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Some properties of the quasicompact-open topology on $C(X)$

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Abstract

This paper introduces quasicompact-open topology on $C(X)$ and compares this topology with the compact-open topology and the topology of uniform convergence. Then it examines submetrizability, metrizability, separability, and second countability of the quasicompact-open topology on $C(X)$.

Keywords: Function space, set-open topology, compact-open topology, quasicompactness, separability, submetrizability, second countability.
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1. Introduction and Preliminaries

There are several natural topologies that can be placed on $C(X)$ of all continuous real-valued functions on space X. The idea of defining a topology on $C(X)$ emerges from the studies of convergence of sequences of functions. The two major classes of topologies on $C(X)$ are the set-open topologies and the uniform topologies. The well-known set-open topologies are the point-open topology (or the topology of pointwise convergence) and the compact-open topology. The compact-open topology was introduced by Fox [6] in 1945 and soon after was developed by Arens in [2] and by Arens and Dugundji in [3]. It is shown in [12] that this topology is the proper setting to study sequences of functions converging uniformly on compact subsets. Thus, the compact-open topology is sometimes called the topology of uniform convergence on compact sets. Therefore, there have been many topologies that lie between the compact-open topology and

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the topology of uniform convergence, such as the σ-compact-open topology [9], the bounded-open topology [16], the pseudocompact-open topology [15], and the C-compact-open topology [20].

In the present paper, we introduce quasicompact-open topology on $C(X)$ and compare this topology with the compact-open topology and the topology of uniform convergence. We investigate the properties of the quasicompact-open topology on $C(X)$ such as submetrizability, metrizability, separability, and second countability.

A topological space X is called *functionally Hausdorff* (or *completely Hausdorff*) if for any distinct points $x, y \in X$ there exists a continuous real function f on X such that $f(x) = 0$ and $f(y) = 1$, equivalently $f(x) \neq f(y)$. This property lies strictly between the Hausdorffness and the complete regularity.

Unless otherwise stated clearly, throughout this paper, all spaces are assumed to be functionally Hausdorff.

If X and Y are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X.

We denote $\bar{A}$ and A° the closure and the interior of a set A, respectively. If $A \subseteq X$ and $f \in C(X)$, then we use the notation $f|_A$ for the restriction of the function f to the set A. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the preimage of the set A under the mapping f, respectively. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the real line with the natural topology. Finally, the constant zero function in $C(X)$ is denoted by f_0.

2. The quasicompact-open topology and its comparison with other topologies

In this section, we define the quasicompact-open topology on $C(X)$ and also give some equivalent definitions. Then we compare the quasicompact-open topology with the compact-open topology and the topology of uniform convergence.

A subset A of X is called a *zero-set* if there is a continuous real-valued function f defined on X such that $A = \{x \in X : f(x) = 0\}$. The complement of a zero-set is called a cozero-set. A space X is said to be *quasicompact* [7] if every covering of X by cozero-sets admits a finite subcollection which covers X, also known as z-compact space. For more information see [7].

We recall that any compact space is quasicompact and the continuous image of a quasicompact space is quasicompact[4]. We also note that the closure of a quasicompact subset is quasicompact and any quasicompact space is pseudocompact [4].

Let α be a nonempty collection of subsets of a space X. Then various topologies on $C(X)$ has a subbase consisting of the sets $S(A, V) = \{f \in C(X) : f(A) \subseteq V\}$, where $A \in \alpha$ and V is an open subset of real line $\mathbb{R}$, and the function space $C(X)$ endowed with these topologies is denoted by $C_\alpha(X)$. The topology defined in this way is called the *set-open topology*.

Now let $QC(X)$ denote the collection of all quasicompact subsets of X. For the quasicompact-open topology on $C(X)$, we take as subbase, the collection $\{S(A, V) : A \in QC(X),\ V \text{ is open in } \mathbb{R}\}$ and we denote the corresponding space by $C_q(X)$. Let $K(X)$ denote the collection of all compact subsets of X. The compact-open topology on $C(X)$ is defined similarly and is denoted by $C_k(X)$.

