithm 3.1. For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated iteratively by

$$\begin{cases} u_n = T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})}(I - r_{M-1,n}A_{M-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)x_n, \\ v_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n}B_{N-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ x_{n+1} = P_C[\alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n V)W_nGv_n], \quad \text{for all } n \geq 1, \end{cases} \tag{3.1}$$

where $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $i \in \{1, 2, ..., N\}$, $k \in \{1, 2, ..., M\}$, $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$, and W_n is the W-mapping defined by (2.2).

This multi-step iterative algorithm is based on Korpelevich's extragradient method, viscosity approximation method, projection method, and strongly positive bounded linear operator and W-mapping approaches.

Now we present the strong convergence of the sequence generated by Algorithm 3.1 to an element of Ω.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and for each $k \in \{1, 2, ..., M\}$, $\Theta_k : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4), $A_k : H \to H$ be η_i-inverse strongly monotone and $\varphi_k : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. For each $i \in \{1, 2, ..., N\}$, let $R_i : C \to 2^H$ be a maximal monotone mapping and $B_i : C \to H$ be μ_k-inverse strongly monotone. For $j = 1, 2$, let the mapping $F_j : C \to H$ be ζ_j-inverse strongly monotone. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ be an l-Lipschitz continuous mapping with $0 \le \gamma l < \bar{\gamma}$. Assume that $\Omega = \emptyset$, and W_n be the W-mapping defined by (2.2). Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such the following conditions hold:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$*;*
(iii) $\sum_{n=1}^{\infty} |\lambda_{i,n+1} - \lambda_{i,n}| < \infty$ *for each* $i \in \{1, 2, ..., N\}$ *and* $\sum_{n=1}^{\infty} |r_{k,n+1} - r_{k,n}| < \infty$ *for each* $k \in \{1, 2, ..., M\}$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (V - \gamma f))x^*$ is a unique solution of the following VIP:

$$\langle (V - \gamma f)x^*, x^* - x\rangle \le 0, \quad \text{for all } x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Vx, x\rangle - h(x),$$

where h is a potential function for γf.

Proof. Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, we may assume without loss of generality that for some $0 < c, d < 1$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$ and $\alpha_n \le (1 - \beta_n)\|V\|^{-1}$ for all $n \ge 1$. Since V is a $\bar{\gamma}$-strongly positive bounded linear operator on H, we know that

$$\|V\| = \sup\{\langle Vu, u\rangle : u \in H, \|u\| = 1\}.$$

Observe that

$$\begin{aligned}\langle ((1 - \beta_n)I - \alpha_n V)u, u\rangle &= 1 - \beta_n - \alpha_n\langle Vu, u\rangle \\ &\ge 1 - \beta_n - \alpha_n\|V\| \\ &\ge 0.\end{aligned}$$

It follows that

$$\begin{aligned}\|(1 - \beta_n)I - \alpha_n V\| &= \sup\{\langle ((1 - \beta_n)I - \alpha_n V)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Vu, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1 - \beta_n - \alpha_n\bar{\gamma}.\end{aligned}$$

Put

$$\Delta_n^k = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})}(I - r_{k-1,n}A_{k-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)x_n,$$

for all $k \in \{1, 2, ..., M\}$ and $n \ge 1$,

$$\Lambda_n^i = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I - \lambda_{i-1,n}B_{i-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1),$$

for all $i \in \{1, 2, ..., N\}$ and $n \ge 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then, we have $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$. We observe that $P_\Omega(\gamma f + (I - A))$ is a contraction.

Indeed, for all $x, y \in H$ we have

$$\begin{aligned}\|P_{\Omega}(\gamma f+(I-V))x-P_{\Omega}(\gamma f+(I-V))y\| &\le \|(\gamma f+(I-V))x-(\gamma f+(I-V))y\|\\ &\le \gamma\|f(x)-f(y)\|+\|I-V\|\|x-y\|\\ &\le \gamma l\|x-y\|+(1-\bar{\gamma})\|x-y\|\\ &=(1-(\bar{\gamma}-\gamma l))\|x-y\|.\end{aligned}$$

By Banach contraction principle, we deduce that $P_{\Omega}(\gamma f+(I-V))$ has a unique fixed point $x^* \in H$, that is, $x^*=P_{\Omega}(\gamma f+(I-V))x^*$.

We divide the rest of the proof into six steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take a fixed $p\in\Omega$ arbitrarily. Utilizing (2.1) and Proposition 2.13 (b), we have

$$\begin{aligned}\|u_n-p\| &= \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_n^{M-1}x_n-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}B_M)\Delta_n^{M-1}p\|\\ &\le \|(I-r_{M,n}B_M)\Delta_n^{M-1}x_n-(I-r_{M,n}B_M)\Delta_n^{M-1}p\|\\ &\le \|\Delta_n^{M-1}x_n-\Delta_n^{M-1}p\|\\ &\ \vdots\\ &\le \|\Delta_n^0x_n-\Delta_n^0p\|\\ &=\|x_n-p\|.\end{aligned} \tag{3.2}$$

Utilizing (2.1) and Lemma 2.17, we have

$$\begin{aligned}\|v_n-p\| &= \|J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}u_n-J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}p\|\\ &\le \|(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}u_n-(I-\lambda_{N,n}A_N)\Lambda_n^{N-1}p\|\\ &\le \|\Lambda_n^{N-1}u_n-\Lambda_n^{N-1}p\|\\ &\ \vdots\\ &\le \|\Lambda_n^0u_n-\Lambda_n^0p\|\\ &=\|u_n-p\|.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\|v_n-p\|\le\|x_n-p\|. \tag{3.4}$$

Since $p=Gp=P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)p$, F_j is ζ_j-inverse-strongly monotone for $j=1,2$, and $0<\nu_j\le 2\zeta_j$ for $j=1,2$, we deduce that, for any $n\ge 1$,

$$\begin{aligned}\|Gv_n-p\|^2 &= \|P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)v_n-P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)p\|^2\\ &\le \|(I-\nu_1F_1)P_C(I-\nu_2F_2)v_n-(I-\nu_1F_1)P_C(I-\nu_2F_2)p\|^2\\ &= \|[P_C(I-\nu_2F_2)v_n-P_C(I-\nu_2F_2)p]-\nu_1[F_1P_C(I-\nu_2F_2)v_n-F_1P_C(I-\nu_2F_2)p]\|^2\\ &\le \|P_C(I-\nu_2F_2)v_n-P_C(I-\nu_2F_2)p\|^2\\ &\quad+\nu_1(\nu_1-2\zeta_1)\|F_1P_C(I-\nu_2F_2)v_n-F_1P_C(I-\nu_2F_2)p\|^2\\ &\le \|P_C(I-\nu_2F_2)v_n-P_C(I-\nu_2F_2)p\|^2\\ &\le \|(I-\nu_2F_2)v_n-(I-\nu_2F_2)p\|^2\\ &= \|(v_n-p)-\nu_2(F_2v_n-F_2p)\|^2\\ &\le \|v_n-p\|^2+\nu_2(\nu_2-2\zeta_2)\|F_2v_n-F_2p\|^2\\ &\le \|v_n-p\|^2.\end{aligned} \tag{3.5}$$

This shows that G is nonexpansive. Thus, from (3.1), (3.4), (3.5) and $W_np = p$, we get

$$\begin{aligned}
\|x_{n+1}-p\| &= \|P_C[\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n V)W_nGv_n]-p\| \\
&\le \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n V)W_nGv_n-p\| \\
&= \|\alpha_n(\gamma f(x_n)-Vp)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n V)(W_nGv_n-p)\| \\
&\le \|(1-\beta_n)I-\alpha_n V\|\|W_nGv_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Vp\| \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})\|Gv_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Vp\| \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})\|v_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Vp\| \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Vp\| \\
&\le (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n(\gamma\|f(x_n)-f(p)\|+\|\gamma f(p)-Vp\|) \\
&\le (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n(\gamma l\|x_n-p\|+\|\gamma f(p)-Vp\|) \\
&= [1-(\bar{\gamma}-\gamma l)\alpha_n]\|x_n-p\|+\alpha_n\|\gamma f(p)-Vp\| \\
&= [1-(\bar{\gamma}-\gamma l)\alpha_n]\|x_n-p\|+(\bar{\gamma}-\gamma l)\alpha_n\frac{\|\gamma f(p)-Vp\|}{\bar{\gamma}-\gamma l} \\
&\le \max\{\|x_n-p\|,\frac{\|\gamma f(p)-Vp\|}{\bar{\gamma}-\gamma l}\}.
\end{aligned}$$

By induction, we obtain

$$\|x_n-p\|\le \max\{\|x_0-p\|,\frac{\|\gamma f(p)-Vp\|}{\bar{\gamma}-\gamma l}\}.$$

Therefore, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$.

