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A novel approach to Banach contraction principle in extended quasi-metric spaces

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Abstract

The purpose of this note is to give a natural approach to the extensions of the Banach contraction principle in metric spaces endowed with a partial order, a directed graph or a binary relation in terms of extended quasi-metric. This novel approach is new and may open the door to other new fixed point theorems. The case of multivalued mappings is also discussed and an analogue result to Nadler's fixed point theorem in extended quasi-metric spaces is given.

Keywords: Banach contraction principle, contraction mapping, extended quasi-metric space, fixed point, monotone mapping, multivalued mapping.
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1. Introduction

Since the very beginning, the Banach contraction principle [3] became a basic tool of Analysis; so that, it was the subject of many extensions. Among these, the case of ambient metric space being endowed with an ordering relation was a very promising one. Some early attempts to extend the Banach contraction principle in this way were performed in the 1986 papers by Turinici [18, 19]. Two decades later, these results have been re-obtained and refined by Ran and Reurings [15], who also gave some interesting applications of them to matrix equations theory. Shortly after, Nieto and Rodriguez-Lopez [14] improved the main result of [15] and used their new extension to solve a lot of practical questions belonging to differential equations theory. Further extensions of these results have also been given by

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(i) Jachymski [10], who proposed a natural extension of problem setting from partial orders to directed graphs,
(ii) Ben-El-Mechaiekh [5], who gave a formulation of Ran-Reurings fixed point theorem over metric spaces endowed with a binary relation,
(iii) Turinici [20], who extended the metrical convergence of the ambient space to (abstract) sequential convergence structures.

In this paper, we propose a novel approach to Banach contraction principle by considering its extension to the so-called extended quasi-metric spaces which allows the quasi-distance to take infinite values [6].

In terms of content, this paper overlaps in places with the following popular books on fixed point theory by Goebel and Kirk [9], by Khamsi and Kirk [11].

2. Basic results

First, we define the concept of an extended quasi-metric space.

Definition 2.1. [6] Let X be an abstract set. The function $\bar{d} : X \times X \to [0, \infty]$ is called an extended quasi-distance if the following conditions are satisfied:

(i) $\bar{d}(x, y) = 0 \Leftrightarrow x = y$;
(ii) $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$, for all $x, y, z \in X$ (oriented triangle inequality).

In this case, the pair $(X, \bar{d})$ is called an extended quasi-metric space.

The reader may find an extensive list of examples on quasi-metric spaces in [6]. One of the problems dealing with quasi-metric spaces is the issue of a topology and more specifically the concepts of convergence, Cauchy and completeness [7, 16, 17].

Definition 2.2. Let $(X, \bar{d})$ be an extended quasi-metric space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to+\infty} \bar{d}(x_n, x) = 0$. X is said to be separated if and only if the limit of a sequence is unique,i.e., whenever $\lim_{n\to+\infty} \bar{d}(x_n, x) = \lim_{n\to+\infty} \bar{d}(x_n, y) = 0$ implies $x = y$. We will say that a subset Y of X is closed in $(X, \bar{d})$ if Y contains the limit of any convergent sequence from Y.

Throughout this paper, we will assume that any extended quasi-metric space $(X, \bar{d})$ is separated. The concept of Cauchy sequences is little more complicated. We refer the interested reader to the paper [7] for more details.

Definition 2.3 ([7])**.** Let $(X, \bar{d})$ be an extended quasi-metric space. A sequence $\{x_n\}$ in X is said to be Cauchy if and only if there exists a sequence $\{y_m\}$ such that for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $n, m > N$ we have

$$\bar{d}(x_n, y_m) < \varepsilon,$$

i.e., $\lim_{n,m\to+\infty} \bar{d}(x_n, y_m) = 0$. The sequence $\{y_m\}$ is called a cosequence to $\{x_n\}$. The extended quasi-metric space $(X, \bar{d})$ is said to be complete if and only if any Cauchy sequence in $(X, \bar{d})$ is convergent.

Although this definition is kind of complicated, it allows the following to be true:

Proposition 2.4 ([7])**.**

(i) Every convergent sequence in $(X, \bar{d})$ is a Cauchy sequence.

(ii) Every subsequence of a Cauchy sequence is a Cauchy sequence.

(iii) If $(X, \bar{d})$ is an extended metric space, then Definition 2.3 is equivalent to the usual definition of Cauchy sequence.

