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Variational approach to second-order damped Hamiltonian systems with impulsive effects

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Abstract

In this paper, we consider the existence of second-order damped vibration Hamiltonian systems with impulsive effects. We obtain some new existence theorems of solutions by using variational methods.

Keywords: Hamiltonian systems, variational method, impulsive effects, damped vibration.
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1. Introduction and Preliminaries

The pioneering research of impulsive differential equations via variational methods was initiated by Nieto and O'Regan [5], and study on impulsive differential equations with derivative dependence via variational methods was introduced by Nieto. In [4] the following boundary value problem was investigated

$$\begin{cases} -\ddot{u}(t) + g(t)\dot{u}(t) + \lambda u(t) = \sigma(t), \quad \text{a.e. } t \in [0,T], \\ \Delta\ddot{u}(t_j) = d_j, \quad j = 1,2,\dots,p, \\ u(0) = u(T) = 0, \end{cases}$$

where $\lambda, d_j \in \mathbb{R}$, $\sigma \in C[0,1]$. The author introduced a variational formulation for the damped linear Dirichlet problem with impulses and the concept of a weak solution for this problem. Since then there is a trend to study differential equation via variational methods which leads to many meaningful results, see [14, 15, 16, 17] and the references therein.

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Motivated by the above mentioned work, in [2], we obtained the existence of at least one classical solution, at least two classical solutions and infinitely many classical solutions of the following damped boundary value problem with impulses

$$\begin{cases} -\ddot{u}(t) + g(t)\dot{u}(t) + \lambda u(t) = f(t,u), \quad \text{a.e. } t \in [0,T], \\ -\Delta \dot{u}(t_i) = I_i(u(t_i)), \quad i = 1,2,\dots,p, \\ u(0) = 0, \quad \alpha u(T) + \beta \dot{u}(T) = 0, \end{cases}$$

where λ is a parameter, $T > 0$, $g \in C[0,T]$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $I_i : \mathbb{R} \to \mathbb{R} (i = 1,2,\dots,p)$ are continuous, $\alpha \geq 0, \beta > 0$ (or $\beta = 0$).

Just as what Nieto said in [5], "This approach is novel and it may open a new approach to deal with nonlinear problems with some type of discontinuities such as impulses". From then on, the second-order Hamiltonian systems with impulsive effects are also surveyed by variational methods, we refer the readers to [7, 8, 18] and the references therein.

Zhou and Li [18] studied the existence of period solutions of the following impulsive second-order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t,u), \quad a.e.\ t \in [0,T], \\ \triangle \dot{u}^i(t_j) = I_{ij}(u^i(t_j)), \quad i = 1,2,...,N; j = 1,2,...,p, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $\triangle \dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = \lim_{t \to t_i^+} \dot{u}(t) - \lim_{t \to t_i^-} \dot{u}(t)$ and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ and $I_{ij} : \mathbb{R} \to \mathbb{R}$ are continuous.

In this paper, we consider the second-order Hamiltonian systems with impulsive damped vibration

$$\begin{cases} \ddot{u}(t) + g(t)\dot{u}(t) - \lambda A(t)u(t) = -\nabla F(t,u), \quad a.e.\ t \in [0,T], \\ \triangle \dot{u}^i(t_j) = I_{ij}(u^i(t_j)), \quad i = 1,2,...,N; j = 1,2,...,p, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0, T > 0$, $g \in L^1[0,T], \int_0^T g(s)ds = 0, A : [0,T] \to \mathbb{R}^{N \times N}$ is a continuous map from $[0,T]$ to the set of N-order symmetric matrices, $I_{ij} : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,...,N; j = 1,2,...,p)$ are continuous, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $\triangle \dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = \lim_{t \to t_i^+} \dot{u}(t) - \lim_{t \to t_i^-} \dot{u}(t)$ and $F \in C([0,T] \times \mathbb{R}^N, \mathbb{R})$ satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for $a.e.\ t \in [0,T]$, and there exists $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T], \mathbb{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \ |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and $a.e.\ t \in [0,T]$.

When $g(t) \equiv 0$, $A(t)$ is a zero matrix and $I_{ij} \equiv 0$, Hamiltonian systems (1.1) has been studied extensively, see [3, 9, 10, 11, 12, 13] and the references therein.

More precisely, Mawhin and Willem [3] studied the following Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t,u), \quad a.e.\ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

they established existence of at least one period solution when satisfies the following assumption:

$F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for $a.e.\ t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \ |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and *a.e.* $t \in [0,T]$ and there exists $h \in L(0,T)$ such that $|\nabla F(t,x)| \leq h(t)$, and furthermore $\int_0^T F(t,x) \to +\infty$ when $|x| \to \infty$(Ahmad-Lazer-Paul type coercive condition).

Tang [10] studied the same Hamiltonian systems by using the least action principle and minimax methods when satisfying

$$|x|^{-2\alpha} \int_0^T F(s,x)ds \to +\infty(|x| \to \infty)$$

or

$$|x|^{-2\alpha} \int_0^T F(s,x)ds \to -\infty(|x| \to \infty).$$

For $g(t) \not\equiv 0$, $A(t)$ is not a zero matrix and $I_{ij} \not\equiv 0$ $(i = 1,2,...,N; j = 1,2,...,p)$, the Hamiltonian systems (1.1) is an impulsive damped vibration Hamiltonian systems. In this paper, our aim is to study the existence of second-order Hamiltonian systems with impulsive damped vibration differential equations. The rest of the paper is organized as follows. In Section 2, we give several important lemmas and variational structure. The main theorems are formulated and proved in Section 3. In Section 4, some examples are presented to illustrate our results.