Let $\alpha = QC(X)$ and $\overline{\alpha} = \{\overline{A} : A \in \alpha\}$. Then note that the quasicompact-open topology is obtained if α is replaced by $\overline{\alpha}$. This is because for each $f \in C(X)$ we have $f(\overline{A}) \subseteq \overline{f(A)} = f(A)$.

The topology of uniform convergence on members of α has as base at each point $f \in C(X)$ the family of all sets of the form $B_A(f, \epsilon) = \{g \in C(X) : \sup|f(x) - g(x)| < \epsilon \text{ for all } x \in A\}$, where $A \in \alpha$ and $\epsilon > 0$. The space $C(X)$ having the topology of uniform convergence on α is denoted by $C_{\alpha,u}(X)$. For $\alpha = QC(X)$, we denote the corresponding space by $C_{q,u}(X)$. In the case that $\alpha = \{X\}$, the topology on $C(X)$ is called *the topology of uniform convergence* or *uniform topology* and denoted by $C_u(X)$.

There is another way to consider the quasicompact-open topology on $C(X)$. For each $A \in QC(X)$ and $\epsilon > 0$, we define the seminorm p_A on $C(X)$ and $V_{A,\epsilon}$, as follow: $p_A(f) = \sup\{|f(x)| : x \in A\}$ and

$V_{A,\epsilon} = \{f \in C(X) : p_A(f) < \epsilon\}$. Let $\mathcal{V} = \{V_{A,\epsilon} : A \in QC(X), \epsilon > 0\}$. Then for each $f \in C(X)$, $f + \mathcal{V} = \{f + V : V \in \mathcal{V}\}$ forms a neighborhood base at f. This topology is locally convex since it is generated by a collections of seminorms and it is the same as the quasicompact-open topology on $C(X)$. It is also easy to see that this topology is Hausdorff. $C_q(X)$, being a locally convex Hausdorff space, is a Tychonoff space.

Now, we can compare the topologies. We have $C_k(X) \leq C_q(X)$ since $K(X) \subseteq QC(X)$. But to compare the quasicompact-open topology and the topology of uniform convergence, we need the following theorem.

Theorem 2.1. *For any space X, the quasicompact-open topology on $C(X)$ is the same as the topology of uniform convergence on the quasicompact subsets of X, that is, $C_q(X) = C_{q,u}(X)$.*

Proof. Assume that $S(A, V)$ is a subbasic open set in $C_q(X)$ and $f \in S(A, V)$. Recall that compact and quasicompact subsets of $\mathbb{R}$ are equivalent. Since $f(A)$ is compact and $f(A) \subseteq V$, there exists $\epsilon > 0$ such that $(f(A)-\epsilon, f(A)+\epsilon) \subseteq V$ (see [5, Corollay 4.1.14]). If $g \in B_A(f, \epsilon)$ and $x \in A$, then we obtain $g(x) \in (f(x)-\epsilon, f(x) + \epsilon)$. Hence, we find $g(A) \subseteq V$, i.e. $g \in S(A, V)$. It follows that $B_A(f, \epsilon) \subseteq S(A, V)$. Consequently, $C_q(X) \leq C_{q,u}(X)$.