Step 2. We prove that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. Indeed, we write $y_n=\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n V)W_nGv_n$. Then, $x_{n+1}=P_Cy_n$ for each $n\ge 1$. Define $y_n=\beta_n x_n+(1-\beta_n)w_n$ for each $n\ge 1$. Then from the definition of w_n, we obtain

$$\begin{aligned}
&w_{n+1}-w_n \\
&= \frac{y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{y_n-\beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}V)W_{n+1}Gv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n V)W_nGv_n}{1-\beta_n} \\
&= \gamma(\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}f(x_n))+(I-\frac{\alpha_{n+1}V}{1-\beta_{n+1}})W_{n+1}Gv_{n+1}-(I-\frac{\alpha_n V}{1-\beta_n})W_nGv_n \\
&= \gamma[(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})f(x_{n+1})+\frac{\alpha_n}{1-\beta_n}(f(x_{n+1})-f(x_n))] \\
&\quad +((I-\frac{\alpha_{n+1}V}{1-\beta_{n+1}})-(I-\frac{\alpha_n V}{1-\beta_n}))W_{n+1}Gv_{n+1}+(I-\frac{\alpha_n V}{1-\beta_n})(W_{n+1}Gv_{n+1}-W_nGv_n) \\
&= (\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n})(\gamma f(x_{n+1})-VW_{n+1}Gv_{n+1})+\frac{\alpha_n}{1-\beta_n}\gamma(f(x_{n+1})-f(x_n)) \\
&\quad +(I-\frac{\alpha_n}{1-\beta_n}V)(W_{n+1}Gv_{n+1}-W_nGv_n).
\end{aligned}$$

It follows from Lemma 2.3 that

$$\begin{aligned}\|w_{n+1}-w_n\| &\le |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}-\tfrac{\alpha_n}{1-\beta_n}|\|\gamma f(x_{n+1})-VW_{n+1}Gv_{n+1}\|+\tfrac{\alpha_n}{1-\beta_n}\gamma\|f(x_{n+1})-f(x_n)\| \\ &\quad +\|(I-\tfrac{\alpha_n}{1-\beta_n}V)(W_{n+1}Gv_{n+1}-W_nGv_n)\| \\ &\le |\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}-\tfrac{\alpha_n}{1-\beta_n}|\|\gamma f(x_{n+1})-VW_{n+1}Gv_{n+1}\|+\tfrac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\| \\ &\quad +\|I-\tfrac{\alpha_n}{1-\beta_n}V\|\|W_{n+1}Gv_{n+1}-W_nGv_n\| \\ &\le |\tfrac{(\alpha_{n+1}-\alpha_n)(1-\beta_n)+\alpha_n(\beta_{n+1}-\beta_n)}{(1-\beta_n)(1-\beta_{n+1})}|\|\gamma f(x_{n+1})-VW_{n+1}Gv_{n+1}\|+\tfrac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\| \\ &\quad +\|I-\tfrac{\alpha_n}{1-\beta_n}V\|(\|W_{n+1}Gv_{n+1}-W_{n+1}Gv_n\|+\|W_{n+1}Gv_n-W_nGv_n\|) \\ &\le \tfrac{|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1})-VW_{n+1}Gv_{n+1}\|+\tfrac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\| \\ &\quad +(1-\tfrac{\alpha_n}{1-\beta_n}\bar{\gamma})(\|Gv_{n+1}-Gv_n\|+\|W_{n+1}Gv_n-W_nGv_n\|) \\ &\le \tfrac{|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1})-VW_{n+1}Gv_{n+1}\|+\tfrac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\| \\ &\quad +(1-\tfrac{\alpha_n}{1-\beta_n}\bar{\gamma})(\|v_{n+1}-v_n\|+\|W_{n+1}Gv_n-W_nGv_n\|).\end{aligned} \tag{3.6}$$

Since W_n, T_n and $U_{n,i}$ are all nonexpansive, from (2.2), we have

$$\begin{aligned}\|W_{n+1}Gv_n-W_nGv_n\| &= \|\lambda_1T_1U_{n+1,2}Gv_n-\lambda_1T_1U_{n,2}Gv_n\| \\ &\le \lambda_1\|U_{n+1,2}Gv_n-U_{n,2}Gv_n\| \\ &= \lambda_1\|\lambda_2T_2U_{n+1,3}Gv_n-\lambda_2T_2U_{n,3}Gv_n\| \\ &\le \lambda_1\lambda_2\|U_{n+1,3}Gv_n-U_{n,3}Gv_n\| \\ &\ \ \vdots \\ &\le \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}Gv_n-U_{n,n+1}Gv_n\| \\ &\le \widehat{M}\prod_{i=1}^{n}\lambda_i,\end{aligned} \tag{3.7}$$

where $\sup_{n\ge1}\{\|U_{n+1,n+1}Gv_n\|+\|U_{n,n+1}Gv_n\|\}\le\widehat{M}$ for some $\widehat{M}>0$. Utilizing (2.1) and (2.4), we get

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|\Lambda_{n+1}^Nu_{n+1}-\Lambda_n^Nu_n\| \\ &= \|J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n+1}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\| \\ &\le \|J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n+1}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}\| \\ &\quad +\|J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\| \\ &\le \|(I-\lambda_{N,n+1}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}\| \\ &\quad +\|(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\|+|\lambda_{N,n+1}-\lambda_{N,n}|\times \\ &\quad \times(\tfrac{1}{\lambda_{N,n+1}}\|J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\| \\ &\quad +\tfrac{1}{\lambda_{N,n}}\|(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\|) \\ &\le |\lambda_{N,n+1}-\lambda_{N,n}|(\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\|+\widetilde{M})+\|\Lambda_{n+1}^{N-1}u_{n+1}-\Lambda_n^{N-1}u_n\| \\ &\le |\lambda_{N,n+1}-\lambda_{N,n}|(\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\|+\widetilde{M}) \\ &\quad +|\lambda_{N-1,n+1}-\lambda_{N-1,n}|(\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\|+\widetilde{M})+\|\Lambda_{n+1}^{N-2}u_{n+1}-\Lambda_n^{N-2}u_n\| \\ &\ \ \vdots \\ &\le |\lambda_{N,n+1}-\lambda_{N,n}|(\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\|+\widetilde{M}) \\ &\quad +|\lambda_{N-1,n+1}-\lambda_{N-1,n}|(\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\|+\widetilde{M}) \\ &\quad +\cdots+|\lambda_{1,n+1}-\lambda_{1,n}|(\|B_1\Lambda_{n+1}^{0}u_{n+1}\|+\widetilde{M})+\|\Lambda_{n+1}^0u_{n+1}-\Lambda_n^0u_n\| \\ &\le \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,n+1}-\lambda_{i,n}|+\|u_{n+1}-u_n\|,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}\sup_{n\ge0}\{&\tfrac{1}{\lambda_{N,n+1}}\|J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\| \\ &+\tfrac{1}{\lambda_{N,n}}\|(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1}-J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\|\}\le\widetilde{M},\end{aligned}$$

for some $\widetilde{M} > 0$ and $\sup_{n\geq 0}\{\sum_{i=1}^N \|B_i\Lambda_{n+1}^{i-1}u_{n+1}\| + \widetilde{M}\} \leq \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$.