2. Preliminaries and variational structure

Let $H := H_T^1(0,T;\mathbb{R}^N) = \{u : [0,T] \to \mathbb{R}^N | u$ be absolutely continuous, $\dot{u} \in L^2(0,T;\mathbb{R}^N)$ and $u(0) = u(T)\}$ with the inner product

$$\langle u,v\rangle_0 = \int_0^T (\dot{u}(t),\dot{v}(t))dt + \int_0^T (u(t),v(t))dt$$

for all $u,v \in H$, where $(.,.)$ denotes the inner product in $\mathbb{R}^N$.

Set $H^2(0,T;\mathbb{R}^N) = \{u : [0,T] \to \mathbb{R} | u, \dot{u}$ are absolutely continuous, $\ddot{u} \in L^2(0,T;\mathbb{R}^N)\}$.

For all $u \in H^2(0,T;\mathbb{R}^N)$, we have $\triangle\dot{u}(t) = \dot{u}(t^+) - \dot{u}(t^-) = 0$ for any $t \in (0,T)$. If $u \in H$, we have that u is absolutely continuous and $\dot{u} \in L^2(0,T;\mathbb{R}^N)$, thus the one side derivatives $\dot{u}(t^+)$, $\dot{u}(t^-)$ may not exist, which leads to the impulsive effects.

Let $G(t) = \int_0^t g(s)ds$. Noting that $g \in L^1[0,T]$, one has $G'(t) = (\int_0^t g(s)ds)' = g(t), a.e.\ t \in [0,T]$, thus $G(t)$ is absolutely continuous, which leads to the boundedness of $G(t)$, hence $\max_{t\in[0,T]} e^{G(t)}$ and $\min_{t\in[0,T]} e^{G(t)}$ exist, thus set $M = \max_{t\in[0,T]} e^{G(t)},\ m = \min_{t\in[0,T]} e^{G(t)}$.

Set

$$\langle u,v\rangle = \int_0^T e^{G(t)}(\dot{u}(t),\dot{v}(t))dt + \int_0^T e^{G(t)}(\lambda A(t)u(t),v(t))dt \ \ \forall\ u,v \in H$$

and the corresponding norm is defined by

$$\|u\| = \left(\int_0^T e^{G(t)}|\dot{u}(t)|^2dt + \int_0^T e^{G(t)}(\lambda A(t)u(t),u(t))dt\right)^{\frac{1}{2}}.$$

By the similar proof of the corresponding parts in [1], one has the above norm is equivalent to the usual one

$$\|u\|_0 = \left(\int_0^T |\dot{u}(t)|^2dt + \int_0^T |u(t)|^2dt\right)^{\frac{1}{2}}.$$

Now we recall that

$$\|u\|_{L^2} = \left(\int_0^T |u(t)|^2dt\right)^{\frac{1}{2}}$$

and

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

The Sobolev space H has some properties as follows.

Lemma 2.1 ([3]). *There exists $C_1 > 0$ such that, if $u \in H$, then*

$$\|u\|_\infty \leq C_1 \|u\|.$$

Moreover, if $\int_0^T u(t)dt = 0$, then

$$\|u\|_\infty \leq C_1 \|\dot{u}\|_{L^2}.$$

Lemma 2.2 ([3]). *If the sequence $\{u_k\}$ converges weakly to u in H, then $\{u_k\}$ converges uniformly to u on $[0, T]$.*

Lemma 2.3 ([10]). *If $u \in H$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)dt$, $\tilde{u} = u - \bar{u}$. Then one has*

$$\|\tilde{u}\|_{L^2}^2 \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt \quad (Wirtinger's\ inequality)$$

and

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \quad (Sobolev\ inequality).$$

In the following we give the variational structure.

Multiply the two sides of the first equality (1.1) by $e^{G(t)}$ to get

$$e^{G(t)}\ddot{u}(t) + e^{G(t)}g(t)\dot{u}(t) - \lambda e^{G(t)}A(t)u(t) = -e^{G(t)}\nabla F(t, u).$$

By Remark 3 in P.7 of [3], one has $\dot{u}$ is classical derivative of u, thus the above equality implies

$$(e^{G(t)}\dot{u}(t))' = \lambda e^{G(t)}A(t)u(t) - e^{G(t)}\nabla F(t, u).$$

Now multiply by $v(t) \in H$ at both sides and integrate from 0 to T,

$$\int_0^T (e^{G(t)}\dot{u}(t))', v(t))dt = \int_0^T e^{G(t)}(\lambda A(t)u(t) - \nabla F(t, u), v(t))dt. \tag{2.1}$$