Now, let $B_A(f, \epsilon)$ be a basic neighborhood of f in $C_{q,u}(X)$. Then, there exist $f(x_1), f(x_2), \ldots, f(x_n)$ in $f(A)$ such that $f(A) \subseteq \cup_{i=1}^{n}(f(x_i)-\frac{\epsilon}{3}, f(x_i)+\frac{\epsilon}{3})$ since $f(A)$ is compact. If we take $V_i = (f(x_i)-\frac{\epsilon}{3}, f(x_i)+\frac{\epsilon}{3})$ and $W_i = (f(x_i) - \frac{2\epsilon}{3}, f(x_i) + \frac{2\epsilon}{3})$, we find $\overline{V_i} \subseteq W_i$. Also $f(A) \subseteq \cup_{i=1}^{n} V_i \subseteq \cup_{i=1}^{n}\overline{V_i}$. Let $A_i = A \cap f^{-1}(\overline{V_i})$, where clearly each A_i is quasicompact and $A = \cup_{i=1}^{n} A_i$. We have $f(A_i) \subseteq \overline{V_i} \subseteq W_i$ and so $f \in \cap_{i=1}^{n} S(A_i, W_i)$. Now we need to show that $\cap_{i=1}^{n} S(A_i, W_i) \subseteq B_A(f, \epsilon)$. Suppose that $g \in \cap_{i=1}^{n} S(A_i, W_i)$ and $x \in A$. Thus, there exists an i such that $x \in A_i$ and consequently, $f(x) \in \overline{V_i}$ and $g(x) \in W_i$. Since $|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x)| < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$, then $g \in B_A(f, \epsilon)$. Hence, $C_{q,u}(X) \leq C_q(X)$. □

Corollary 2.2. *For any space X, $C_q(X) = C_{q,u}(X) \leq C_u(X)$.*

From this result, we obtain the following.

Corollary 2.3. *For any space X, $C_k(X) \leq C_q(X) \leq C_u(X)$.*

Note that in a perfectly normal space, every open set is a cozero-set and consequently, a quasicompact space is compact. Thus, for a perfectly normal space X, $C_k(X) = C_q(X)$.

Theorem 2.4. *For any space X, $C_q(X) = C_u(X)$ if and only if X is quasicompact.*

Proof. Let $C_q(X) = C_u(X)$. We know that $C_q(X) = C_{q,u}(X)$ by Theorem 2.1. So, $C_u(X) = C_{q,u}(X)$. Thus, $B_X(f, \epsilon)$ in $C_u(X)$ is also basic neighborhood of f in $C_{q,u}(X)$ and so X is quasicompact.

Conversely, suppose that X is quasicompact. It follows that for each $f \in C(X)$ and each $\epsilon > 0$, $B_X(f, \epsilon)$ is a basic open set in $C_q(X)$. Consequently, $C_q(X) = C_u(X)$. □

We know that for a compact space X, $C_k(X) = C_u(X)$. Then we can give the following example.

Example 2.5. For any compact space X, $C_k(X) = C_q(X) = C_u(X)$.

If X is both realcompact and pseudocompact, then it is compact [8, Problem 5H]. Also every Lindelöf space is realcompact [8, Theorem 8.2]. Thus, we get the following result.

Theorem 2.6. *For any Lindelöf space X, $C_k(X) = C_q(X)$.*

Proof. We know that every quasicompact space is pseudocompact. Considering the above description, Lindelöf quasicompact space is compact and consequently, $C_k(X) = C_q(X)$ by Example 2.5. □

Since every countable or second countable space is Lindelöf, we obtain the following result.

Corollary 2.7. *For any countable or second countable space X, $C_k(X) = C_q(X)$.*

Example 2.8. Let X denote the set of positive integers endowed with the particular point topology [22, Example 9]. The space X is a quasicompact, but not compact. Thus, we obtain $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.9. Let X be the prime integer topology [22, Example 61]. The space X is a quasicompact, but not compact [1]. This yields $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.10. Let $X = \mathbb{R}$ and define a topology on X by requiring that a neighborhood of a point x is any set containing x which contains all the rationals in an open interval around x [21]. The space X is quasicompact, but not compact [4]. It follows that $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.11. Hewitt's example [11] of a regular space X on which every continuous real-valued function is constant is a quasicompact space which is not compact [13]. For this space X, we have $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.12. Let X be the skyline space [10]. The space X is a quasicompact, but not compact [14]. Hence, we obtain $C_k(X) \leq C_q(X) = C_u(X)$.