Utilizing Proposition 2.13 (b), (e), we deduce

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \|\Delta_{n+1}^M x_{n+1} - \Delta_n^M x_n\| \\
&= \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)\Delta_n^{M-1}x_n\| \\
&\leq \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)\Delta_n^{M-1}x_n\| \\
&\leq \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I - r_{M,n}A_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|(I - r_{M,n}A_M)\Delta_{n+1}^{M-1}x_{n+1} - (I - r_{M,n}A_M)\Delta_n^{M-1}x_n\| \\
&\leq \frac{|r_{M,n+1} - r_{M,n}|}{r_{M,n+1}}\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - (I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + |r_{M,n+1} - r_{M,n}|\|A_M\Delta_{n+1}^{M-1}x_{n+1}\| + \|\Delta_{n+1}^{M-1}x_{n+1} - \Delta_n^{M-1}x_n\| \qquad (3.9)\\
&= |r_{M,n+1} - r_{M,n}|[\|A_M\Delta_{n+1}^{M-1}x_{n+1}\| + \frac{1}{r_{M,n+1}}\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - (I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1}\|] + \|\Delta_{n+1}^{M-1}x_{n+1} - \Delta_n^{M-1}x_n\| \\
&\vdots \\
&\leq |r_{M,n+1} - r_{M,n}|[\|A_M\Delta_{n+1}^{M-1}x_{n+1}\| + \frac{1}{r_{M,n+1}}\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - (I - r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1}\|] + \cdots + |r_{1,n+1} - r_{1,n}|[\|A_1\Delta_{n+1}^0x_{n+1}\| \\
&\quad + \frac{1}{r_{1,n+1}}\|T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)}(I - r_{1,n+1}A_1)\Delta_{n+1}^0x_{n+1} - (I - r_{1,n+1}A_1)\Delta_{n+1}^0x_{n+1}\|] \\
&\quad + \|\Delta_{n+1}^0x_{n+1} - \Delta_n^0x_n\| \\
&\leq \widetilde{M}_1\sum_{k=1}^M|r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\|,
\end{aligned}
$$

where $\widetilde{M}_1 > 0$ is a constant such that for each $n \geq 0$ and so we have

$$
\sum_{k=1}^M[\|A_k\Delta_{n+1}^{k-1}x_{n+1}\| + \frac{1}{r_{k,n+1}}\|T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)}(I - r_{k,n+1}A_k)\Delta_{n+1}^{k-1}x_{n+1} - (I - r_{k,n+1}A_k)\Delta_{n+1}^{k-1}x_{n+1}\|] \leq \widetilde{M}_1.
$$

Combining (3.6)–(3.9), we obtain

$$
\begin{aligned}
\|w_{n+1} - w_n\| &\leq \frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1}) - VW_{n+1}Gv_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1} - x_n\| \\
&\quad + (1 - \frac{\alpha_n}{1-\beta_n}\bar{\gamma})(\|v_{n+1} - v_n\| + \|W_{n+1}Gv_n - W_nGv_n\|) \\
&\leq \frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1}) - VW_{n+1}Gv_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1} - x_n\| \qquad (3.10)\\
&\quad + (1 - \frac{\alpha_n}{1-\beta_n}\bar{\gamma})\{\widetilde{M}_0\sum_{i=1}^N|\lambda_{i,n+1} - \lambda_{i,n}| + \|u_{n+1} - u_n\| + \widehat{M}\prod_{i=1}^n\lambda_i\} \\
&\leq \frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1}) - VW_{n+1}Gv_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1} - x_n\|
\end{aligned}
$$

$$
\begin{aligned}
&+ (1 - \frac{\alpha_n}{1-\beta_n}\bar{\gamma})\{\widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\| + \widehat{M} b^n\} \\
\le\ & \frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{(1-d)(1-\beta_n)} \|\gamma f(x_{n+1}) - VW_{n+1}Gv_{n+1}\| + (1 - \frac{\alpha_n}{1-\beta_n}(\bar{\gamma} - \gamma l))\|x_{n+1} - x_n\| \\
&+ \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \widehat{M} b^n.
\end{aligned}
$$

Note that

$$
y_{n+1} - y_n = \beta_n(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)(x_{n+1} - w_{n+1}) + (1-\beta_n)(w_{n+1} - w_n).
$$

Hence it follows from (3.10) that

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|P_C y_{n+1} - P_C y_n\| \\
&\le \|y_{n+1} - y_n\| \\
&\le \beta_n \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| + (1-\beta_n)\|w_{n+1} - w_n\| \\
&\le \beta_n \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| \\
&\quad + (1-\beta_n)\{\frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{(1-d)(1-\beta_n)} \|\gamma f(x_{n+1}) - VW_{n+1}Gv_{n+1}\| \\
&\quad + (1 - \frac{\alpha_n}{1-\beta_n}(\bar{\gamma} - \gamma l))\|x_{n+1} - x_n\| + \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| \\
&\quad + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \widehat{M} b^n\} \\
&\le [1 - (\bar{\gamma} - \gamma l)\alpha_n]\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| \\
&\quad + \frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{1-d} \|\gamma f(x_{n+1}) - VW_{n+1}Gv_{n+1}\| \\
&\quad + \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \widehat{M} b^n \\
&\le [1 - (\bar{\gamma} - \gamma l)\alpha_n]\|x_{n+1} - x_n\| + \widetilde{M}_2(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| \\
&\quad + \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + b^n),
\end{aligned}
$$

where $\sup_{n\ge1}\{\frac{\gamma f(x_n) - VW_nGv_n\|}{1-d} + \|x_n - w_n\| + \widetilde{M}_0 + \widetilde{M}_1 + \widehat{M}\} \le \widetilde{M}_2$ for some $\widetilde{M}_2 > 0$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{i,n+1} - \lambda_{i,n}| < \infty$ and $\sum_{n=1}^{\infty} |r_{k,n+1} - r_{k,n}| < \infty$ where $i \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$, from $b \in (0,1)$ and Lemma 2.21 we conclude that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}
$$

Step 3. We prove that $\lim_{n\to\infty} \|v_n - Gv_n\| = 0$. Indeed, for simplicity, we write $\tilde{v}_n = P_C(I - \nu_2 F_2)v_n$, $z_n = P_C(I - \nu_1 F_1)\tilde{v}_n$ and $\tilde{p} = P_C(I - \nu_2 F_2)p$. Then $z_n = Gv_n$ and

$$
p = P_C(I - \nu_1 F_1)\tilde{p} = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)p = Gp.
$$

From (3.1), (3.4), (3.5) and Proposition 2.10 (a) and Lemma 2.2 (b), we obtain that for $p \in \Omega$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|\alpha_n(\gamma f(x_n) - VW_nGv_n) + \beta_n(x_n-p) + (1-\beta_n)(W_nGv_n - p)\|^2 \\
&= \|\beta_n(x_n-p) + (1-\beta_n)(W_nGv_n-p)\|^2 + \alpha_n^2\|\gamma f(x_n) - VW_nGv_n\|^2 \\
&\quad + 2\alpha_n\langle(\gamma f(x_n) - VW_nGv_n), \beta_n(x_n-p) + (1-\beta_n)(W_nGv_n-p)\rangle \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|W_nGv_n-p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|[2\|\beta_n(x_n-p) + (1-\beta_n)(W_nGv_n-p)\| \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|] \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|Gv_n-p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|[2(\beta_n\|x_n-p\| + (1-\beta_n)\|v_n-p\|) \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|] \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|Gv_n-p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n-p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)[\|v_n-p\|^2 + \nu_2(\nu_2-2\zeta_2)\|F_2v_n - F_2p\|^2 \\
&\quad + \nu_1(\nu_1-2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n-p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)[\|v_n-p\|^2 + \nu_2(\nu_2-2\zeta_2)\|F_2v_n - F_2p\|^2 \\
&\quad + \nu_1(\nu_1-2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n-p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)[\|x_n-p\|^2 + \nu_2(\nu_2-2\zeta_2)\|F_2v_n - F_2p\|^2 \\
&\quad + \nu_1(\nu_1-2\zeta_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n-p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&= \|x_n-p\|^2 - (1-\beta_n)[\nu_2(2\zeta_2-\nu_2)\|F_2v_n - F_2p\|^2 \\
&\quad + \nu_1(2\zeta_1-\nu_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n-p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|),
\end{aligned} \tag{3.12}$$

which immediately implies that

$$\begin{aligned}
&(1-d)[\nu_2(2\zeta_2-\nu_2)\|F_2v_n - F_2p\|^2 + \nu_1(2\zeta_1-\nu_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] + c(1-d)\|x_n - W_nGv_n\|^2 \\
&\le (1-\beta_n)[\nu_2(2\zeta_2-\nu_2)\|F_2v_n - F_2p\|^2 + \nu_1(2\zeta_1-\nu_1)\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\
&\quad + \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\alpha_n\|x_n-p\|\|\gamma f(x_n) - VW_nGv_n\| \\
&\quad + \alpha_n^2\|\gamma f(x_n) - VW_nGv_n\|^2 \\
&\le \|x_n - x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + 2\alpha_n\|x_n-p\|\|\gamma f(x_n) - VW_nGv_n\| \\
&\quad + \alpha_n^2\|\gamma f(x_n) - VW_nGv_n\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\nu_i \in (0, 2\zeta_i)$, $i = 1, 2$, we deduce from the boundedness of $\{x_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$ that

$$\lim_{n\to\infty}\|F_2v_n - F_2p\| = 0, \quad \lim_{n\to\infty}\|F_1\tilde{v}_n - F_1\tilde{p}\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|x_n - W_nGv_n\| = 0. \tag{3.13}$$

