Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3519–3526
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Generalized $k$-Mittag-Leffler function and its composition with pathway integral operators

K. S. Nisar[a], S. D. Purohit[b,*], M. S. Abouzaid[c], M. Al Qurashi[d], D. Baleanu[e,f]

[a]*Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Al-Dawaser, Saudi Arabia.*
[b]*Department of HEAS (Mathematics), Rajasthan Technical University, Kota 324010, Rajasthan, India.*
[c]*Department of Mathematics, Faculty of Science, Kafrelshiekh University, Egypt.*
[d]*Department of Mathematics, King Saud University, P. O. Box 22452, Riyadh 11495, Saudi Arabia.*
[e]*Department of Mathematics, Cankaya University, Balgat 06530, Ankara, Turkey.*
[f]*Institute of Space Sciences, Magurele-Bucharest, Romania.*

Communicated by A. Atangana

## Abstract

Our purpose in this paper is to consider a more generalized form of the Mittag-Leffler function. For this newly defined function, we obtain certain composition formulas with pathway fractional integral operators. We also point out some important special cases of the main results. 

*Keywords:* Mittag-Leffler functions, pathway integral operator.
*2010 MSC:* 33E12, 05C38, 26A33.

## 1. Introduction

Mittag-Leffler functions are important for obtaining solutions of fractional differential and integral equations, and they are connected with an extensive variety of problem in diverse areas of mathematics and mathematical physics. In addition, from exponential behavior, the deviations of physical phenomena could also be represented by physical laws via Mittag-Leffler functions. Therefore, the uses of Mittag-Leffler functions are constantly increasing, specially in physics. For more details about the recent works in the

*Corresponding author
*Email addresses:* ksnisar1@gmail.com (K. S. Nisar), sunil_a_purohit@yahoo.com (S. D. Purohit), moheb_abouzaid@hotmail.com (M. S. Abouzaid), maysaa@ksu.edu.sa (M. Al Qurashi), dumitru@cankaya.edu.tr (D. Baleanu)

*Received 2016-04-08*

field of dynamical systems theory, stochastic systems, non-equilibrium statistical mechanics and quantum mechanics, interesting readers can refer [3, 14, 16–18, 25] and the references cited therein.

Below, let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of positive integers, real numbers and complex numbers respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

By means of power series, the Mittag-Leffler function $E_\alpha(z)$, defined by [10, 11]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{1.1}$$

where $z \in \mathbb{C}$, $\Gamma$ represents well known Gamma function and $\alpha \geq 0$.

The generalization of $E_\alpha(z)$, also known as Wiman function, is given by Wiman [26]:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

Further, in 1971, Prabhakar [15] proposed the more general function $E^{\gamma}_{\alpha,\beta}(z)$ as:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\Gamma(\alpha n + \beta)} z^n, \tag{1.3}$$

for which $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$. The importance and great considerations of Mittag-Leffler function have led many researchers in the theory of special functions for exploring the possible generalizations and applications. Many more extensions or unifications for these functions are found in large number of papers [6, 21–24]. A useful generalization of the Mittag-Leffler function called as $k$-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$, introduced in [4], and it is given by

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.4}$$

where $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $k \in \mathbb{R}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $(\gamma)_{n,k}$ is the $k$-Pochhammer symbol defined as:

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k) \cdots (\gamma + (n-1)k) \quad (\gamma \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}). \tag{1.5}$$

Lately, a generalized form of $k$-Mittag-Leffler function was introduced and studied in [5] as:

Let $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $k \in \mathbb{R}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $q \in (0,1) \cup \mathbb{N}$, then

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.6}$$

where $(\gamma)_{nq,k}$ is defined as (1.5) and the generalized Pochhammer symbol is defined as (see [19]):

$$(\gamma)_{nq} = \frac{\Gamma(\gamma + nq)}{\Gamma(\gamma)} = q^{qn} \prod_{r=1}^{q} \left( \frac{\gamma + r - 1}{q} \right)_n, \text{if } q \in \mathbb{N}. \tag{1.7}$$