Combining $\int_0^T g(s)ds = 0, \dot{u}(0) = \dot{u}(T)$ and $v(0) = v(T)$, one has

$$\begin{aligned}
\int_0^T ((e^{G(t)}\dot{u}(t))', v(t))dt &= \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} ((e^{G(t)}\dot{u}(t))', v(t))dt \\
&= \sum_{j=0}^{p} e^{G(t)}\dot{u}(t)v(t) \Big|_{t_j}^{t_{j+1}} - \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} e^{G(t)}\dot{u}(t)\dot{v}(t)dt \\
&= \sum_{j=0}^{p} \left[\sum_{i=1}^{N} (e^{G(t)}\dot{u}^i(t), v(t)) \Big|_{t_j}^{t_{j+1}} - \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} e^{G(t)}\dot{u}(t)\dot{v}(t)dt \right] \\
&= -\sum_{j=1}^{p} \left[\sum_{i=1}^{N} e^{G(t_j)} \triangle \dot{u}^i(t_j)v^i(t_j) - \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} e^{G(t)}\dot{u}(t)\dot{v}(t)dt \right] \\
&= -\sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} I_{ij}(u^i(t_j))v^i(t_j) - \int_0^T e^{G(t)}\dot{u}(t)\dot{v}(t)dt.
\end{aligned}$$

Combining with (2.1), one has

$$\int_0^T e^{G(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{G(t)}(\lambda A(t)u(t) - \nabla F(t, u), v(t))dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} I_{ij}(u^i(t_j))v^i(t_j) = 0. \tag{2.2}$$

Thus, a weak solution to (1.1) is given below and it is inspired by the weak solution defined in [4].

Definition 2.4. $u \in H$ is a weak solution of (1.1) if

$$\int_0^T e^{G(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{G(t)}(\lambda A(t)u(t) - \nabla F(t,u)), v(t))dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} I_{ij}(u^i(t_j))v^i(t_j) = 0$$

holds for any $v \in H$.

For the sake of convenience, we define $\mathcal{A} = \{1, 2, ..., N\}$, $\mathcal{B} = \{1, 2, ..., p\}$. Consider the functional $\varphi : H \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2 dt - \frac{\lambda}{2}\int_0^T e^{G(t)}(A(t)u(t), u(t))dt \\ &\quad - \int_0^T e^{G(t)} F(t,u)dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} \int_0^{u^i(t_j)} I_{ij}(t)dt \\ &= \frac{1}{2}\|u\|^2 + \int_0^T e^{G(t)} F(t,u)dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} \int_0^{u^i(t_j)} I_{ij}(t)dt,\end{aligned}$$

we have the following two lemmas.

Lemma 2.5. *The functional φ is continuously differentiable.*

Proof. Let

$$\begin{aligned}\phi(u) &= \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2 dt - \frac{\lambda}{2}\int_0^T e^{G(t)}(A(t)u(t), u(t))dt - \int_0^T e^{G(t)} F(t,u)dt, \\ L(t,x,y) &= \frac{1}{2}e^{G(t)}|y|^2 - \frac{\lambda}{2}e^{G(t)}(A(t)x, x) - e^{G(t)}F(t,x)\end{aligned}$$

and

$$\psi(u) = \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} \int_0^{u^i(t_j)} I_{ij}(t)dt.$$

By assumption (A), $L(t, x, y)$ satisfies all assumptions of Theorem 1.4 in [3]. Hence, by Theorem 1.4 in [3], we know that ϕ is continuously differentiable. By the continuity of I_{ij}, $i \in \mathcal{A}$, $j \in \mathcal{B}$, we know that ψ is continuously differentiable. Thus φ is continuously differentiable and $\varphi'(u)$ is defined by

$$\langle \varphi'(u), v\rangle = \int_0^T e^{G(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{G(t)}(\lambda A(t)u(t) - \nabla F(t,u), v(t))dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} I_{ij}(u^i(t_j))v^i(t_j).$$

□

Lemma 2.6. *If $u \in H$ is a solution of the Euler equation $\varphi'(u) = 0$, then u is a weak solution of* (1.1).

Proof. Since $\varphi'(u) = 0$, thus for any $v \in H$,

$$\begin{aligned}0 = \langle \varphi'(u), v\rangle &= \int_0^T e^{G(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{G(t)}(\lambda A(t)u(t) - \nabla F(t,u), v(t))dt \\ &\quad + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} I_{ij}(u^i(t_j))v^i(t_j),\end{aligned}$$

thus by Definition 2.4, u is a weak solution of (1.1). □

3. Main results

Theorem 3.1. *Suppose that (A) and the following conditions are satisfied*

(i) There exist $f, e \in L^1(0,T;\mathbb{R}^+)$ *and* $\alpha \in [0,1)$ *such that*

$$|\nabla F(t,x)| \leq f(t)|x|^\alpha + e(t)$$

for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$;

(ii) $A(t)x \cdot x \geq 0$ *for all* $x \in \mathbb{R}^N$, *a.e.* $t \in [0,T]$; $I_{i,j}(t)t \geq 0 \quad \forall i \in \mathcal{A},\ j \in \mathcal{B},\ t \in \mathbb{R}$;

(iii) $|x|^{-2\alpha}\int_0^T F(s,x)ds \to -\infty$, *as* $|x| \to \infty$.

Then (1.1) *has at least one weak solution in* H *when* $\lambda > 0$.