Example 2.13. Let $X = \mathbb{N}$ and define a topology on X by taking every odd integer to be open and a set U is open if for every even integer $p \in U$, the predecessor and the successor of p are also in U [14]. From this it follows that $C_k(X) \leq C_q(X) = C_u(X)$.

3. Main Results on $C_q(X)$

In this section, we study the submetrizability, metrizability, separability, and second countability of $C_q(X)$. First, we provide some natural functions which play a useful role in studying the topological properties of function spaces.

If $f : X \to Y$ is a continuous function, then the induced function of f, denoted by $f^* : C(Y) \to C(X)$ is defined by $f^*(g) = g \circ f$ for all $g \in C(Y)$.

Given a nonempty set X a topological space Y, a function $f : X \to Y$ is called almost onto if $f(X)$ is dense in Y.

Theorem 3.1. *Let $f : X \to Y$ be a continuous function between two spaces X and Y. Then we have the following.*

1. *$f^* : C_q(Y) \to C_q(X)$ is continuous;*
2. *for normal space Y, if f is one-to-one, then $f^* : C_q(Y) \to C_q(X)$ is almost onto;*
3. *$f^* : C(Y) \to C(X)$ is one-to-one if and only if f is almost onto* [19].

Proof. (1) Let $g \in C_q(Y)$ and $S(A, V)$ be a basic neighborhood of $f^*(g)$ in $C_q(X)$. It is easily seen that $f^*(g) = g \circ f \in S(A, V)$ if and only if $g \in S(f(A), V)$. Then $f^*(S(f(A), V)) = S(A, V)$ and consequently, f^* is continuous.

The proof of (2) is similar to 2(a) in [18]. □

Another kind of useful function on function spaces is the sum function. Let $\{X_i : i \in I\}$ be a family of topological spaces. If $\oplus X_i$ denotes their topological sum, then the sum function s is defined by $s : C(\oplus X_i) \to \prod\{C(X_i) : i \in I\}$ where $s(f) = f|_{X_i}$ for each $f \in C(\oplus X_i)$.

Theorem 3.2. *Let $\{X_i : i \in I\}$ be a family of spaces. Then the sum function $s : C(\oplus X_i) \to \prod\{C(X_i) : i \in I\}$ is a homeomorphism.*

Proof. The proof is similar to Theorem 4.10 in [15]. □

A space X is said to be *submetrizable* if it has a weaker metrizable topology, equivalently if there exists a metrizable space Y and a continuous bijection $f : X \to Y$ from the space X onto Y.

In a topological space a G_δ*-set* is a set which can be written as the intersection of a countable collection of open sets.

Remark 3.3.

1. For any space X, if the set $\{(x,x) : x \in X\}$ is a G_δ-set (resp. zero-set) in the product space $X \times X$, then X is said to have a G_δ-diagonal (resp. zero-set diagonal). Every submetrizable space X has a G_δ-diagonal. Consequently, every submetrizable space X has a zero-set diagonal since a zero-set is a G_δ-set.
2. A space X is called an E_0-*space* if every point in the space is a G_δ-set. The submetrizable spaces are E_0-spaces.

Proposition 3.4. *If X is a submetrizable space then all quasicompact subsets of X are G_δ-sets.*

Proof. Let X be submetrizable. Then there exists a continuous bijection $f : X \to Y$ from the space X onto a metrizable space Y. Let A be a quasicompact subset of X. Then $f(A)$ is compact in the metric space Y. Since a closed set in a metric space is a G_δ-set, $f(A)$ is a G_δ-set in Y. In other words, $f(A) = \cap_{n=1}^{\infty} G_n$, where G_n is an open subset of Y for each n. It follows that $A = \cap_{n=1}^{\infty} f^{-1}(G_n)$ and so A is a G_δ-set. □

A space X is called σ-*quasicompact* if there exists a sequence $\{A_n\}$ of quasicompact sets in X such that $X = \cup_{n=1}^{\infty} A_n$. By using this fact we obtain the following result.