According to [7], the Sorgenfrey line is a complete quasi-metric space, the so called Kofner plane and Pixley-Roy space, considered as quasi-metric spaces, are also complete.

3. Banach contraction principle in extended quasi-metric spaces

In order to discuss the Banach contraction principle in extended quasi-metric spaces, we will need to introduce the concept of Lipschitzian mappings in these spaces.

Definition 3.1. Let $(X, \bar{d})$ be an extended quasi-metric space. A mapping $T : X \to X$ is said to be Lipschitzian if there exists $k > 0$ such that

$$\bar{d}(T(x), T(y)) \leq k\ \bar{d}(x, y),$$

for all $x, y \in X$. If $k < 1$, then T is said to be a contraction mapping. A point $x \in X$ is called a fixed point of T whenever $T(x) = x$.

Remark 3.2. Let $(X, \bar{d})$ be an extended quasi-metric space and $T : X \to X$ a $\bar{d}$-contraction mapping. Then for any fixed points x and y of T, we have $x = y$ whenever $\bar{d}(x, y) < \infty$.

The following technical lemma will be useful when studying the Banach contraction principle in extended metric spaces.

Theorem 3.3. *Let $(X, \bar{d})$ be a complete extended quasi-metric space. Let $T : X \to X$ be a contraction. Set $X_T = \{x \in X;\ \bar{d}(x, T(x)) < \infty\}$. For any $x_0 \in X_T$, the orbit $\{T^n(x_0)\}$ is Cauchy. Moreover if $\{T^n(x_0)\}$ converges to $x \in X$, then $T(x) = x$, i.e., x is a fixed point of T.*

Proof. Since T is a contraction mapping, there exists $k \in (0, 1)$ such that

$$\bar{d}(T(x), T(y)) \leq k\ \bar{d}(x, y),\ \ for\ any\ x, y \in X.$$

Hence for any $x, y \in X$ and $n \in \mathbb{N}$, we have

$$\bar{d}(T^n(x), T^n(y)) \leq k^n\ \bar{d}(x, y).$$

Let $x_0 \in X_T$, i.e., $\bar{d}(x_0, T(x_0)) < \infty$. Then

$$\bar{d}(T^n(x_0), T^{n+1}(x_0)) \leq k^n\ \bar{d}(x_0, T(x_0)),$$

for any $n \in \mathbb{N}$. Hence for any $n, h \in \mathbb{N}$, we have

$$\bar{d}(T^n(x_0), T^{n+h+1}(x_0)) \leq k^n\ \bar{d}(x_0, T^{h+1}(x_0)) \leq k^n \sum_{i=0}^{i=h-1} \bar{d}(T^i(x_0), T^{i+1}(x_0)),$$

which implies

$$\bar{d}(T^n(x_0), T^{n+h+1}(x_0)) \leq \frac{k^n}{1-k}\ \bar{d}(x_0, T(x_0)).$$

Hence $\{T^n(x_0)\}$ is Cauchy, since $k < 1$ and $\bar{d}(x_0, T(x_0)) < \infty$, with $\{T^m(x_0)\}$ as its cosequence. Since $(X, \bar{d})$ is a complete extended quasi-metric space, $\{T^n(x_0)\}$ converges to $x \in X$. We have

$$\bar{d}(T^{n+1}(x_0), T(x)) \leq k\ \bar{d}(T^n(x_0), x),\ \ \ for\ any\ n \in \mathbb{N}.$$

Hence $\{T^{n+1}(x_0)\}$ converges to $T(x)$ and x. Since $(X, \bar{d})$ is separated, we conclude that $T(x) = x$. □

In the next example, we discuss how a partially ordered metric space may be endowed with an extended quasi-metric structure. Though this example is given for a metric space endowed with a partial order but the same ideas may be used for metric spaces endowed with a directed graph or a binary relation. In fact, this example will shed some light on a better understanding of the fixed point theorems of Ran and Reurings [15], Nieto and Rodríguez-López [14], Jachymski [10], and Ben-El-Mechaiekh [5] in connection with Theorem 3.3.