Proof. It follows from (i) and Lemma 2.5 that

$$\begin{aligned}
\left|\int_0^T (F(t,u(t)) - F(t,\bar{u}))dt\right| &\leq \left|\int_0^T \int_0^1 (\nabla F(t,\bar{u}+s\tilde{u}(t))\tilde{u}(t)dsdt\right| \\
&\leq \int_0^T \int_0^1 f(t)|\bar{u}+s\tilde{u}(t)|^\alpha|\tilde{u}(t)|dsdt + \int_0^T \int_0^1 e(t)|\tilde{u}(t)|dsdt \\
&\leq 2(|\bar{u}|^\alpha + \|\tilde{u}\|_\infty^\alpha)\|\tilde{u}\|_\infty \int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T e(t)dt \\
&\leq \frac{3m}{MT}\|\tilde{u}\|_\infty^2 + \frac{MT}{3m}|\bar{u}|^{2\alpha}\left(\int_0^T f(t)dt\right)^2 \\
&\quad + 2\|\tilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T e(t)dt \\
&\leq \frac{m}{4M}\int_0^T |\dot{u}(t)|^2dt + C_2|\bar{u}|^{2\alpha} + C_3\left(\int_0^T |\dot{u}(t)|^2dt\right)^{\frac{\alpha+1}{2}} \\
&\quad + C_4\left(\int_0^T |\dot{u}(t)|^2dt\right)^{\frac{1}{2}},
\end{aligned}$$

where

$$C_2 = \frac{MT}{3m}\left(\int_0^T f(t)dt\right)^2, \quad C_3 = 2\left(\frac{T}{12}\right)^{\frac{\alpha+1}{2}}\int_0^T f(t)dt, \quad C_4 = \left(\frac{T}{12}\right)^{\frac{1}{2}}\int_0^T e(t)dt.$$

By (ii), one has $\sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)}\int_0^{u^i(t_j)} I_{ij}(t)dt > 0$. Thus

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2dt + \frac{\lambda}{2}\int_0^T e^{G(t)}(A(t)u(t),u(t))dt \\
&\quad - \frac{1}{2}\int_0^T e^{G(t)}F(t,u)dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)}\int_0^{u^i(t_j)} I_{ij}(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2 dt + \frac{\lambda}{2}\int_0^T e^{G(t)}(A(t)u(t), u(t))dt \\
&\quad - \int_0^T e^{G(t)}(F(t,u) - F(t,\bar{u}))dt - \int_0^T e^{G(t)}F(t,\bar{u})dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)}\int_0^{u^i(t_j)} I_{ij}(t)dt \\
&\geq \frac{m}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{m}{4}\int_0^T |\dot{u}(t)|^2 dt - C_2 M|\bar{u}|^{2\alpha} \qquad (3.1) \\
&\quad - C_3 M\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} - C_4 M\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} - M\int_0^T F(t,\bar{u})dt \\
&\geq \frac{m}{4}\int_0^T |\dot{u}(t)|^2 dt - C_3 M\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} - C_4 M\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad - (|\bar{u}|^{-2\alpha} M\int_0^T F(t,\bar{u})dt + C_2 M)|\bar{u}|^{2\alpha}. \qquad (3.2)
\end{aligned}
$$

Noting that

$$
\left(\int_0^T |\dot{u}(t)|^2 dt + |\bar{u}|^2\right)^{\frac{1}{2}} = \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{T^2}\left(\int_0^T |u(t)|dt\right)^2)^{\frac{1}{2}},
$$

$$
\|u\|_0 = \left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T |u(t)|^2 dt\right)^{\frac{1}{2}}
$$

and $\|u\|_0 \to \infty$ if and only if $(\int_0^T |\dot{u}(t)|^2 dt + |\bar{u}|^2)^{\frac{1}{2}} \to \infty$([10]), furthermore, $\|u\|_0$ is equivalent to $\|u\|$, thus $\|u\| \to \infty$ if and only if $(\int_0^T |\dot{u}(t)|^2 dt + |\bar{u}|^2)^{\frac{1}{2}} \to \infty$, thus (3.1) and (iii) imply that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. By Theorem 1.1 and Corollary 1.1 in [3], φ has a minimum point on H, which is a critical point of φ and also is a weak solution of (1.1). □

Theorem 3.2. *Suppose that assumption (A) and (i) of Theorem* 3.1 *hold, furthermore, the following conditions are satisfied*

(iv) *There exist* $b_{ij}, c_{ij} > 0$ *and* $\beta_{ij} \in [0,1)$, *such that*

$$
|I_{ij}(t)| \leq b_{ij}|t|^{\alpha\beta_{ij}} + c_{ij} \ \forall i \in \mathcal{A},\ j \in \mathcal{B},\ t \in \mathbb{R};
$$

(v) $I_{ij}(t)t \leq 0 \quad \forall i \in \mathcal{A},\ j \in \mathcal{B},\ t \in \mathbb{R}$;

(vi) $|x|^{-2}\int_0^T F(s,x)ds \to +\infty$, *as* $|x| \to \infty$;

(vii) $(A(t)x, x) \leq F(t,x)\ \forall x \in \mathbb{R}^N$, *a.e.* $t \in [0,T]$.

Proof. Let $\{u_n\}$ be a sequence in H such that $\{\varphi(u_n)\}$ is bounded and $\varphi(u_n') \to 0$, as $n \to +\infty$, then we will prove $\{u_n\}$ possesses a convergent subsequence. We first prove that $\{u_n\}$ is bounded.