In the integral representation, the generalized $k$-Gamma function is defined as:

$$\Gamma_k(z) = \int_0^\infty e^{\frac{-t^k}{k}} t^{z-1} dt, \quad (k \in \mathbb{R}, z \in \mathbb{C}, \Re(z) > 0). \tag{1.8}$$

By inspection we conclude the following relations:

$$\Gamma_k(x + k) = x\Gamma_k(x), \tag{1.9}$$

$$\Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1}\,\Gamma\left(\frac{\gamma}{k}\right). \tag{1.10}$$

The beta integral is defined by

$$\mathcal{B}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\Re(\alpha), \Re(\beta) > 0), \tag{1.11}$$

and its relationship to the familiar gamma function is

$$\mathcal{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.12}$$

Here we aim at introducing a more generalized $k$-Mittag-Leffler function and we defined it as under:

$$E_{k,\alpha,\beta,\delta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{(\delta)_n}, \tag{1.13}$$

provided $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{C}$, $k \in \mathbb{R}$, $\{\Re(\alpha), \Re(\beta)\} > 0, \delta \neq 0, -1, -2, \cdots$ and $nq$ is a positive integer. Particular cases:

(i) For $\delta = 1$, then (1.13) reduces to the generalized $k$-Mittag-Leffler function (1.6) (see [5]).

(ii) Again, if $\delta = q = 1$ then (1.13) reduces to $E_{k,\alpha,\beta}^{\gamma}(z)$ (see [4]).

Recently, by using the pathway idea of Mathai [7] and developed further by Mathai and Haubold [8, 9], Nair [12] introduced a pathway fractional integral operator and it is given as follows:

Suppose $f(x) \in L(a,b), \eta \in \mathbb{C}, \Re(\eta) > 0, a > 0$ and the pathway parameter $\sigma < 1$ as (cf. [2]), then

$$\left(P_{0+}^{(\eta,\sigma)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\sigma)}\right]} \left[1 - \frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{(1-\sigma)}} f(t)\,dt. \tag{1.14}$$

For given real scalar $\sigma$ and scalar random variables, the pathway model is denoted by the following probability density function (p.d.f.), namely

$$f(x) = c\,|x|^{\upsilon-1}\left[1 - a(1-\sigma)|x|^{\xi}\right]^{\frac{\lambda}{(1-\sigma)}}, \tag{1.15}$$

provided that $-\infty < x < \infty, \xi > 0, \lambda \geq 0, \left[1 - a(1-\sigma)|x|^{\xi}\right] > 0$, and $\upsilon > 0$. Here $c$ and $\sigma$, respectively denotes the normalizing constant and the pathway parameter.

Further, the normalizing constants for real $\sigma$, are the following:

$$c_1 = \frac{1}{2} \frac{\xi\,[a(1-\sigma)]^{\frac{\upsilon}{\xi}}\,\Gamma\left(\frac{\upsilon}{\xi} + \frac{\lambda}{1-\sigma} + 1\right)}{\Gamma\left(\frac{\upsilon}{\xi}\right)\Gamma\left(\frac{\lambda}{1-\sigma} + 1\right)}, \text{for } \sigma < 1, \tag{1.16}$$

$$c_2 = \frac{1}{2} \frac{\xi\,[a(\sigma-1)]^{\frac{\upsilon}{\xi}}\,\Gamma\left(\frac{\lambda}{\sigma-1}\right)}{\Gamma\left(\frac{\upsilon}{\xi}\right)\Gamma\left(\frac{\lambda}{\sigma-1} - \frac{\upsilon}{\xi}\right)}, \text{for } \frac{1}{\sigma-1} - \frac{\upsilon}{\xi} > 0, \sigma > 1, \tag{1.17}$$

$$c_3 = \frac{1}{2} \frac{(a\lambda)^{\frac{\upsilon}{\xi}}}{\Gamma\left(\frac{\upsilon}{\xi}\right)}, \ \sigma \to 1. \tag{1.18}$$

It is noted that for $\sigma < 1$, we have $\left[1 - a(1-\sigma)|x|^{\xi}\right] > 0$ and (1.15) can be considered as member of the extended generalized type-1 beta family. Also, the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f. are particular cases of the pathway density function in (1.15), for $\sigma < 1$.