Example 3.4. Let $(X, d, \preceq)$ be a partially ordered metric space. Define $\bar{d} : X \times X \to [0, +\infty]$ by

$$\bar{d}(x, y) = \begin{cases} d(x, y), & \text{if } x \preceq y; \\ +\infty, & \textit{otherwise}. \end{cases}$$

It is easy to check that $(X, \bar{d})$ is an extended quasi-metric space. Next, we discuss the convergence of sequences in $(X, \bar{d})$ and how it connects to (X, d). First, it is clear that if $\{x_n\}$ converges to x in $(X, \bar{d})$, then it also converges to x in (X, d). This will imply that $(X, \bar{d})$ is separated. Next, we investigate Cauchy sequences in $(X, \bar{d})$. Let $\{x_n\}$ be a Cauchy sequence in $(X, \bar{d})$. Then there exists a cosequence $\{y_m\}$ in X such that $\lim_{n,m\to+\infty} \bar{d}(x_n, y_m) = 0$. There exists $N_0 \geq 1$ such that for any $n, m \geq N_0$ we have $\bar{d}(x_n, y_m) < +\infty$. In particular, we have $x_n \preceq y_m$, for any $n, m \geq N_0$. Moreover, the condition $\lim_{n,m\to+\infty} \bar{d}(x_n, y_m) = 0$ implies that both sequences $\{x_n\}$ and $\{y_m\}$ are Cauchy in (X, d). Therefore, if (X, d) is complete, then $\{x_n\}$ and $\{y_m\}$ are convergent to the same limit $x \in X$. In order to show that $\{x_n\}$ converges to x in $(X, \bar{d})$, we will need to assume that order intervals are closed. Recall that order intervals are any subset

$$(\leftarrow, x] = \{y \in X; y \preceq x\} \textit{ or } [x, \rightarrow) = \{y \in X; x \preceq y\}$$

for any $x \in X$. In this case, we will have

$$x_n \preceq x$$

for any $n \geq N_0$. Hence we have $\bar{d}(x_n, x) = d(x_n, x)$, for any $n \geq N_0$. Clearly, we will have $\lim_{n\to+\infty} \bar{d}(x_n, x) = 0$, i.e., $\{x_n\}$ converges to x in $(X, \bar{d})$. Therefore, if (X, d) is complete and the order intervals are closed, then $(X, \bar{d})$ is a complete quasi-metric space. Next, we investigate Lipschitzian mappings in $(X, \bar{d})$. Let $T : X \to X$ be a Lipschitzian mapping. Then there exists $k > 0$ such that

$$\bar{d}(T(x), T(y)) \leq k\ \bar{d}(x, y),$$

for any $x, y \in X$. Fix $x, y \in X$ such that $\bar{d}(x, y) < +\infty$. Then $\bar{d}(T(x), T(y)) < +\infty$ holds. In other words, if $x \preceq y$, then $T(x) \preceq T(y)$. This is the definition of a monotone increasing mapping. Moreover, we have

$$\bar{d}(T(x), T(y)) = d(T(x), T(y)) \leq k\ d(x, y) = k\ \bar{d}(x, y).$$

Therefore if T is a Lipschitizan mapping in $(X, \bar{d})$, then T is a Lipchitzian monotone mapping in $(X, \preceq, d)$ with the same Lipchitz constant. The converse is also true. In particular, T is a monotone contraction mapping in $(X, \preceq, d)$ if and only if T is a contraction mapping in $(X, \bar{d})$. Putting all these results together, we get an analogue result to Theorem 3.3 in partially ordered metric spaces.

Theorem 3.5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space, such that*

(*) *the order intervals are closed.*

Furthermore, let $T : X \to X$ be a contraction mapping with $X_T = \{x \in X;\ x \preceq T(x)\}$ is nonempty. Then T has a fixed point in X.

Remark 3.6. Note that the conclusion of Theorem 3.5 will still hold if we assume the following weaker property:

(**) if $\{x_n\}$ is monotone nondecreasing (resp. nonincreasing) and converges to x, then we have $x_n \preceq x$ (resp. $x \preceq x_n$), for any $n \in \mathbb{N}$.

This property was introduced by Nieto and Rodriguez-Lopez [14]. Indeed, in the proof of Theorem 3.3, the fixed point was the limit of a sequence which is monotone. Therefore, the property $(**)$ is enough to prove the conclusion of Theorem 3.5.

In the next section, we discuss Nadler's fixed point theorem for multivalued contractions in extended quasi-metric spaces.

4. Nadler's fixed point theorem in extended metric spaces

The multivalued version of the Banach contraction principle was given by Nadler [13].