Let

$$
\tilde{C}_2 = \frac{1}{TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty}\left(\int_0^T f(t)dt\right)^2.
$$

In a way similar to the proof of Theorem 3.1, for all n, one has

$$
|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt| \ \leq TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty \|\tilde{u}_n\|_\infty^2 + \frac{1}{TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty}|\bar{u}_n|^{2\alpha}\left(\int_0^T f(t)dt\right)^2
$$

$$
\begin{aligned}
&+ 2\|\tilde{u}_n\|_\infty^{\alpha+1}\int_0^T f(t)dt + \|\tilde{u}_n\|_\infty \int_0^T e(t)dt\\
&\le \frac{T^2}{12}M\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty \int_0^T |\dot{u}_n(t)|^2dt + \tilde{C_2}|\bar{u}_n|^{2\alpha}\\
&+ C_3\left(\int_0^T |\dot{u}_n(t)|^2dt\right)^{\frac{\alpha+1}{2}} + C_4\left(\int_0^T |\dot{u}_n(t)|^2dt\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
|\int_0^T e^{G(t)}(A(t)u_n(t),\tilde{u}_n)dt| &\le \int_0^T e^{G(t)}|(A(t)u_n(t),\tilde{u}_n)|dt\\
&\le \sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty \int_0^T e^{G(t)}|(\tilde{u}_n+\bar{u}_n)\tilde{u}_n|dt\\
&\le TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\|\tilde{u}_n\|_\infty^2 + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|\|\tilde{u}_n\|_\infty\\
&\le \frac{T}{12}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n|^2dt + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\frac{1}{2}\|\tilde{u}_n\|_\infty^2\\
&\quad + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\frac{1}{2}|\bar{u}_n|^2\\
&\le \frac{T}{12}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n|^2dt\\
&\quad + \frac{1}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\frac{T}{12}\int_0^T|\dot{u}_n(t)|^2dt + \frac{1}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2\\
&= \frac{T}{8}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n|^2dt + \frac{1}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2.
\end{aligned}
\tag{3.4}
$$

Let $b = \max_{i\in\mathcal{A},j\in\mathcal{B}} b_{ij}$ and $c = \max_{i\in\mathcal{A},j\in\mathcal{B}} c_{ij}$. According to (iv), one has

$$
\begin{aligned}
|\sum_{j=1}^{p}\sum_{i=1}^{N}e^{G(t_j)}\int_0^{u^i(t_j)}I_{ij}(t)dt| &\le M\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)}(b_{ij}|t|^{\alpha\beta_{ij}}+c_{ij})dt\\
&\le cpNM\|u\|_\infty + bM\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|_\infty^{\alpha\beta_{ij}+1}\\
&\le cpNM\sqrt{\frac{T}{12}}(\int_0^T|\dot{u}|^2dt)^{\frac{1}{2}} + (\frac{T}{12})^{\frac{\alpha\beta_{ij}+1}{2}}bM\sum_{j=1}^{p}\sum_{i=1}^{N}\left(\int_0^T|\dot{u}|^2dt\right)^{\frac{\alpha\beta_{ij}+1}{2}}.
\end{aligned}
$$

When $\lambda < \frac{24m}{5T^2M\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty}$, by (3.3), (3.4), (iv) and Young's inequality, for large n one has

$$
\begin{aligned}
\|\tilde{u}_n\| &\ge \langle\varphi'(u_n),\tilde{u}_n\rangle\\
&= \int_0^T e^{G(t)}|\dot{u}_n(t)|^2dt + \int_0^T e^{G(t)}(\lambda A(t)u_n(t) - \nabla F(t,u_n),\tilde{u}_n)dt
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{p}\sum_{i=1}^{N}e^{G(t_j)}I_{ij}(u_n^i(t_j))\tilde{u}_n^i(t_j)\\
\geq&\int_0^T e^{G(t)}|\dot{u}_n(t)|^2dt-\frac{T}{12}\lambda TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt-\tilde{C}_2|\bar{u}_n|^{2\alpha}\\
&-C_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}-C_4(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-\lambda\frac{T}{8}TM\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt\\
&-\frac{\lambda}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2+\sum_{j=1}^{p}\sum_{i=1}^{N}e^{G(t_j)}I_{ij}(u_n^i(t_j))\tilde{u}_n^i(t_j)\\
\geq&\int_0^T e^{G(t)}|\dot{u}_n(t)|^2dt-\frac{T}{12}\lambda TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt-\tilde{C}_2|\bar{u}_n|^{2\alpha}\\
&-C_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}-C_4(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-\frac{\lambda T}{8}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt\\
&-\frac{\lambda}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2-\sum_{j=1}^{p}\sum_{i=1}^{N}e^{G(t_j)}(b_{ij}|\bar{u}_n^i(t_j)+\tilde{u}_n^i(t_j)|^{\alpha\beta_{ij}}+c_{ij})|\tilde{u}_n^i(t_j)|\\
\geq&\int_0^T e^{G(t)}|\dot{u}_n(t)|^2dt-\frac{T}{12}\lambda TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt-\tilde{C}_2|\bar{u}_n|^{2\alpha}\\
&-C_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}-C_4(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-\frac{T}{8}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt \qquad (3.5)\\
&-\frac{\lambda}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2-cpNM\|\tilde{u}_n\|_\infty-2bM\sum_{j=1}^{p}\sum_{i=1}^{N}(|\bar{u}_n|^{\alpha\beta_{ij}}+\|\tilde{u}_n\|_\infty^{\alpha\beta_{ij}})\|\tilde{u}_n\|_\infty\\
\geq&\int_0^T e^{G(t)}|\dot{u}_n(t)|^2dt-\frac{T}{12}\lambda TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt-\tilde{C}_2|\bar{u}_n|^{2\alpha}\\
&-C_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}-C_4(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-\lambda\frac{T}{8}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt\\
&-\frac{\lambda}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2-cpNM\sqrt{\frac{T}{12}}(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-2bM\sum_{j=1}^{p}\sum_{i=1}^{N}\frac{\alpha\beta_{ij}}{2}|\bar{u}_n|^{2\alpha}\\
&-2bM\sum_{j=1}^{p}\sum_{i=1}^{N}\|\tilde{u}_n\|_\infty^{\frac{2}{2-\alpha\beta_{ij}}}-2bM\sum_{j=1}^{p}\sum_{i=1}^{N}\|\tilde{u}_n\|_\infty^{\alpha\beta_{ij}+1}\\
\geq&\int_0^T e^{G(t)}|\dot{u}_n(t)|^2dt-\frac{T}{12}\lambda TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt-\tilde{C}_2|\bar{u}_n|^{2\alpha}\\
&-C_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}-C_4(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-\frac{\lambda T}{8}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt\\
&-\frac{\lambda}{2}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2-cpNM\sqrt{\frac{T}{12}}(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-2bM\sum_{j=1}^{p}\sum_{i=1}^{N}\frac{\alpha\beta_{ij}}{2}|\bar{u}_n|^{2\alpha}
\end{aligned}
$$