For instance, $\sigma > 1$, writing $(1-\sigma) = -(\sigma - 1)$ in (1.14) gives

$$\left(P_{0+}^{(\eta,\sigma)} f\right)(x) = x^{\eta} \int_{0}^{\left[\frac{x}{-a(1-\sigma)}\right]} \left[1 + \frac{a(\sigma-1)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}} f(t)\,dt, \tag{1.19}$$

and

$$f(x) = c\,|x|^{\upsilon-1}\left[1 + a(\sigma-1)|x|^{\xi}\right]^{-\frac{\lambda}{(\sigma-1)}}, \tag{1.20}$$

provided that $-\infty < x < \infty, \xi > 0, \lambda \geq 0$, and $\sigma > 1$ which represents the extended generalized type-2 beta model for real $x$. Further, the type-2 beta density, the $F$ density, the Student-$t$ density, the Cauchy density and many more are special cases of the density function (1.20).

Moreover, the operator (1.14) includes Laplace integral transform, when $\sigma \to 1_{-}$, and the the Riemann-Liouville fractional integral operator, when $\sigma = 0, a = 1$ and $\eta$ replacing by $\eta - 1$. For more details on the pathway model and its particular cases, the reader is referred to the recent papers of Mathai and Haubold [8, 9] and Nair [12].

It is observed that the pathway fractional integral operator (1.14), can lead to other interesting examples of fractional calculus operators, related to some probability density functions and applications in statistics. This has led number of workers in the theory of fractional calculus for exploring the possible generalization of the known results. For example, the composition of the integral transform operator (1.14) with the product of generalized Bessel function of the first kind is given in [2]. Recently Nisar et al. studied the pathway fractional integral operator associated with Struve function of first kind [13]. The results provided in [2] are extensions of the results given by Agarwal and Purohit [1] and Nair [12]. The main objective of this work is to obtain the composition formula of pathway integral transform operator due to Nair, with the more generalized $k$-Mittag-Leffler function introduced above in (1.13).

## 2. Pathway fractional integration of generalized $k$-Mittag-Leffler function.

Below, we derive the pathway image formulas involving the generalized $k$-Mittag-Leffler function from (1.13). The main results of this section are obtained by using the concept of changing the order of integral and summation. The following theorems are presented as our main results.

**Theorem 2.1.** *Let $\rho,\ \beta,\ \gamma, \delta, \eta \in \mathbb{C},\ k \in \mathbb{R},\ \{\Re(\rho), \Re(\beta), \Re(\eta)\} > 0, \Re\left(\frac{\eta}{1-\sigma}\right) > -1, \sigma < 1, k, w \in \mathbb{R}$, $\delta \neq 0, -1, -2, \cdots$ and $q > 0$. Then the following image formula holds true:*

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1} E_{k,\rho,\beta,\delta}^{\gamma,q}\left(wt^{\frac{\rho}{k}}\right)\right](x) = \frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)\ x^{\eta+\frac{\beta}{k}} k^{\left(1+\frac{\eta}{1-\sigma}\right)}}{\left[a(1-\sigma)\right]^{\frac{\beta}{k}}}\ E_{k,\rho,\beta+k\left(1+\frac{\eta}{1-\sigma}\right),\delta}^{\gamma,q}\left[w\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right]. \tag{2.1}$$

*Proof.* By applying (1.13) and (1.14), we have

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1} E_{k,\rho,\beta,\delta}^{\gamma,q}\left(wt^{\frac{\rho}{k}}\right)\right] = x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\sigma)}\right]} t^{\frac{\beta}{k}-1}\left[1 - \frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}}\ E_{k,\rho,\beta,\delta}^{\gamma,q}\left(wt^{\frac{\rho}{k}}\right)dt.$$