Also, in terms of the firm nonexpansivity of P_C and the ζ_j-inverse strong monotonicity of F_j for $j = 1, 2$, we obtain from $\nu_j \in (0, 2\zeta_j)$, $j = 1, 2$ and (3.5) that

$$\begin{aligned}
\|\tilde{v}_n - \tilde{p}\|^2 &= \|P_C(I-\nu_2F_2)v_n - P_C(I-\nu_2F_2)p\|^2 \\
&\le \langle(I-\nu_2F_2)v_n - (I-\nu_2F_2)p, \tilde{v}_n - \tilde{p}\rangle \\
&= \tfrac{1}{2}[\|(I-\nu_2F_2)v_n - (I-\nu_2F_2)p\|^2 + \|\tilde{v}_n - \tilde{p}\|^2 \\
&\quad - \|(I-\nu_2F_2)v_n - (I-\nu_2F_2)p - (\tilde{v}_n - \tilde{p})\|^2] \\
&\le \tfrac{1}{2}[\|v_n-p\|^2 + \|\tilde{v}_n - \tilde{p}\|^2 - \|(v_n - \tilde{v}_n) - \nu_2(F_2v_n - F_2p) - (p - \tilde{p})\|^2] \\
&= \tfrac{1}{2}[\|v_n-p\|^2 + \|\tilde{v}_n - \tilde{p}\|^2 - \|(v_n - \tilde{v}_n) - (p - \tilde{p})\|^2 \\
&\quad + 2\nu_2\langle(v_n - \tilde{v}_n) - (p - \tilde{p}), F_2v_n - F_2p\rangle - \nu_2^2\|F_2v_n - F_2p\|^2],
\end{aligned}$$

and

$$\begin{aligned} ll\|z_n - p\|^2 &= \|P_C(I-\nu_1 F_1)\tilde{v}_n - P_C(I-\nu_1 F_1)\tilde{p}\|^2 \\ &\le \langle (I-\nu_1 F_1)\tilde{v}_n - (I-\nu_1 F_1)\tilde{p}, z_n - p\rangle \\ &= \frac{1}{2}[\|(I-\nu_1 F_1)\tilde{v}_n - (I-\nu_1 F_1)\tilde{p}\|^2 + \|z_n - p\|^2 \\ &\quad - \|(I-\nu_1 F_1)\tilde{v}_n - (I-\nu_1 F_1)\tilde{p} - (z_n - p)\|^2] \\ &\le \frac{1}{2}[\|\tilde{v}_n - \tilde{p}\|^2 + \|z_n - p\|^2 - \|(\tilde{v}_n - z_n) + (p - \tilde{p})\|^2 \\ &\quad + 2\nu_1\langle F_1\tilde{v}_n - F_1\tilde{p}, (\tilde{v}_n - z_n) + (p-\tilde{p})\rangle - \nu_1^2\|F_1\tilde{v}_n - F_1\tilde{p}\|^2] \\ &\le \frac{1}{2}[\|v_n - p\|^2 + \|z_n - p\|^2 - \|(\tilde{v}_n - z_n) + (p - \tilde{p})\|^2 \\ &\quad + 2\nu_1\langle F_1\tilde{v}_n - F_1\tilde{p}, (\tilde{v}_n - z_n) + (p-\tilde{p})\rangle]. \end{aligned}$$

Thus, we have

$$\|\tilde{v}_n - \tilde{p}\|^2 \le \|v_n - p\|^2 - \|(v_n - \tilde{v}_n) - (p-\tilde{p})\|^2 + 2\nu_2\langle (v_n - \tilde{v}_n) - (p - \tilde{p}), F_2 v_n - F_2 p\rangle - \nu_2^2\|F_2 v_n - F_2 p\|^2, \tag{3.14}$$

and

$$\|z_n - p\|^2 \le \|v_n - p\|^2 - \|(\tilde{v}_n - z_n) + (p - \tilde{p})\|^2 + 2\nu_1\|F_1\tilde{v}_n - F_1\tilde{p}\|\|(\tilde{v}_n - z_n) + (p-\tilde{p})\|. \tag{3.15}$$

Consequently, from (3.4), (3.5), (3.12) and (3.14), it follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Gv_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGv_n\|^2 \\ &\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\tilde{v}_n - \tilde{p}\|^2 \\ &\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|v_n - p\|^2 - \|(v_n - \tilde{v}_n) - (p-\tilde{p})\|^2 \\ &\quad + 2\nu_2\|(v_n - \tilde{v}_n) - (p - \tilde{p})\|\|F_2 v_n - F_2 p\|] \\ &\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\ &\le \|x_n - p\|^2 - (1-\beta_n)\|(v_n - \tilde{v}_n) - (p-\tilde{p})\|^2 + 2\nu_2\|(v_n - \tilde{v}_n) - (p-\tilde{p})\|\|F_2 v_n - F_2 p\| \\ &\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|), \end{aligned}$$

which yields

$$\begin{aligned} &(1-d)\|(v_n - \tilde{v}_n) - (p - \tilde{p})\|^2 \\ &\quad\le (1-\beta_n)\|(v_n - \tilde{v}_n) - (p-\tilde{p})\|^2 \\ &\quad\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\nu_2\|(v_n - \tilde{v}_n) - (p-\tilde{p})\|\|F_2 v_n - F_2 p\| \\ &\qquad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\ &\quad\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\nu_2\|(v_n - \tilde{v}_n) - (p-\tilde{p})\|\|F_2 v_n - F_2 p\| \\ &\qquad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|). \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|F_2 v_n - F_2 p\| = 0$, we deduce from the boundedness of $\{x_n\}$, $\{v_n\}$, $\{\tilde{v}_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$ that

$$\lim_{n\to\infty}\|(v_n - \tilde{v}_n) - (p - \tilde{p})\| = 0. \tag{3.16}$$

Furthermore, from (3.4), (3.5), (3.12) and (3.15), it follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|Gv_n-p\|^2 \\
&\quad +\alpha_n\|\gamma f(x_n)-VW_nGv_n\|(2\|x_n-p\|+\alpha_n\|\gamma f(x_n)-VW_nGv_n\|) \\
&= \beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2 \\
&\quad +\alpha_n\|\gamma f(x_n)-VW_nGv_n\|(2\|x_n-p\|+\alpha_n\|\gamma f(x_n)-VW_nGv_n\|) \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\|v_n-p\|^2-\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|^2 \\
&\quad +2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|] \\
&\quad +\alpha_n\|\gamma f(x_n)-VW_nGv_n\|(2\|x_n-p\|+\alpha_n\|\gamma f(x_n)-VW_nGv_n\|) \\
&\le \|x_n-p\|^2-(1-\beta_n)\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|^2+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-z_n)+(p-\tilde{p})\| \\
&\quad +\alpha_n\|\gamma f(x_n)-VW_nGv_n\|(2\|x_n-p\|+\alpha_n\|\gamma f(x_n)-VW_nGv_n\|),
\end{aligned}$$

which leads to

$$\begin{aligned}
&(1-d)\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|^2 \\
&\qquad \le (1-\beta_n)\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|^2 \\
&\qquad \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-z_n)+(p-\tilde{p})\| \\
&\qquad\quad +\alpha_n\|\gamma f(x_n)-VW_nGv_n\|(2\|x_n-p\|+\alpha_n\|\gamma f(x_n)-VW_nGv_n\|) \\
&\qquad \le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\nu_1\|F_1\tilde{v}_n-F_1\tilde{p}\|\|(\tilde{v}_n-z_n)+(p-\tilde{p})\| \\
&\qquad\quad +\alpha_n\|\gamma f(x_n)-VW_nGv_n\|(2\|x_n-p\|+\alpha_n\|\gamma f(x_n)-VW_nGv_n\|).
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|F_1\tilde{v}_n-F_1\tilde{p}\|=0$, we deduce from the boundedness of $\{x_n\}$, $\{z_n\}$, $\{\tilde{v}_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$ that

$$\lim_{n\to\infty}\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|=0. \tag{3.17}$$

Note that

$$\|v_n-z_n\|\le\|(v_n-\tilde{v}_n)-(p-\tilde{p})\|+\|(\tilde{v}_n-z_n)+(p-\tilde{p})\|.$$