$$- 2bM \sum_{j=1}^{p}\sum_{i=1}^{N}(\frac{T}{12}\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2-\alpha\beta_{ij}}} - 2bM \sum_{j=1}^{p}\sum_{i=1}^{N}(\frac{T}{12}\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{\alpha\beta_{ij}+1}{2}}.$$

It follows from Wirtinger's inequality that

$$\|\tilde{u}_n\|^2 \le (1+\frac{T^2}{4\pi^2})\int_0^T |\dot{u}_n(t)|^2 dt. \tag{3.6}$$

By (3.5) and (3.6), one has

$$\int_0^T |\dot{u}_n(t)|^2 dt \le C_5|\bar{u}_n|^2 + C_6|\bar{u}_n|^{2\alpha} + C_7|\bar{u}_n|^{\alpha} + C_8 \tag{3.7}$$

for all large n and some positive constants $C_5,\ C_6,\ C_7$ and C_8. It follows from (i) and Lemma 2.3 that

$$\begin{aligned}
|\int_0^T e^{G(t)}(A(t)u_n(t), u_n(t))dt| &\le \int_0^T e^{G(t)}|(A(t)u_n(t), u_n(t))|dt \\
&\le \sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty \int_0^T e^{G(t)}|(\tilde{u}_n+\bar{u}_n)(\tilde{u}_n+\bar{u}_n)|dt \\
&\le TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\|\tilde{u}_n\|_\infty^2 + 2TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|\|\tilde{u}_n\|_\infty \\
&\quad + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2 \\
&\le \frac{T}{12}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt + 2TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\frac{1}{2}\|\tilde{u}_n\|_\infty^2 \\
&\quad + 2TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\frac{1}{2}|\bar{u}_n|^2 \\
&\quad + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2 \\
&\le \frac{T}{12}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt \\
&\quad + \frac{T}{12}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt \\
&\quad + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2 + TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2 \\
&= \frac{T}{6}TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}_n(t)|^2dt + 2TM\sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty|\bar{u}_n|^2
\end{aligned} \tag{3.8}$$

and

$$|\int_0^T (F(t,u_n(t)) - F(t,\tilde{u}_n))dt| \le |\int_0^T\int_0^1 (\nabla F(t,\bar{u}_n+s\tilde{u}_n)\tilde{u}_n ds dt|$$

$$
\begin{aligned}
&\le \int_0^T \int_0^1 f(t)|\bar{u}_n + s\tilde{u}_n|^\alpha |\tilde{u}_n| ds dt + \int_0^T \int_0^1 e(t)|\tilde{u}_n(t)| ds dt \\
&\le 2(|\bar{u}_n|^\alpha + \|\tilde{u}_n\|_\infty^\alpha)\|\tilde{u}_n\|_\infty \int_0^T f(t)dt + \|\tilde{u}_n\|_\infty \int_0^T e(t)dt \\
&\le TM \sum_{j=1}^p \sum_{i=1}^N \|a_{ij}\|_\infty \|\tilde{u_n}\|_\infty^2 + \frac{1}{TM \sum_{j=1}^p \sum_{i=1}^N \|a_{ij}\|_\infty} |\bar{u}_n|^{2\alpha} \left(\int_0^T f(t)dt\right)^2 \\
&\quad + 2\|\tilde{u}\|_\infty^{\alpha+1} \int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T e(t)dt \qquad (3.9)\\
&\le \frac{T^2}{12} M \sum_{j=1}^p \sum_{i=1}^N \|a_{ij}\|_\infty \int_0^T |\dot{u}(t)|^2 dt + \tilde{C}_2 |\bar{u}|^{2\alpha} \\
&\quad + C_3 \left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} + C_4 \left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}},
\end{aligned}
$$