We denote, for convenience, the right hand integral of the above term by $I_1$, then

$$I_1 = x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\sigma)}\right]} t^{\frac{\beta}{k}-1}\left[1 - \frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}}\ \sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k}}{\Gamma_k(\rho n+\beta)}\frac{\left(wt^{\frac{\rho}{k}}\right)^n}{(\delta)_n},$$

and interchange the order of integration and summation to get

$$I_1 = x^\eta \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\rho n+\beta)} \frac{(w)^n}{(\delta)_n} \int_0^{\left[\frac{x}{a(1-\sigma)}\right]} \left[1 - \frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}} t^{\frac{\beta}{k}+\frac{\rho}{n}-1} dt.$$

Now, by evaluating the inner integral using beta function formula (1.12) , we get

$$I_1 = x^\eta \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\rho n+\beta)} \frac{w^n}{(\delta)_n} \left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}n+\frac{\beta}{k}} \frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}+1+\frac{\eta}{1-\sigma}\right)}.$$

Using (1.9), we obtain

$$\begin{aligned} I_1 &= \frac{x^{\eta+\frac{\beta}{k}}}{[a(1-\sigma)]^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\left(w\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right)^n}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}\right)k^{\frac{\rho}{k}n+\frac{\beta}{k}-1}(\delta)_n} \frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}+1+\frac{\eta}{1-\sigma}\right)} \\ &= \frac{x^{x^{\eta+\frac{\beta}{k}}}\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}\left(w\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right)^n}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}+1+\frac{\eta}{1-\sigma}\right)k^{\frac{\rho}{k}n+\frac{\beta}{k}-1}(\delta)_n}. \end{aligned}$$

Again, on applying (1.9), we get

$$I_1 = \frac{x^{\eta+\frac{\beta}{k}}k^{\left(1+\frac{\eta}{1-\sigma}\right)}\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}} E^{\gamma,q}_{k,\rho,\beta+k\left(1+\frac{\eta}{1-\sigma}\right),\delta}\left[w\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right],$$

which yields the desired proof of Theorem 2.1. □

**Corollary 2.2.** *If we put $\delta = 1$, then Theorem 2.1 leads to the known image formula given in [20]:*

$$P^{(\eta,\sigma)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\gamma,q}_{k,\rho,\beta,1}\left(wt^{\frac{\rho}{k}}\right)\right](x) = x^{\eta+\frac{\beta}{k}}k^{\left(1+\frac{\eta}{1-\sigma}\right)} \frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}} E^{\gamma,q}_{k,\rho,\beta+k\left(1+\frac{\eta}{1-\sigma}\right)}\left[w\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right].$$

**Corollary 2.3.** *If we put $\delta = q = 1$ in Theorem 2.1, then we get the following known result contains $k-$Mittag-Leffler function defined in (1.4) (see[4]):*

$$P^{(\eta,\sigma)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\gamma,1}_{k,\rho,\beta,1}\left(wt^{\frac{\rho}{k}}\right)\right](x) = x^{\eta+\frac{\beta}{k}}k^{\left(1+\frac{\eta}{1-\sigma}\right)} \frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}} E^{\gamma}_{k,\rho,\beta+k\left(1+\frac{\eta}{1-\sigma}\right)}\left[w\left(\frac{x}{a(1-\sigma)}\right)^{\frac{\rho}{k}}\right].$$

**Corollary 2.4.** *For $\delta = q = k = 1$ in Theorem 2.1, we get the following known image formula of Nair [12]:*

$$P^{(\eta,\sigma)}_{0+}\left[t^{\beta-1}E^{\gamma,1}_{k,\rho,1,1}(wt^\rho)\right](x) = x^{\eta+\beta}\frac{\Gamma\left(1+\frac{\eta}{1-\sigma}\right)}{[a(1-\sigma)]^\beta} E^{\gamma}_{\rho,\beta+1+\frac{\eta}{1-\sigma}}\left[\frac{(wx)^\rho}{(a(1-\sigma))^\rho}\right].$$

Now we establish the following theorem, by considering the case $\sigma > 1$ and using the equation (1.19).