Theorem 3.5. *For any space X, the following are equivalent.*

1. *$C_q(X)$ is submetrizable.*
2. *Every quasicompact subset of $C_q(X)$ is a G_δ-set in $C_q(X)$.*
3. *Every compact subset of $C_q(X)$ is a G_δ-set in $C_q(X)$.*
4. *$C_q(X)$ is an E_0-space.*
5. *X is σ-quasicompact.*
6. *$C_q(X)$ has a zero-set-diagonal.*
7. *$C_q(X)$ has a G_δ-diagonal.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) follow from Proposition 3.4.

(4) ⇒ (5) If $C_q(X)$ is an E_0-space, then the constant zero function f_0 defined on X is a G_δ-set. Suppose that $\cap_{n=1}^{\infty} B_{A_n}(f_0, \epsilon_n) = \{f_0\}$ where each A_n is quasicompact subset in X and $\epsilon_n > 0$. We need to show that $X = \cup_{n=1}^{\infty} A_n$. Assume that $x_0 \in X \setminus \cup_{n=1}^{\infty} A_n$. Hence there exists a continuous function $f_1 : X \to [0,1]$ such that $f_1(x) = 0$ for all $x \in \cup_{n=1}^{\infty} A_n$ and $f_1(x_0) = 1$. Since $f_1(x) = 0$ for all $x \in A_n$, $f_1 \in B_{A_n}(f_0, \epsilon_n)$ for all n and thus, $f_1 \in \cap_{n=1}^{\infty} B_{A_n}(f_0, \epsilon_n) = \{f_0\}$, that is, f_1 is the zero function on X. But $f_1(x_0) = 1$. This contradicts the hypothesis, hence X is σ-quasicompact.

(5) ⇒ (4) Assume that X is σ-quasicompact and $f \in C_q(X)$. Now we need to prove that $\{f\} = \cap_{n=1}^{\infty} B_{A_n}(f, \frac{1}{n})$. Let $g \in \cap_{n=1}^{\infty} B_{A_n}(f, \frac{1}{n})$ and $x \in X$. Then there exists $m \in \mathbb{N}$ such that $x \in A_n$ for all $n \geq m$. Then we find $|g(x) - f(x)| \leq \frac{1}{n}$ for all $n \geq m$. Thus $g(x) = f(x)$ and consequently $C_q(X)$ is an E_0-space.

(5) ⇒ (1) Suppose that $X = \cup_{n=1}^{\infty} A_n$, where each A_n is quasicompact. Let $S = \oplus\{A_n : n \in \mathbb{N}\}$ be the topological sum of the A_n and let $\phi : S \to X$ be the natural function. Thus, the induced function $\phi^* : C_q(X) \to C_q(S)$ defined by $\phi^*(f) = f \circ \phi$ is continuous. We need to show that ϕ^* is one-to-one. Let $\phi^*(g_1) = \phi^*(g_2)$. So, g_1 and g_2 are equal on $\cup_{n=1}^{\infty} A_n$. So $g_1 - g_2 \in \cap_{n=1}^{\infty} B_{A_n}(f_0, \epsilon_n) = \{f_0\}$. Hence, $g_1 = g_2$ and consequently, ϕ^* is one-to-one. By Theorem 3.2, $C_q(\oplus\{A_n : n \in \mathbb{N}\})$ is homeomorphic to $\prod\{C_q(A_n) : n \in \mathbb{N}\}$. But each $C_q(A_n)$ is metrizable by Theorem 2.4. Since $C_q(S)$ is metrizable and ϕ^* is a continuous injection, $C_q(X)$ is submetrizable.