Hence, from (3.16) and (3.17), we get

$$\lim_{n\to\infty}\|v_n-z_n\|=\lim_{n\to\infty}\|v_n-Gv_n\|=0. \tag{3.18}$$

Step 4. We prove that $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|x_n-v_n\|=0$ and $\lim_{n\to\infty}\|v_n-Wv_n\|=0$. Indeed, observe that

$$\begin{aligned}
\|\Delta_n^k x_n-p\|^2 &= \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)\Delta_n^{k-1}x_n-T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)p\|^2 \\
&\le \|(I-r_{k,n}A_k)\Delta_n^{k-1}x_n-(I-r_{k,n}A_k)p\|^2 \\
&\le \|\Delta_n^{k-1}x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\|A_k\Delta_n^{k-1}x_n-A_kp\|^2 \\
&\le \|x_n-p\|^2+r_{k,n}(r_{k,n}-2\mu_k)\|A_k\Delta_n^{k-1}x_n-A_kp\|^2,
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
\|\Lambda_n^i u_n-p\|^2 &= \|J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n-J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)p\|^2 \\
&\le \|(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n-(I-\lambda_{i,n}B_i)p\|^2 \\
&\le \|\Lambda_n^{i-1}u_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|B_i\Lambda_n^{i-1}u_n-B_ip\|^2 \\
&\le \|u_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|B_i\Lambda_n^{i-1}u_n-B_ip\|^2 \\
&\le \|x_n-p\|^2+\lambda_{i,n}(\lambda_{i,n}-2\eta_i)\|B_i\Lambda_n^{i-1}u_n-B_ip\|^2,
\end{aligned} \tag{3.20}$$

for $i \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$. Combining (3.4), (3.5), (3.12), (3.19) and (3.20), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|Gv_n - p\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\Lambda_n^i u_n - p\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2] \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|\Delta_n^k x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2] \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\|A_k\Delta_n^{k-1}x_n - A_kp\|^2 \\
&\quad + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2] \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&= \|x_n - p\|^2 + (1 - \beta_n)[r_{k,n}(r_{k,n} - 2\mu_k)\|A_k\Delta_n^{k-1}x_n - A_kp\|^2 \\
&\quad + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2] \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|),
\end{aligned}$$

which hence implies that

$$\begin{aligned}
&(1 - d)[r_{k,n}(2\mu_k - r_{k,n})\|A_k\Delta_n^{k-1}x_n - A_kp\|^2 + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2] \\
&\le (1 - \beta_n)[r_{k,n}(2\mu_k - r_{k,n})\|A_k\Delta_n^{k-1}x_n - A_kp\|^2 + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2] \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\|x_n - p\|\|\gamma f(x_n) - VW_nGv_n\| \\
&\quad + \alpha_n^2\|\gamma f(x_n) - VW_nGv_n\|^2 \\
&\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n\|x_n - p\|\|\gamma f(x_n) - VW_nGv_n\| \\
&\quad + \alpha_n^2\|\gamma f(x_n) - VW_nGv_n\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [c_k, d_k] \subset (0, 2\mu_k)$ where $i \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$, we deduce from the boundedness of $\{x_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$ that

$$\lim_{n\to\infty} \|A_k\Delta_n^{k-1}x_n - A_kp\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|B_i\Lambda_n^{i-1}u_n - B_ip\| = 0, \tag{3.21}$$

where $i \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$.

Furthermore, by Proposition 2.13 (b) and Lemma 2.2 (a), we have

$$\begin{aligned}
\|\Delta_n^k x_n - p\|^2 &= \|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)p\|^2 \\
&\le \langle (I - r_{k,n}A_k)\Delta_n^{k-1}x_n - (I - r_{k,n}A_k)p, \Delta_n^k x_n - p\rangle \\
&= \frac{1}{2}(\|(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - (I - r_{k,n}A_k)p\|^2 + \|\Delta_n^k x_n - p\|^2 \\
&\quad - \|(I - r_{k,n}A_k)\Delta_n^{k-1}x_n - (I - r_{k,n}A_k)p - (\Delta_n^k x_n - p)\|^2) \\
&\le \frac{1}{2}(\|\Delta_n^{k-1}x_n - p\|^2 + \|\Delta_n^k x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n - r_{k,n}(A_k\Delta_n^{k-1}x_n - A_kp)\|^2),
\end{aligned}$$

which implies that

$$
\begin{aligned}
\|\Delta_n^k x_n - p\|^2 &\le \|\Delta_n^{k-1}x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n - r_{k,n}(A_k\Delta_n^{k-1}x_n - A_kp)\|^2 \\
&= \|\Delta_n^{k-1}x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 - r_{k,n}^2\|A_k\Delta_n^{k-1}x_n - A_kp\|^2 \\
&\quad + 2r_{k,n}\langle \Delta_n^{k-1}x_n - \Delta_n^k x_n, A_k\Delta_n^{k-1}x_n - A_kp\rangle \qquad (3.22)\\
&\le \|\Delta_n^{k-1}x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|A_k\Delta_n^{k-1}x_n - A_kp\| \\
&\le \|x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2 + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|A_k\Delta_n^{k-1}x_n - A_kp\|.
\end{aligned}
$$

By Lemma 2.2 (a) and Lemma 2.17, we obtain

$$
\begin{aligned}
\|\Lambda_n^i u_n - p\|^2 &= \|J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)p\|^2 \\
&\le \langle (I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p, \Lambda_n^i u_n - p\rangle \\
&= \frac{1}{2}(\|(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p\|^2 + \|\Lambda_n^i u_n - p\|^2 \\
&\quad - \|(I-\lambda_{i,n}B_i)\Lambda_n^{i-1}u_n - (I-\lambda_{i,n}B_i)p - (\Lambda_n^i u_n - p)\|^2) \\
&\le \frac{1}{2}(\|\Lambda_n^{i-1}u_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ip)\|^2) \\
&\le \frac{1}{2}(\|u_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ip)\|^2) \\
&\le \frac{1}{2}(\|x_n - p\|^2 + \|\Lambda_n^i u_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ip)\|^2),
\end{aligned}
$$

which immediately leads to

$$
\begin{aligned}
\|\Lambda_n^i u_n - p\|^2 &\le \|x_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n - \lambda_{i,n}(B_i\Lambda_n^{i-1}u_n - B_ip)\|^2 \\
&= \|x_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^k u_n\|^2 - \lambda_{i,n}^2\|B_i\Lambda_n^{i-1}u_n - B_ip\|^2 \\
&\quad + 2\lambda_{i,n}\langle \Lambda_n^{i-1}u_n - \Lambda_n^i u_n, B_i\Lambda_n^{i-1}u_n - B_ip\rangle \qquad (3.23)\\
&\le \|x_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|B_i\Lambda_n^{i-1}u_n - B_ip\|.
\end{aligned}
$$

Combining (3.12) and (3.23), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - p\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\Lambda_n^i u_n - p\|^2 \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|x_n - p\|^2 - \|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\
&\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|B_i\Lambda_n^{i-1}u_n - B_ip\|] \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|) \\
&\le \|x_n - p\|^2 - (1-\beta_n)\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\
&\quad + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|B_i\Lambda_n^{i-1}u_n - B_ip\| \\
&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&(1-d)\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\
&\quad \le (1-\beta_n)\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|B_i\Lambda_n^{i-1}u_n - B_ip\| \\
&\qquad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|)
\end{aligned}
$$

$$\begin{aligned}&\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\lambda_{i,n}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\|\|B_i\Lambda_n^{i-1}u_n - B_ip\|\\&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|).\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ where $i \in \{1, 2, ..., N\}$, we deduce from (3.21) and the boundedness of $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$ that

$$\lim_{n\to\infty}\|\Lambda_n^{i-1}u_n - \Lambda_n^i u_n\| = 0, \quad \forall i \in \{1, 2, ..., N\}. \tag{3.24}$$

Combining (3.3), (3.12) and (3.22), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2\\&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|)\\&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2\\&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|)\\&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\Delta_n^k x_n - p\|^2\\&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|)\\&\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|x_n - p\|^2 - \|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2\\&\quad + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|A_k\Delta_n^{k-1}x_n - A_kp\|]\\&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|)\\&\le \|x_n - p\|^2 - (1 - \beta_n)\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2\\&\quad + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|A_k\Delta_n^{k-1}x_n - A_kp\|\\&\quad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|),\end{aligned}$$

which leads to

$$\begin{aligned}&(1 - d)\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2\\&\quad\le (1 - \beta_n)\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|^2\\&\quad\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|A_k\Delta_n^{k-1}x_n - A_kp\|\\&\qquad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|)\\&\quad\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2r_{k,n}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\|\|A_k\Delta_n^{k-1}x_n - A_kp\|\\&\qquad + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|(2\|x_n - p\| + \alpha_n\|\gamma f(x_n) - VW_nGv_n\|).\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\{r_{k,n}\} \subset [c_k, d_k] \subset (0, 2\mu_k)$ where $k \in \{1, 2, ..., M\}$, we conclude from (3.21) and the boundedness of $\{x_n\}$, $\{v_n\}$, $\{f(x_n)\}$ and $\{W_nGv_n\}$ that

$$\lim_{n\to\infty}\|\Delta_n^{k-1}x_n - \Delta_n^k x_n\| = 0, \quad \forall k \in \{1, 2, ..., M\}. \tag{3.25}$$