**Theorem 2.5.** *Suppose $\rho, \beta, \gamma, \delta, \eta \in \mathbb{C}$, $k \in \mathbb{R}$, $\{\Re(\rho), \Re(\beta), \Re(\eta)\} > 0, \Re\left(1-\frac{\eta}{\sigma-1}\right) > 0, \sigma > 1, k, w \in \mathbb{R}$, $\delta \neq 0, -1, -2, \cdots$ and $q > 0$. Then the pathway fractional integral representation of (1.13) is given by*

$$P^{(\eta,\sigma)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\gamma,q}_{k,\rho,\beta,\delta}\left(wt^{\frac{\rho}{k}}\right)\right](x) = \frac{\Gamma\left(1-\frac{\eta}{\sigma-1}\right)x^{\eta+\frac{\beta}{k}}k^{\left(1-\frac{\eta}{1-\sigma}\right)}}{[-a(1-\sigma)]^{\frac{\beta}{k}}} E^{\gamma,q}_{k,\rho,\beta+k\left(1-\frac{\eta}{\sigma-1}\right),\delta}\left[w\left(\frac{x}{-a(1-\sigma)}\right)^{\frac{\rho}{k}}\right].$$

*Proof.* By applying (1.13) and (1.19), we have

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}E_{k,\rho,\beta,\delta}^{\gamma,q}\left(wt^{\frac{\rho}{k}}\right)\right](x)=x^{\eta}\int_{0}^{\left[\frac{x}{-a(1-\sigma)}\right]}t^{\frac{\beta}{k}-1}\left[1+\frac{a\left(\sigma-1\right)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}}E_{k,\rho,\beta,\delta}^{\gamma,q}\left(wt^{\frac{\rho}{k}}\right)dt.$$

Again we denote, for convenience, the right hand integral of the above term by $I_2$, then

$$I_2=x^{\eta}\int_{0}^{\left[\frac{x}{-a(\rho-1)}\right]}t^{\frac{\beta}{k}-1}\left[1+\frac{a\left(\sigma-1\right)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k}}{\Gamma_k\left(\rho n+\beta\right)}\frac{\left(wt^{\frac{\rho}{k}}\right)^n}{(\delta)_n}.$$

Now, on interchanging the order of integration and summation, we have

$$I_2=x^{\eta}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k}}{\Gamma_k\left(\rho n+\beta\right)}\frac{w^n}{(\delta)_n}\int_{0}^{\left[\frac{x}{-a(\rho-1)}\right]}\left[1+\frac{a\left(\sigma-1\right)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}}t^{\frac{\beta}{k}+\frac{\rho}{k}n-1}dt.$$

By putting $\frac{-a(\sigma-1)t}{x}=u$, we use the beta function formula (1.12) and the relation (1.9) to evaluate the above inner integral, and get

$$\begin{aligned}I_2&=\frac{x^{\eta+\frac{\beta}{k}}}{\left[-a\left(1-\sigma\right)\right]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k}\left[w\left(\frac{x}{-a(\sigma-1)}\right)^{\frac{\rho}{k}}\right]^n}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}\right)k^{\frac{\rho}{k}n+\frac{\beta}{k}-1}(\delta)_n}\frac{\Gamma\left(1-\frac{\eta}{1-\sigma}\right)\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}+1-\frac{\eta}{\rho-1}\right)}\\&=x^{\eta+\frac{\beta}{k}}\frac{\Gamma\left(1-\frac{\eta}{\sigma-1}\right)}{\left[-a\left(\sigma-1\right)\right]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,k}\,w\left[\left(\frac{x}{-a(\sigma-1)}\right)^{\frac{\rho}{k}}\right]^n}{\Gamma\left(\frac{\rho}{k}n+\frac{\beta}{k}+1-\frac{\eta}{\sigma-1}\right)k^{\frac{\rho}{k}n+\frac{\beta}{k}-1}(\delta)_n}.\end{aligned}$$