The implications (1) ⇒ (6) ⇒ (7) ⇒ (4) are immediate from Remark 3.3. □

Lemma 3.6. *In a completely regular submetrizable space, the notions of compactness and quasicompactness coincide.*

Proof. Since pseudocompact completely regular submetrizable space is metrizable [17, Corollary 2.7] and every quasicompact space is pseudocompact, then the notions of compactness and quasicompactness coincide. □

Corollary 3.7. *Let X be σ-quasicompact. Then compact and quasicompact subsets of $C_q(X)$ are equivalent.*

Proof. If X is σ-quasicompact, then $C_q(X)$ is submetrizable by Theorem 3.5. Also we know that $C_q(X)$ is Tychonoff (completely regular Hausdorff). Hence, compact and quasicompact subsets of $C_q(X)$ are equivalent by Lemma 3.6. □

A space X is called a *q-space* if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. This fact yields the following theorem.

Theorem 3.8. *For any space X, the following are equivalent.*

1. *$C_q(X)$ is metrizable.*
2. *$C_q(X)$ is first countable.*
3. *$C_q(X)$ is a q-space.*
4. *X is hemiquasicompact; that is, there exists a sequence of quasicompact sets $\{A_n\}$ in X such that for any quasicompact subset A of X, $A \subseteq A_n$ holds for some n.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are all immediate.

(3) $\Rightarrow$ (4) Suppose that $C_q(X)$ is a q-space. Hence, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the zero function f_0 in $C_q(X)$ such that if $g_n \in U_n$ for each n, then $\{g_n : n \in \mathbb{N}\}$ has a cluster point in $C_q(X)$. Now for each n, there exists a quasicompact subset A_n of X and $\epsilon_n > 0$ such that $f_0 \in B_{A_n}(f_0, \epsilon_n) \subseteq U_n$. Let A be a quasicompact subset of X. If possible, suppose that A is not a subset of A_n for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A \setminus A_n$. So for each $n \in \mathbb{N}$, there exists a continuous function $g_n : X \to \mathbb{R}$ such that $g_n(a_n) = n$ and $g_n(x) = 0$ for all $x \in A_n$. It is clear that $g_n \in B_{A_n}(f_0, \epsilon_n)$. Suppose that this sequence has a cluster point g in $C_q(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $g_{n_k} \in B_A(g, 1)$. Thus, $g(a_{n_k}) > g_{n_k}(a_{n_k}) - 1 = n_k - 1 \geq k$ for all $k \in \mathbb{N}$. But this means that g is unbounded on the quasicompact set A. Hence, the sequence $\{g_n\}_{n \in \mathbb{N}}$ cannot have a cluster point in $C_q(X)$ and consequently, $C_q(X)$ fails to be a q-space. Thus, X must be hemiquasicompact.

(4) $\Rightarrow$ (1) Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable(see page 119 in [23]). Now the locally convex topology on $C(X)$ generated by the countable family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$ is metrizable and weaker than the quasicompact-open topology. But since for each quasicompact set A in X, there exists A_n such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms $\{p_A : A \in QC(X)\}$, that is, the quasicompact-open topology is weaker than the topology generated by the family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$. Hence, $C_q(X)$ is metrizable. □

Proposition 3.9. *Let X be locally compact and second countable. Then $C_q(X)$ is second countable.*

Proof. Since regular second countable space X is metrizable by Urysohn's Metrization Theorem, then $C_k(X) = C_q(X)$. We know that $C_k(X)$ is second countable by [18] it follows that $C_q(X)$ is second countable. □

Theorem 3.10. *For any space X, the following are equivalent.*

1. *$C_q(X)$ is separable.*
2. *$C_k(X)$ is separable.*
3. *X has a weaker separable metrizable topology.*

Proof. (1) ⇒ (2) is straightforward and proof of (2) ⇒ (3) was given in [18].