Combining $(v), (vi), (3.7), (3.8)$ with (3.9), one has

$$
\begin{aligned}
\varphi(u_n) &= \frac{1}{2}\int_0^T e^{G(t)} |\dot{u}_n(t)|^2 dt + \frac{1}{2}\int_0^T e^{G(t)} (\lambda A(t)u_n(t), u_n(t)) dt \\
&\quad - \int_0^T e^{G(t)} (F(t, u_n(t)) - F(t, \bar{u}_n)) dt - \int_0^T e^{G(t)} F(t, \bar{u}_n) dt \\
&\quad + \sum_{j=1}^p \sum_{i=1}^N e^{G(t_j)} \int_0^{u_n^i(t_j)} I_{ij}(t) dt \qquad (3.10)\\
&\le |\bar{u}_n|^2 (C_9 - |\bar{u}_n|^{-2} \int_0^T e^{G(t)} F(t, \bar{u}_n) dt) + C_{10}
\end{aligned}
$$

for large n and some positive constants C_9 and C_{10}. We claim that $\{\bar{u}_n\}$ is bounded. Otherwise, suppose $|\bar{u}_n| \to +\infty (n \to \infty)$. It follows from (vi) and (3.10) that

$$\varphi(u_n) \to -\infty \ (n \to \infty),$$

which contradicts the boundedness of $\varphi(u_n)$.

Hence, it follows from (3.6), (3.7), and the boundedness of $\{\bar{u}_n\}$ that $\{u_n\}$ is bounded in H, then there exists a subsequence denoted by $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to some u in H, then the sequence $\{u_{n_k}\}$ converges uniformly to u in $C[0, T]$. Hence

$$
\begin{gathered}
(\varphi'(u_{n_k}) - \varphi'(u))(u_{n_k} - u) \to 0, \\
\int_0^T e^{G(t)} A(t)(\nabla F(t, u_{n_k}(t)) - \nabla F(t, u(t)))(u_{n_k} - u(t)) dt \to 0, \\
[I_{ij}(u_{n_k}^i(t_i)) - I_{ij}(u^i(t_i))](u_{n_k}^i(t_j) - u^i(t_j)) \to 0,
\end{gathered}
$$

as $n \to +\infty$. Thus, we have

$$
\begin{aligned}
\langle \varphi'(u_{n_k}) - \varphi'(u), u_{n_k} - u\rangle &= \int_0^T e^{G(t)} |\dot{u}_{n_k}(t) - \dot{u}(t)|^2 dt + \lambda \int_0^T e^{G(t)} A(t) |u_{n_k}(t) - u(t)|^2 dt \\
&\quad + \int_0^T e^{G(t)} A(t)(\nabla F(t, u_{n_k}(t)) - \nabla F(t, u(t)))(u_{n_k}(t) - \dot{u}(t)) dt \\
&\quad + [I_{ij}(u_{n_k}^i(t_i)) - I_{ij}(u^i(t_i))](u_{n_k}^i(t_j) - u^i(t_j)),
\end{aligned}
$$

which follows that $u_{n_k} \to u$ in H. Thus, φ satifies the Palais-Smale condition.

Let $\tilde{H} = \{u \in H | \bar{u} = 0\}$. Then $H = \tilde{H} \bigoplus \mathbb{R}^N$.

To use the saddle point theorem (Theorem 4.6 in [6]), we only need to verify

(H_1) $\varphi(u) \to -\infty$ as $|u| \to \infty$ in $\mathbb{R}^N$;

(H_2) $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{H}$.

Firstly, we show that (H_1) holds. By (vi), one has $\int_0^T e^{G(t)} F(t,u)dt \to +\infty$ as $|u| \to \infty$ in $\mathbb{R}^N$. When $\lambda < 2$, by (vii), one has

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2 dt + \frac{1}{2}\int_0^T e^{G(t)}(\lambda A(t)u(t), u(t))dt \\ &\quad - \int_0^T e^{G(t)} F(t,u)dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)} \int_0^{u^i(t_j)} I_{ij}(t)dt \\ &\le (\frac{\lambda}{2} - 1)\int_0^T e^{G(t)}(F(t,u))dt \to -\infty,\end{aligned}$$

as $|u| \to \infty$ in $\mathbb{R}^N$.

Secondly, we verify (H_2) holds. For all $u \in \tilde{H}$, then $\bar{u} = 0$, by (i) and Sobolev inequality, for all n, one has

$$\begin{aligned}|\int_0^T (F(t,u(t)) - F(t,0)dt| &= |\int_0^T \int_0^1 (\nabla F(t, su(t)), u(t))dsdt| \\ &\le \int_0^T f(t)|u(t)|^{\alpha+1}dt + \int_0^T e(t)|u(t)|dt \\ &\le \int_0^T f(t)\|u\|_\infty^{\alpha+1}dt + \int_0^T e(t)\|u\|_\infty dt \qquad (3.11) \\ &\le \int_0^T f(t)dt(\frac{T}{12})^{\frac{\alpha+1}{2}}\left(\int_0^T |\dot{u}|^2 dt\right)^{\frac{\alpha+1}{2}} + \int_0^T e(t)dt(\frac{T}{12})^{\frac{1}{2}}\left(\int_0^T |\dot{u}|^2 dt\right)^{\frac{1}{2}}.\end{aligned}$$