Theorem 4.1 ([13])**.** *Let (M, d) be a complete metric space. Denote by $\mathcal{CB}(\mathcal{M})$ the set of all nonempty closed bounded subsets of M. Let $T : M \to \mathcal{CB}(\mathcal{M})$ be a multivalued contraction mapping, i.e., there exists $k \in [0, 1)$ such that*

$$H(T(x), T(y)) \leq k\ d(x, y)$$

for all $x, y \in M$, where H is the Pompeiu-Hausdorff metric on $CB(M)$. Then T has a fixed point in M.

Following the publication of Nadler's fixed point theorem, many mathematicians gave a number of its extensions and generalizations; see for instance [8, 12] and references cited therein. In this section, we discuss an extension of Theorem 4.1 to extended metric spaces. In light of the Example 3.4, our extension gives a simpler and novel approach to the recent interest around monotone multivalued contraction mappings [1, 4].

Definition 4.2. Let $(X, \bar{d})$ be an extended quasi-metric space and $\mathcal{C}(\mathcal{X})$ be the class of all nonempty closed subsets of $(X, \bar{d})$. A multivalued map $T : X \to \mathcal{C}(\mathcal{X})$ is called a contraction mapping if there exists $k \in (0, 1)$ such that for any $x, y \in X$ and any $u \in T(x)$, there exists $v \in T(y)$ such that

$$\bar{d}(u, v) \leq k\ \bar{d}(x, y).$$

A point $x \in X$ is called a fixed point of T if $x \in T(x)$.

The following property will be needed to prove the existence of fixed points for multivalued contractive mappings in quasi-metric spaces.

Definition 4.3. Let $(X, \bar{d})$ be an extended quasi-metric space. We will say that X satisfies the property (P_1) if for any $\{x_n\}$ in X which converges to x, and $\{y_n\} \in X$ such that $\lim_{n\to\infty} \bar{d}(x_n, y_n) = 0$, we have $\lim_{n\to\infty} \bar{d}(y_n, x) = 0$.

Now we are ready to give the multivalued version of Theorem 3.3.

Theorem 4.4. *Let $(X, \bar{d})$ be a complete extended quasi-metric space. Assume that X satisfies the property (P_1). Let $T : X \to \mathcal{C}(X)$ be a contraction mapping. Assume that $X_T := \{x \in X;\ \bar{d}(x, u) < \infty$ for some $u \in T(x)\} \neq \emptyset$. Then T has a fixed point.*

Proof. Let $x_0 \in X_T$. Then there exists $x_1 \in T(x_0)$ such that $\bar{d}(x_0, x_1) < \infty$. Since T is a contraction mapping, there exists $k \in (0, 1)$ such that for any $x, y \in X$ and any $u \in T(x)$, there exists $v \in T(y)$ such that

$$\bar{d}(u, v) \leq k\ \bar{d}(x, y).$$

Hence there exists $x_2 \in T(x_1)$ such that

$$\bar{d}(x_1, x_2) \leq k\ \bar{d}(x_0, x_1).$$

By induction, we construct a sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$ and

$$\bar{d}(x_n, x_{n+1}) \leq k\ \bar{d}(x_n, x_{n-1}) \leq k^n\ \bar{d}(x_0, x_1)$$

for any $n \in \mathbb{N}$. Hence for any $n, h \in \mathbb{N}$, we have

$$\bar{d}(x_n, x_{n+h}) \leq \frac{k^n}{1-k} \bar{d}(x_0, x_1).$$

Hence $\{x_n\}$ is Cauchy, since $k < 1$ and $\bar{d}(x_0, x_1) < \infty$, with $\{x_m\}$ as its cosequence. Since $(X, \bar{d})$ is a complete extended quasi-metric space, $\{x_n\}$ converges to $x \in X$. Since T is a contraction, there exists $y_n \in T(x)$ such that

$$\bar{d}(x_{n+1}, y_n) \leq k\ \bar{d}(x_n, x)$$

for any $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \bar{d}(x_{n+1}, y_n) = 0$ holds. Since X satisfies the property (P_1), and $\{x_{n+1}\}$ converges to x, we conclude that $\lim_{n\to\infty} \bar{d}(y_n, x) = 0$. Since $T(x)$ is closed, we conclude that $x \in T(x)$, i.e., x is a fixed point of T as claimed. □

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