Therefore, from (3.24) and (3.25), we get

$$\begin{aligned}\|x_n - u_n\| &= \|\Delta_n^0 x_n - \Delta_n^M x_n\|\\&\le \|\Delta_n^0 x_n - \Delta_n^1 x_n\| + \|\Delta_n^1 x_n - \Delta_n^2 x_n\| + \cdots + \|\Delta_n^{M-1}x_n - \Delta_n^M x_n\|\\&\to 0 \quad \text{as } n \to \infty,\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\|u_n - v_n\| &= \|\Lambda_n^0 u_n - \Lambda_n^N u_n\|\\&\le \|\Lambda_n^0 u_n - \Lambda_n^1 u_n\| + \|\Lambda_n^1 u_n - \Lambda_n^2 u_n\| + \cdots + \|\Lambda_n^{N-1}u_n - \Lambda_n^N u_n\|\\&\to 0 \quad \text{as } n \to \infty,\end{aligned} \tag{3.27}$$

respectively. Thus, from (3.26) and (3.27), we obtain

$$\begin{aligned}\|x_n - v_n\| &\le \|x_n - u_n\| + \|u_n - v_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.28}$$

In the meantime, we observe that

$$\|Gv_n - W_nGv_n\| \le \|Gv_n - v_n\| + \|v_n - x_n\| + \|x_n - W_nGv_n\|.$$

From (3.13), (3.18) and (3.28), it follows that

$$\lim_{n\to\infty} \|Gv_n - W_nGv_n\| = 0. \tag{3.29}$$

Also, note that

$$\begin{aligned}\|v_n - Wv_n\| &\le \|v_n - Gv_n\| + \|Gv_n - W_nGv_n\| + \|W_nGv_n - W_nv_n\| + \|W_nv_n - Wv_n\| \\ &\le 2\|v_n - Gv_n\| + \|Gv_n - W_nGv_n\| + \|W_nv_n - Wv_n\|.\end{aligned}$$

From (3.18), (3.29), Remark 2.8 and the boundedness of $\{v_n\}$, we immediately obtain

$$\lim_{n\to\infty} \|v_n - Wv_n\| = 0. \tag{3.30}$$

Step 5. We prove that

$$\limsup_{n\to\infty} \langle(\gamma f - V)x^*, x_n - x^*\rangle \le 0,$$

where $x^* = P_\Omega(\gamma f + (I - V))x^*$. Indeed, as previously, we have proven that x^* is the unique fixed point of the mapping $P_\Omega(\gamma f + (I - V))$, that is, x^* is the unique solution in Ω to the following VIP:

$$\langle(V - \gamma f)x^*, x^* - x\rangle \le 0, \quad \forall x \in \Omega.$$

Equivalently, x^* is the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Vx, x\rangle - h(x),$$

where h is a potential function for γf. Now, observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle(\gamma f - V)x^*, x_n - x^*\rangle = \lim_{i\to\infty} \langle(\gamma f - V)x^*, x_{n_i} - x^*\rangle. \tag{3.31}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. From (3.24)–(3.26) and (3.31), we have that $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$ and $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, where $m \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$. Utilizing Lemma 2.14, we deduce from (3.18) and (3.30) that $w \in \Gamma$ and $w \in \mathrm{Fix}(W) = \cap_{n=1}^\infty \mathrm{Fix}(T_n)$ (due to Lemma 2.9).

Next, we prove that $w \in \cap_{m=1}^N \mathrm{I}(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 2.20 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$, i.e., $g - B_m v \in R_m v$. Again, since $\Lambda^m_n u_n = J_{R_m,\lambda_{m,n}}(I - \lambda_{m,n} B_m)\Lambda^{m-1}_n u_n, n \ge 1, m \in \{1, 2, ..., N\}$, we have

$$\Lambda^{m-1}_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n \in (I + \lambda_{m,n} R_m)\Lambda^m_n u_n,$$

that is,

$$\frac{1}{\lambda_{m,n}}(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n} B_m \Lambda^{m-1}_n u_n) \in R_m \Lambda^m_n u_n.$$

In terms of the monotonicity of R_m, we get

$$\langle v - \Lambda_n^m u_n, g - B_m v - \frac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n)\rangle \geq 0,$$

and hence,

$$\begin{aligned} \langle v - \Lambda_n^m u_n, g\rangle &\geq \langle v - \Lambda_n^m u_n, B_m v + \tfrac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n - \lambda_{m,n}B_m\Lambda_n^{m-1}u_n)\rangle \\ &= \langle v - \Lambda_n^m u_n, B_m v - B_m\Lambda_n^m u_n + B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n + \tfrac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n)\rangle \\ &\geq \langle v - \Lambda_n^m u_n, B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n\rangle + \langle v - \Lambda_n^m u_n, \tfrac{1}{\lambda_{m,n}}(\Lambda_n^{m-1}u_n - \Lambda_n^m u_n)\rangle. \end{aligned}$$

In particular,

$$\langle v - \Lambda_{n_i}^m u_{n_i}, g\rangle \geq \langle v - \Lambda_{n_i}^m u_{n_i}, B_m\Lambda_{n_i}^m u_{n_i} - B_m\Lambda_{n_i}^{m-1}u_{n_i}\rangle + \langle v - \Lambda_{n_i}^m u_{n_i}, \frac{1}{\lambda_{m,n_i}}(\Lambda_{n_i}^{m-1}u_{n_i} - \Lambda_{n_i}^m u_{n_i})\rangle.$$

Since $\|\Lambda_n^m u_n - \Lambda_n^{m-1}u_n\| \to 0$ (due to (3.24)) and $\|B_m\Lambda_n^m u_n - B_m\Lambda_n^{m-1}u_n\| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda_{n_i}^m u_{n_i} \rightharpoonup w$ and $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ that

$$\lim_{i\to\infty}\langle v - \Lambda_{n_i}^m u_{n_i}, g\rangle = \langle v - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$, i.e., $w \in \mathrm{I}(B_m, R_m)$. Therefore, $w \in \cap_{m=1}^N \mathrm{I}(B_m, R_m)$.

Next we prove that $w \in \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)\Delta_n^{k-1}x_n, n \geq 1, k \in \{1, 2, ..., M\}$, we have

$$\Theta_k(\Delta_n^k x_n, y) + \varphi_k(y) - \varphi_k(\Delta_n^k x_n) + \langle A_k\Delta_n^{k-1}x_n, y - \Delta_n^k x_n\rangle + \frac{1}{r_{k,n}}\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1}x_n\rangle \geq 0.$$

By (A2), we have

$$\varphi_k(y) - \varphi_k(\Delta_n^k x_n) + \langle A_k\Delta_n^{k-1}x_n, y - \Delta_n^k x_n\rangle + \frac{1}{r_{k,n}}\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1}x_n\rangle \geq \Theta_k(y, \Delta_n^k x_n).$$

Let $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned} \langle z_t - \Delta_n^k x_n, A_k z_t\rangle &\geq \varphi_k(\Delta_n^k x_n) - \varphi_k(z_t) + \langle z_t - \Delta_n^k x_n, A_k z_t\rangle - \langle z_t - \Delta_n^k x_n, A_k\Delta_n^{k-1}x_n\rangle \\ &\quad - \langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1}x_n}{r_{k,n}}\rangle + \Theta_k(z_t, \Delta_n^k x_n) \\ &= \varphi_k(\Delta_n^k x_n) - \varphi_k(z_t) + \langle z_t - \Delta_n^k x_n, A_k z_t - A_k\Delta_n^k x_n\rangle \\ &\quad + \langle z_t - \Delta_n^k x_n, A_k\Delta_n^k x_n - A_k\Delta_n^{k-1}x_n\rangle - \langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1}x_n}{r_{k,n}}\rangle + \Theta_k(z_t, \Delta_n^k x_n). \end{aligned} \tag{3.32}$$