(3) ⇒ (1). If X has a weaker separable metrizable topology, then X is embeddable into Hilbert cube I^ω (see [5, Theorem 4.2.10]). Let $f : X \to I^\omega$ be a continuous injection. Then the induced function $f^* : C(I^\omega) \to C_q(X)$ is almost onto by Theorem 3.1. Since $C(I^\omega)$ is second countable by Proposition 3.9, then $C_q(X)$ must be separable. □

Corollary 3.11. *Let X be completely regular space. If $C_q(X)$ is separable, then $C_k(X) = C_q(X)$.*

Proof. If $C_q(X)$ is separable, X is submetrizable. Since X is completely regular and submetrizable, compact and quasicompact subsets of X are equivalent by Lemma 3.6. Consequently, $C_k(X) = C_q(X)$. □

Example 3.12. Since $\mathbb{R}$ is a separable metric space, $C_q(\mathbb{R})$ is separable. Thus, we have $C_k(\mathbb{R}) = C_q(\mathbb{R})$.

Example 3.13. Let X be a countable discrete space. Then $C_q(X)$ is separable and so $C_k(X) = C_q(X)$.

Corollary 3.14. *Let X be quasicompact space. If X is metrizable, then $C_q(X)$ is separable.*

Proof. If X is metrizable and quasicompact, then X is compact. Since X is compact and metrizable, then X is separable and consequently, $C_q(X)$ is separable. □

Note that converse of Corollary 3.14 is not always true. If $C_q(X)$ is separable, then X is submetrizable. But a quasicompact submetrizable space need not be metrizable. An example of this, the space $E \cap [0, 1]$ of [8, Problem 3J] is quasicompact and submetrizable, but not metrizable. If X is completely regular, then is metrizable by Corollary 2.7 in [17]. Then we can give the following theorem.

Theorem 3.15. *Let X be quasicompact and completely regular space. $C_q(X)$ is separable if and only if X is compact and metrizable.*

Proof. If $C_q(X)$ is separable, then X is submetrizable by Theorem 3.10. Since quasicompact completely regular submetrizable space is metrizable, X is metrizable and by Lemma 3.6, X is compact.

The sufficiency part follows from Corollary 3.14. □

A topological space is said to be *hemicompact* if it has a sequence of compact subsets such that every compact subset of the space lies inside some compact set in the sequence.

Theorem 3.16. *For a locally compact space X, the following are equivalent.*

1. *$C_q(X)$ is second countable.*
2. *$C_k(X)$ is second countable.*
3. *X is hemicompact and submetrizable.*

Proof. (1) ⇔ (2) If either $C_q(X)$ or $C_k(X)$ is second countable, then it is separable and submetrizable by Theorem 3.10. We know that regular separable space is normal. Consequently, $C_k(X) = C_q(X)$.

(2) ⇒ (3) If $C_k(X)$ is second countable, then it is submetrizable as well as it is separable. Hence, X is hemicompact and submetrizable.

(3) ⇒ (2) If X is hemicompact, then $C_k(X)$ is metrizable. Note that X, being hemicompact, is Lindelöf. Since X is also submetrizable, X has a separable metrizable compression and consequently, $C_k(X)$ is separable. Thus, $C_k(X)$ is second countable. □

Considering Corollary 3.11, we obtain the following result.

Corollary 3.17. *Let X be a completely regular space. If $C_q(X)$ is second countable, then $C_k(X) = C_q(X)$.*

Note that if X is locally compact, then X is hemicompact if and only if X is either Lindelöf or σ-compact in [5, Exercises 3.8.C]. Hence, by using Theorem 3.16 and Proposition 3.9, we have the following result.

Theorem 3.18. *For a locally compact space X, the following statements are equivalent.*

1. $C_q(X)$ *is second countable.*
2. $C_k(X)$ *is second countable.*
3. X *is hemicompact and submetrizable.*
4. X *is σ-compact and submetrizable.*
5. X *is Lindelöf and submetrizable.*
6. X *is second countable.*

Proof. From Theorem 3.16, we obtain (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Also by [5, Exercises 3.8.C], we get (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). It is easy to see that (6) $\Rightarrow$ (1) from Proposition 3.9.