Again, on applying (1.9), we arrive at the following desired result:

$$I_2=\frac{x^{\eta+\frac{\beta}{k}}k^{\left(1-\frac{\eta}{\sigma-1}\right)}\Gamma\left(1-\frac{\eta}{\sigma-1}\right)}{\left[-a\left(\sigma-1\right)\right]^{\frac{\beta}{k}}}E_{k,\rho,\beta+k\left(1-\frac{\eta}{\sigma-1}\right),\delta}^{\gamma,q}\left[w\left(\frac{x}{-a\left(\sigma-1\right)}\right)^{\frac{\rho}{k}}\right].$$

□

**Corollary 2.6.** *Let $\delta=1$, then Theorem 2.5 reduces to the following result given in [20]:*

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}E_{k,\rho,\beta,1}^{\gamma,q}\left(wt^{\frac{\rho}{k}}\right)\right](x)=\frac{x^{\eta+\frac{\beta}{k}}k^{\left(1-\frac{\eta}{\sigma-1}\right)}\Gamma\left(1-\frac{\eta}{\sigma-1}\right)}{\left[-a\left(\sigma-1\right)\right]^{\frac{\beta}{k}}}E_{k,\rho,\beta+k\left(1-\frac{\eta}{\sigma-1}\right)}^{\gamma,q}\left[w\left(\frac{x}{-a\left(\sigma-1\right)}\right)^{\frac{\rho}{k}}\right].$$

**Corollary 2.7.** *If we put $\delta=q=1$ in Theorem 2.5, then we get the following known result contains $k-$Mittag-Leffler function (due to [4]):*

$$P_{0+}^{(\eta,\sigma)}\left[t^{\frac{\beta}{k}-1}E_{k,\rho,\beta,1}^{\gamma,1}\left(wt^{\frac{\rho}{k}}\right)\right](x)=\frac{x^{\eta+\frac{\beta}{k}}k^{\left(1-\frac{\eta}{\sigma-1}\right)}\Gamma\left(1-\frac{\eta}{\sigma-1}\right)}{\left[-a\left(\sigma-1\right)\right]^{\frac{\beta}{k}}}E_{k,\rho,\beta+k\left(1-\frac{\eta}{\sigma-1}\right)}^{\gamma}\left[w\left(\frac{x}{-a\left(\sigma-1\right)}\right)^{\frac{\rho}{k}}\right]$$

**Corollary 2.8.** *If we set $\delta=q=k=1$, Theorem 2.5 leads to the well known image formula of [12]:*

$$P_{0+}^{(\eta,\sigma)}\left[t^{\beta-1}E_{1,\rho,\beta}^{\gamma,1}\left(wt^{\rho}\right)\right](x)=\frac{x^{\eta+\beta}\Gamma\left(1-\frac{\eta}{\sigma-1}\right)}{\left[-a\left(\sigma-1\right)\right]^{\beta}}E_{\rho,\beta+\left(1-\frac{\eta}{\sigma-1}\right)}^{\gamma}\left[w\left(\frac{x}{-a\left(\sigma-1\right)}\right)^{\rho}\right].$$

## 3. Conclusion

In this paper we introduced a more generalized special function called as $k$-Mittag-Leffler function. For this unified Mittag-Leffler function, we have presented two pathway fractional integral formulas (PFIF). The obtained result provides an extension of the known results, as mentioned earlier. Our paper is concluded with the remark that, the function introduced and reported results are significant and can lead to yield number of other integral (image) formulas involving various Mittag-Leffler type functions.


## Acknowledgments

The research is supported by a grant from the "Research Center of the Center for Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University. The authors are also thankful to visiting professor program at King Saud University for support.