By (3.25), we have $\|A_k\Delta_n^k x_n - A_k\Delta_n^{k-1}x_n\| \to 0$ as $n \to \infty$. Furthermore, by the monotonicity of A_k, we obtain $\langle z_t - \Delta_n^k x_n, A_k z_t - A_k\Delta_n^k x_n\rangle \geq 0$. Then, by (A4) we obtain

$$\langle z_t - w, A_k z_t\rangle \geq \varphi_k(w) - \varphi_k(z_t) + \Theta_k(z_t, w). \tag{3.33}$$

Utilizing (A1), (A4) and (3.33), we obtain

$$\begin{aligned} 0 &= \Theta_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t) \\ &\leq t\Theta_k(z_t, y) + (1-t)\Theta_k(z_t, w) + t\varphi_k(y) + (1-t)\varphi_k(w) - \varphi_k(z_t) \\ &\leq t[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)] + (1-t)\langle z_t - w, A_k z_t\rangle \\ &= t[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)] + (1-t)t\langle y - w, A_k z_t\rangle, \end{aligned}$$

and hence,

$$0 \le \Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\langle y - w, A_k z_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \le \Theta_k(w, y) + \varphi_k(y) - \varphi_k(w) + \langle y - w, A_k w\rangle.$$

This implies that $w \in \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$, and hence, $w \in \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. Consequently, $w \in \cap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \cap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \cap_{i=1}^N \mathrm{I}(B_i, R_i) \cap \Gamma =: \Omega$. Therefore, from (3.31) and $x^* = P_\Omega(\gamma f + (I - V))x^*$, we have

$$\limsup_{n\to\infty} \langle (\gamma f - V)x^*, x_n - x^*\rangle = \langle (\gamma f - V)x^*, w - x^*\rangle \le 0. \tag{3.34}$$

Step 6. We prove that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Indeed, taking into account that $x_{n+1} = P_C y_n$ and $y_n = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n V)W_n G v_n$, we obtain from (3.4) and Proposition 2.10 (a) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle P_C y_n - y_n, P_C y_n - x^*\rangle + \langle y_n - x^*, x_{n+1} - x^*\rangle \\
&\le \langle y_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle \alpha_n(\gamma f(x_n) - Vx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n V)(W_n G v_n - x^*), x_{n+1} - x^*\rangle \\
&= \langle \alpha_n \gamma(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n V)(W_n G v_n - x^*), x_{n+1} - x^*\rangle \\
&\quad + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&\le \|\alpha_n \gamma(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n V)(W_n G v_n - x^*)\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&\le [\|\alpha_n \gamma\| f(x_n) - f(x^*)\| + \beta_n \|x_n - x^*\| + \|(1-\beta_n)I - \alpha_n V\| \|W_n G v_n - x^*\|] \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&\le [\alpha_n \gamma l \|x_n - x^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|v_n - x^*\|] \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&\le [\alpha_n \gamma l \|x_n - x^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - x^*\|] \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma l))\|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&\le \tfrac{1}{2}(1 - \alpha_n(\bar{\gamma} - \gamma l))(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle,
\end{aligned}$$

which immediately implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \tfrac{1 - \alpha_n(\bar{\gamma} - \gamma l)}{1 + \alpha_n(\bar{\gamma} - \gamma l)} \|x_n - x^*\|^2 + \tfrac{\alpha_n}{1 + \alpha_n(\bar{\gamma} - \gamma l)} \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&= (1 - \tfrac{2\alpha_n(\bar{\gamma} - \gamma l)}{1 + \alpha_n(\bar{\gamma} - \gamma l)})\|x_n - x^*\|^2 + \tfrac{2\alpha_n(\bar{\gamma} - \gamma l)}{1 + \alpha_n(\bar{\gamma} - \gamma l)} \cdot \tfrac{1}{2(\bar{\gamma} - \gamma l)} \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \\
&= (1 - \gamma_n)\|x_n - x^*\|^2 + \sigma_n \gamma_n,
\end{aligned} \tag{3.35}$$

where $\gamma_n = \frac{2\alpha_n(\bar{\gamma} - \gamma l)}{1 + \alpha_n(\bar{\gamma} - \gamma l)}$ and $\sigma_n = \frac{1}{2(\bar{\gamma} - \gamma l)} \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle$. Note that $\sum_{n=1}^\infty \alpha_n = \infty$ implies $\sum_{n=1}^\infty \gamma_n \ge \frac{2(\bar{\gamma} - \gamma l)}{1 + (\bar{\gamma} - \gamma l)} \cdot \sum_{n=1}^\infty \alpha_n = \infty$ and that (3.34) leads to

$$\limsup_{n\to\infty} \sigma_n = \limsup_{n\to\infty} \frac{1}{2(\bar{\gamma} - \gamma l)} \langle (\gamma f - V)x^*, x_{n+1} - x^*\rangle \le 0.$$

Applying Lemma 2.21 to (3.35), we infer that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Now, we present an example in support of our main result.

Example 3.2. Let $M = 1$. Let $H = \mathbf{R}^2$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ which are defined by

$$\langle x, y\rangle = ac + bd \quad \text{and} \quad \|x\| = \sqrt{a^2 + b^2},$$

for all $x, y \in \mathbf{R}^2$ with $x = (a, b)$ and $y = (c, d)$. Let $C = \{(a, a) : a \in \mathbf{R}\}$. Clearly, C is a nonempty closed convex subset of a real Hilbert space $H = \mathbf{R}^2$. Put $\Theta(x, y) = \langle Ax, y - x\rangle$ and $\varphi(x) = 0$ for all $x, y \in C$ where $A = \left\{ \begin{array}{cc} 2/3 & 1/3 \\ 1/3 & 2/3 \end{array} \right\}$. Then $\Theta : C \times C \to \mathbf{R}$ is a bi-function satisfying hypotheses (A1)-(A4). Put $S = \left\{ \begin{array}{cc} 3/5 & 2/5 \\ 2/5 & 3/5 \end{array} \right\}$. Then $\|A\| = \|S\| = 1$, and A and S are both 2×2 positive definite symmetric matrices. Let $R_1 : C \to 2^H$ be a maximal monotone mapping, for instance, putting

$$R_1 v = \left\{ \begin{array}{ll} Sv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{array} \right.$$

where $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$. In terms of Rockafellar [19] we know that R_1 is maximal monotone and $0 \in R_1 v$ if and only if $v \in \mathrm{VI}(C, S)$. For each $n = 1, 2, ...$, we set $T_n = S$. Then T_n is a nonexpansive self-mapping on C for each $n = 1, 2, ...$ Put $F_1 = I - A = \left\{ \begin{array}{cc} 1/3 & -1/3 \\ -1/3 & 1/3 \end{array} \right\}$, $F_2 = I - S = \left\{ \begin{array}{cc} 2/5 & -2/5 \\ -2/5 & 2/5 \end{array} \right\}$, $V = \left\{ \begin{array}{cc} 8/9 & 4/9 \\ 4/9 & 8/9 \end{array} \right\} = \frac{4}{3}A$ and $\gamma f = \left\{ \begin{array}{cc} 3/6 & 1/6 \\ 1/6 & 3/6 \end{array} \right\} = \frac{1}{2}(A + \frac{1}{3}I)$. Then $B_1 (:= F_1)$ and F_j are 1/2-inverse strongly monotone for each $j = 1, 2$, V is strongly positive bounded linear operator, and $\|\gamma f\| \leq 2/3$. It is easy to see that $\Omega = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{I}(B_1, R_1) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \Gamma = \{0\}$ where Γ is the fixed point set of the mapping $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0, 1)$, and $\{r_n\}$ be a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$. In this case, for any given $x_0 \in C$, the iterative scheme (3.11) is equivalent to the following one:

$$\left\{ \begin{array}{l} u_n = P_C(x_n - r_n A u_n) = x_n - r_n u_n, \\ v_n = J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n} B_1)u_n = u_n - \lambda_{1,n} v_n, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n V)W_n G v_n] \\ \qquad = \frac{2}{3}\alpha_n x_n + \beta_n x_n + (1 - \beta_n - \frac{4}{3}\alpha_n)v_n. \end{array} \right.$$