Now, it is sufficient to show that (5) $\Rightarrow$ (6). Since X is locally compact, for each $x \in X$, there exists an open set V_x in X such that $x \in V_x$ and $\overline{V_x}$ is compact. Note that $\{V_x : x \in X\}$ is an open cover of X. But X is Lindelöf and consequently, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of X such that $X = \cup_{n=1}^{\infty} V_{x_n}$. Since X is separable submetrizable by Theorem 3.10 and each $\overline{V_{x_n}}$ is compact, each $\overline{V_{x_n}}$ is metrizable and so each $\overline{V_{x_n}}$ is second countable. Consequently, each V_{x_n} is also second countable and X becomes the union of a countable family of second countable open subsets of X. Hence, X is second countable. □

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References

[1] A. T. Al-Ani, *Countably z-compact spaces*, Arch. Math. (Brno), **50** (2014), 97–100. 2.9

[2] R. F. Arens, *A topology for spaces of transformations*, Ann. of Math., **47** (1946), 480–495. 1

[3] R. Arens, J. Dugundji, *Topologies for function spaces*, Pacific J. Math., **1** (1951), 5–31. 1

[4] A. J. D'Aristotle, *Quasicompactness and functionally Hausdorff spaces*, J. Aust. Math. Soc., **15** (1973), 319–324. 2, 2.10

[5] R. Engelking, *General Topology*, revised and completed ed., Heldermann Verlag, Berlin, (1989). 2, 3, 3, 3

[6] R. H. Fox, *On topologies for function spaces*, Bull. Amer. Math. Soc., **51** (1945), 429–432. 1

[7] Z. Frolik, *Generalization of compact and Lindelöf spaces*, Czechoslovak Math. J., **13** (1959), 172–217 (Russian). 2

[8] L. Gillman, M. Jerison, *Rings of continuous functions*, Springer-Verlag, New York, (1960). 2, 3

[9] D. Gulick, *The σ-compact-open topology and its relatives*, Math. Scand., **30** (1972), 159–176. 1

[10] N. C. Heldermann, *Developability and some new regularity axioms*, Can. J. Math., **33** (1981), 641–663. 2.12

[11] E. Hewitt, *On two problems of Urysohn*, Ann. of Math., **47** (1946), 503–509. 2.11

[12] J. R. Jackson, *Comparison of topologies on function spaces*, Proc. Amer. Math. Soc., **3** (1952), 156–158. 1

[13] J. K. Kohli, A. K. Das, *A class of spaces containing all generalized absolutely closed (almost compact) spaces*, Appl. Gen. Topol., **7** (2006), 233–244. 2.11

[14] J. K. Kohli, D. Singh, *Between compactness and quasicompactness*, Acta Math. Hungar., **106** (2005), 317–329. 2.12, 2.13

[15] S. Kundu, P. Garg, *The pseudocompact-open topology on C(X)*, Topology Proc., **30** (2006), 279–299. 1, 3

[16] S. Kundu, A. B. Raha, *The bounded-open topology and its relatives*, Rend. Istit. Mat. Univ. Trieste, **27** (1995), 61–77. 1

[17] W. G. McArthur, *G_δ-diagonals and metrization theorems*, Pacific J. Math., **44** (1973), 613–617. 3, 3

[18] R. A. McCoy, *Second countable and separable function spaces*, Amer. Math. Monthly, **85** (1978), 487–489. 3, 3, 3

[19] R. A. McCoy, I. Ntantu, *Topological Properties of Spaces of Continuous Functions*, Springer-Verlag, Berlin, (1988). 3

[20] A. V. Osipov, *The C-compact-open topology on function spaces*, Topology Appl., **159** (2012), 3059–3066. 1

[21] W. J. Pervin, *Foundations of general topology,* Academic Press, New York, (1964). 2.10

[22] L. A. Steen, J. A. Seebach, *Counter Examples in Topology*, Springer Verlag, New York, (1978). 2.8, 2.9

[23] A. E. Taylor, D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., John Wiley & Sons, New York, (1980). 3