By (iv), one has

$$\begin{aligned}|\sum_{j=1}^{p}\sum_{i=1}^{N} e^{G(t_j)}\int_0^{u^i(t_j)} I_{ij}(t)dt| &\le M\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u^i(t_j)}(b_{ij}|t|^{\alpha\beta_{ij}} + c_{ij}) \\ &\le cpNM\|u\|_\infty + bM\sum_{j=1}^{p}\sum_{i=1}^{N}\|u\|_\infty^{\alpha\beta_{ij}+1} \qquad (3.12) \\ &\le cpNM\sqrt{\frac{T}{12}}\left(\int_0^T |\dot{u}|^2 dt\right)^{\frac{1}{2}} \\ &\quad + (\frac{T}{12})^{\frac{\alpha\beta_{ij}+1}{2}} bM\sum_{j=1}^{p}\sum_{i=1}^{N}\left(\int_0^T |\dot{u}|^2 dt\right)^{\frac{\alpha\beta_{ij}+1}{2}}.\end{aligned}$$

It follows from $(3.11), (3.12)$ and $\lambda < \frac{24m}{5T^2 M \sum_{j=1}^{p}\sum_{i=1}^{N}\|a_{ij}\|_\infty}$ that

$$\varphi(u) = \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2 dt + \frac{1}{2}\int_0^T e^{G(t)}(\lambda A(t)u_n(t), u(t))dt - \int_0^T e^{G(t)}(F(t,u) - F(t,0))dt$$

$$
\begin{aligned}
&-\int_0^T e^{G(t)}F(t,0)dt+\sum_{j=1}^{p}\sum_{i=1}^{N}e^{G(t_j)}\int_0^{u^i(t_j)}I_{ij}(t)dt\\
&\geq \frac{1}{2}\int_0^T e^{G(t)}|\dot{u}(t)|^2dt-\frac{\lambda}{2}\frac{T}{12}TM\sum_{i,j=1}^{N}\|a_{ij}\|_\infty\int_0^T|\dot{u}(t)|^2dt \qquad (3.13)\\
&-(\frac{T}{12})^{\frac{\alpha+1}{2}}\int_0^T f(t)dt\left(\int_0^T|\dot{u}|^2dt\right)^{\frac{\alpha+1}{2}}-\sqrt{\frac{T}{12}}\int_0^T e(t)dt\left(\int_0^T|\dot{u}|^2dt\right)^{\frac{1}{2}}\\
&-cpNM\sqrt{\frac{T}{12}}\left(\int_0^T|\dot{u}|^2dt\right)^{\frac{1}{2}}-(\frac{T}{12})^{\frac{\alpha\beta_{ij}+1}{2}}bM\sum_{j=1}^{p}\sum_{i=1}^{N}\left(\int_0^T|\dot{u}|^2dt\right)^{\frac{\alpha\beta_{ij}+1}{2}}-\int_0^T e^{G(t)}F(t,0)dt.
\end{aligned}
$$

If $u\in\tilde{H}$, then $\bar{u}=0$, by Wirtinger's inequality, one has

$$
\begin{aligned}
\|u\|_0&=\left(\int_0^T|\dot{u}(t)|^2dt+\int_0^T|u(t)|^2dt\right)^{\frac{1}{2}}\\
&\leq\left(\left(1+\frac{T^2}{4\pi^2}\right)\int_0^T|\dot{u}(t)|^2dt\right)^{\frac{1}{2}},
\end{aligned}
$$

furthermore, $\|u\|_0$ is equivalent to $\|u\|$, thus $\|u\|\to\infty$ if and only if $\int_0^T|\dot{u}(t)|^2dt\to\infty$, then (3.13) implies that $\varphi(u)\to+\infty$ as $\|u\|\to\infty$, i. e. (H_2) is verified. □

4. Examples

Example 4.1. Take $T=1,\ N=5, A(t)$ is a fifth-order unit matrix, $t_1=\frac{1}{2}\in(0,1),\ g(t)=t,\ a(t)\in L(0,T;\mathbb{R}^5),\ F(t,u)=(-t)|u|^{\frac{3}{2}}+a(t),\ I_{ij}(t)=t^{\frac{1}{3}}$ and $\alpha=\frac{1}{2}$.
Consider the second-order Hamiltonian systems with impulsive effects

$$
\begin{cases}
\ddot{u}(t)+g(t)\dot{u}(t)-\lambda A(t)u(t)=-\nabla F(t,u), & a.e.\ t\in[0,T],\\
\triangle\dot{u}^i(t_j)=I_{ij}(u^i(t_j)), & i=1,2,...,N;\ j=1,2,...,p,\\
u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0,
\end{cases}
$$

when $\lambda>0$, the above Hamiltonian systems has at least one solution according to Theorem 3.1.

Example 4.2. Take $T=0.1,\ N=5, A(t)$ is a fifth-order unit matrix, $t_1=\frac{1}{20}\in(0,1),\ g(t)=t,\ a(t)\in L(0,T;\mathbb{R}^5),\ I_{ij}(t)=-t^{\frac{1}{3}},\alpha=\frac{1}{2}$ and

$$
F(t,u)=\begin{cases}|u|^2+|a(t)|, & |u|>1,\\ 1, & |u|\leq 1.\end{cases}
$$

Consider the second-order Hamiltonian systems with impulsive effects

$$
\begin{cases}
\ddot{u}(t)+g(t)\dot{u}(t)-\lambda A(t)u(t)=-\nabla F(t,u), & a.e.\ t\in[0,T],\\
\triangle\dot{u}^i(t_j)=I_{ij}(u^i(t_j)), & i=1,2,...,N;\ j=1,2,...,p,\\
u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0,
\end{cases}
$$

when $\lambda<2$, the above Hamiltonian systems has at least one solution according to Theorem 3.2.

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