Note that, whenever $\Theta(x, y) = \langle Ax, y - x\rangle$ and $\varphi(x) = 0$ for all $x, y \in C$, the inequality $\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0$, $\forall y \in C$, is equivalent to the equality $u_n = P_C(x_n - r_n A u_n) = x_n - r_n u_n$. Hence we get $u_n = \frac{1}{1+r_n} x_n$. Taking into account $v_n = J_{R_1,\lambda_{1,n}} u_n = (I + \lambda_{1,n} R_1)^{-1} u_n$, we obtain $u_n \in v_n + \lambda_{1,n} R_1 v_n$, which leads to $\frac{u_n - v_n}{\lambda_{1,n}} \in R_1 v_n = S v_n + N_C v_n$. So, we have $\langle v_n - u, \frac{u_n - v_n}{\lambda_{1,n}} - S v_n\rangle \geq 0, \forall u \in C$, i.e., $\langle u - v_n, u_n - \lambda_{1,n} S v_n - v_n\rangle \leq 0, \forall u \in C$, which hence yields $v_n = P_C(u_n - \lambda_{1,n} S v_n) = u_n - \lambda_{1,n} v_n$. Thus, $v_n = \frac{1}{1+\lambda_{1,n}} u_n = \frac{1}{1+\lambda_{1,n}} \cdot \frac{1}{1+r_n} x_n$. Assume that $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\beta_n\} \subset [a, b] \subset (0, 1)$. Observe that

$$\begin{aligned} \|x_{n+1}\| &\leq \frac{2}{3}\alpha_n \|x_n\| + \beta_n \|x_n\| + (1 - \beta_n - \frac{4}{3}\alpha_n)\|v_n\| \\ &= \frac{2}{3}\alpha_n \|x_n\| + \beta_n \|x_n\| + (1 - \beta_n - \frac{4}{3}\alpha_n)\frac{1}{1 + \lambda_{1,n}} \cdot \frac{1}{1 + r_n}\|x_n\| \\ &\leq \frac{2}{3}\alpha_n \|x_n\| + \beta_n \|x_n\| + (1 - \beta_n - \frac{4}{3}\alpha_n)\|x_n\| \\ &= (1 - \frac{2}{3}\alpha_n)\|x_n\| \\ &\leq \exp(-\frac{2}{3}\sum_{k=0}^{n} \alpha_k)\|x_0\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, $\{x_n\}$ converges to the unique element 0 in Ω, which solves the VIP in Theorem 3.1.

Remark 3.3. Theorem 3.1 extends and improves Theorem 3.2 in [25] in the following ways.

(a) The iterative scheme in [25, Theorem 3.2] is extended for Problem 1.1. The iterative scheme in Algorithm 3.1 is more advantageous and more flexible than the iterative scheme in [25, Theorem 3.2] because it involves solving four problems: a finite family of GMEPs, a finite family of variational inclusions, a general system of variational inequalities and the fixed point problem of a countable family of nonexpansive mappings.

(b) The iterative scheme in Algorithm 3.1 is very different from the iterative scheme in [25, Theorem 3.2] because the iterative scheme in Theorem 3.1 involves Korpelevich's extragradient method and projection method. In addition, the iterative scheme in [25, Theorem 3.2] is an iterative one involving neither Korpelevich's extragradient method nor projection method but the iterative scheme in Theorem 3.1 is an iterative one involving both Korpelevich's extragradient method and projection method.

(c) The convergence analysis of Theorem 3.1 is based on Korpelevich's extragradient method, projection method, viscosity approximation method, and W-mapping and strongly positive bounded linear operator approaches to solve a finite family of GMEPs, a finite family of variational inclusions, GSVI (1.4) and the fixed point problem of a countable family of nonexpansive mappings.

(d) Theorem 3.1 extends [25, Theorem 3.2] from a GMEP to a finite family of GMEPs and from a variational inclusion to a finite family of variational inclusions, generalizes the domain H of the GMEP and the variational inclusion in [25, Theorem 3.2] to the case of nonempty closed convex subset C of H, and extends the problem of finding a point $x^* \in \cap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{I}(B, R)$ in [25, Theorem 3.2] to the setting of GSVI (1.4).

(e) The argument and technique in Theorem 3.1 are different from the argument ones in [25, Theorem 3.2] because we make use of the properties of the W-mappings W_n (see Lemmas 2.6 and 2.9), the properties of resolvent operators and maximal monotone mappings (see Proposition 2.13, Remark 2.16 and Lemmas 2.15–2.20), the fixed point problem $x^* = Gx^*$ ($\Leftrightarrow$ GSVI (1.5)) (see Proposition 2.12) and the properties of strongly positive boundedness linear operators (see Lemma 2.3).

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References

[1] A. Amini-Harandi, Q. H. Ansari, A. P. Farajzadeh, *Existence of equilibria in complete metric spaces*, Taiwanese J. Math., **16** (2012), 777–785. 1

[2] Q. H. Ansari, *Ekeland's Variational Principle and Its Extensions with Applications*, Topics in Fixed Point Theory, **2014** (2014), 65–100.

[3] Q. H. Ansari, *Metric Spaces: Including Fixed Point Theory and Set-Valued Maps*, Alpha Science International, New Delhi, (2010). 1

[4] Q. H. Ansari, C. S. Lalitha, M. Mehta, *Generalized Convexity, Nonsmooth Variational Inequalities and Nonsmooth Optimization*, CRC Press, Boca Raton, London, (2014). 1

[5] Q. H. Ansari, N.-C. Wong, J.-C. Yao, *The existence of nonlinear inequalities*, Appl. Math. Lett., **12** (1999), 89–92. 1

[6] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Bucharest, (1976). 2.15, 2.17

[7] L.-C. Ceng, Q. H. Ansari, M. M. Wong, J.-C. Yao, *Mann type hybrid extragradient method for variational inequalities*, Fixed Point Theory, **13** (2012), 403–422. 2.18, 2.20

[8] L. C. Ceng, A. Petrusel, J. C. Yao, *Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, J. Optim. Theory Appl., **143** (2009), 37–58. 1

[9] L.-C. Ceng, C.-Y. Wang, J.-C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res., **67** (2008), 375–390. 1, 2.11, 2, 2.12

[10] L.-C. Ceng, J.-C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math., **214** (2008), 186–201. 1, 2.13

[11] V. Colao, G. Marino, H.-K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl., **344** (2008), 340–352. 1

[12] Y.-P. Fang, N.-J. Huang, *H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces*, Appl. Math. Lett., **17** (2004), 647–653. 1

[13] K. Geobel, W. A. Kirk, *Topics on Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, (1990). 1, 2.1, 2.2, 2.10, 2.14

[14] N.-J. Huang, *A new completely general class of variational inclusions with noncompact valued mappings*, Comput. Math. Appl., **35** (1998), 9–14.

[15] A. Latif, L.-C. Ceng, Q. H. Ansari, *Multi-step hybrid viscosity method for systems of variational inequalities defined over sets of solutions of equilibrium problem and fixed point problems*, Fixed Point Theory Appl., **2012** (2012), 26 pages. 1, 1

[16] G. Marino, H.-K. Xu, *A general iterative method for nonexpansive mapping in Hilbert spaces*, J. Math. Anal. Appl., **318** (2006), 43–52. 2.3

[17] J.-W. Peng, J.-C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems*, Taiwanese J. Math., **12** (2008), 1401–1432. 1

[18] K. Rattanaseeha, *The general iterative methods for equilibrium problems and fixed point problems of countable family of nonexpansive mappings in Hilbert spaces*, J. Inequal. Appl., **2013** (2013), 20 pages. 1, 1

[19] R. T. Rockafellar, *Monotone operators and the proximal point algorithms*, SIAM J. Control Optim., **14** (1976), 877–898. 1, 2, 3.2

[20] K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math., **5** (2001), 387–404. 2.6, 2.9

[21] S. Takahashi, W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal., **69** (2008), 1025–1033. 1

[22] R. U. Verma, *On a new system of nonlinear variational inequalities and associated iterative algorithms*, Math. Sci. Res., **3** (1999), 65–68. 1

[23] X.-J. Wang, L.-C. Ceng, H.-Y. Hu, S.-X. Li, *General iterative algorithms for mixed equilibrium problems, variational inequalities and fixed point problems*, Fixed Point Theory Appl., **2014** (2014), 25 pages. 1, 1

[24] H. K. Xu, T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, J. Optim. Theory. Appl., **119** (2003), 185–201. 2.21

[25] Y. Yao, Y. J. Cho, Y.-C. Liou, *Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems*, Euro. J. Oper. Res., **212** (2011), 242–250. 1.2, 1, 3.3

[26] Y. Yao, Y.-C. Liou, J.-C. Yao, *Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings*, Fixed Point Theory Appl., **2007** (2007), 12 pages. 1, 2.7, 2.8

[27] L.-C. Zeng, S.-M. Guu, J.-C. Yao, *Characterization of H-monotone operators with applications to variational inclusions*, Comput. Math. Appl., **50** (2005), 329–337. 1, 2.19