## References


[1] P. Agarwal, S. D. Purohit, *The unified pathway fractional integral formulae,* J. Fract. Calc. Appl., **4** (2013), 1–8. 1

[2] D. Baleanu, P. Agarwal, *A composition formula of the pathway integral transform operator,* Note Mat., **34** (2014), 145–155. 1, 1

[3] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus Models and Numerical Methods,* World Sci., **3** (2012), 10–16. 1

[4] G. A. Dorrego, R. A. Cerutti, *The k-Mittag-Leffler function,* Int. J. Contemp. Math. Sci., **7** (2012), 705–716. 1, 1, 2.3, 2.7

[5] K. S. Gehlot, *The Generalized k-Mittag-Leffler function,* Int. J. Contemp. Math. Sci., **7** (2012), 2213–2219. 1, 1

[6] M. A. Khan, S. Ahmed, *On some properties of the generalized Mittag-Leffler function,* Springer Plus, **2** (2013), 1–9. 1

[7] A. M. Mathai, *A pathway to matrix-variate gamma and normal densities,* Linear Algebra Appl., **396** (2005), 317–328. 1

[8] A. M. Mathai, H. J. Haubold, *Pathway model, superstatistics, Tsallis statistics and a generalize measure of entropy,* Phys. A, **375** (2007), 110–122. 1, 1

[9] A. M. Mathai, H. J. Haubold, *On generalized distributions and path-ways,* Phys. Lett. A, **372** (2008), 2109–2113. 1, 1

[10] G. M. Mittag-Leffler, *Sur la nouvelle fonction $E_\alpha(x)$,* CR Acad. Sci. Paris, **137** (1903), 554–558. 1

[11] G. M. Mittag-Leffler, *Sur la representation analytique d'une branche uniforme d'une function monogene,* Acta Math., **29** (1905), 101–181. 1

[12] S. S. Nair, *Pathway fractional integration operator,* Fract. Calc. Appl. Anal., **12** (2009), 237–252. 1, 1, 2.4, 2.8

[13] K. S. Nisar, S. R. Mondal, P. Agarwal, *Pathway fractional integral operator associated with Struve function of first kind,* Adv. Stud. Contemp. Math., **26** (2016), 63–70. 1

[14] K. S. Nisar, S. D. Purohit, S. R. Mondal, *Generalized fractional kinetic equations involving generalized Struve function of the first kind,* J. King Saud Univ. Sci., **28** (2016), 167–171. 1

[15] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the Kernel,* Yokohama Math. J., **19** (1971), 7–15. 1

[16] I. Podlubny, *Fractional Differential Equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications,* Academic Press, San Diego, CA, (1999). 1

[17] S. D. Purohit, *Solutions of fractional partial differential equations of quantum mechanics,* Adv. Appl. Math. Mech., **5** (2013), 639–651.

[18] S. D. Purohit, S. L. Kalla, *On fractional partial differential equations related to quantum mechanics,* J. Phys. A, **44** (2011), 8 pages. 1

[19] E. D. Rainville, *Special Functions,* The Macmillan Co., New York, (1960). 1

[20] C. Ram, P. Choudhary, *Pathway fractional integration operator as generalized k-mittag-leffler function in its kernel,* Int. J. Math. Arch., **4** (2014), 114–120. 2.2, 2.6

[21] T. O. Salim, *Some properties relating to the generalized Mittag-Leffler function,* Adv. Appl. Math. Anal., **4** (2009), 21–30. 1

[22] T. O. Salim, W. F. Ahmad, *A generalization of integral operator associated with fractional calculus,* J. Fract. Calc. Appl., **3** (2012), 1–13.

[23] A. K. Shukla, J. C. Prajapati, *On a generalization of Mittag-Leffler function and its properties,* J. Math. Anal. Appl., **336** (2007), 797–811.

[24] H. M. Srivastava, Z. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel,* Appl. Math. Comput., **211** (2009), 198–210. 1

[25] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers, Volume I, Background and Theory Volume II Applications,* Nonlinear Physical Science, Springer-Verlag, Berlin-Heidelberg, (2013). 1

[26] A. Wiman, *Uber de fundamental satz in der theorie der funktionen $Ea(x)$,* Acta Math., **29** (1905), 191–201. 1