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Stabilization control of generalized type neural networks with piecewise constant argument

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Abstract

The generalized type neural networks have always been a hotspot of research in recent years. This paper concerns the stabilization control of generalized type neural networks with piecewise constant argument. Through three types of stabilization control rules (single state stabilization control rule, multiple state stabilization control rule and output stabilization control rule), together with the estimate of the state vector with piecewise constant argument, several succinct criteria of stabilization are derived. The obtained results improve and extend some existing results. Two numerical examples are proposed to substantiate the effectiveness of the theoretical results.

Keywords: Generalized type systems, neural networks, state stabilization, output stabilization.
2010 MSC: 34H15, 93D15.

1. Introduction

Due to its potential applications in various fields, generalized type system with piecewise constant argument has been widely investigated in recent years ([1–7]). Different from that the traditional system can only be delayed or advanced, generalized type system can change its type of deviation of the parameters during the motion, and hence can be both delayed and advanced. Because of this property, such system considered as a recurrent neural network model for the first time appeared in [5], in which Akhmet and his team explored fully on the recurrent neural network of this type based on the method of Lyapunov functions.

In recent years, a lot of novel results on the dynamic behaviors of a variety of types of neural networks are reported ([8, 11, 14, 17, 28, 32]). As a special case of the generalized type system, more research has been

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carried out on the generalized type neural network with piecewise constant argument. Several interesting results on stability analysis of this type of neural networks are presented ([5–7, 9]). In [5], the stability of generalized type recurrent neural networks with piecewise constant argument was considered. In [6], sufficient conditions on uniform asymptotic stability and global exponential stability of the cellular neural networks with piecewise constant argument were obtained. It was addressed the stability of the impulsive Hopfield-type neural networks with piecewise constant argument in [7]. The robust stability of the generalized type interval fuzzy Cohen-Grossberg neural networks with piecewise constant argument was discussed in [9] and several robust stability criteria were derived based on the comparison principle in the paper.

It is worth noting that all of the literature on the stability analysis mentioned above require the neural networks to be stable, that is, stability of the neural networks is the prerequisite for the applications in practice ([9, 10, 12, 16, 18, 19, 21, 27, 30, 31, 33–36]). Nevertheless, many neural networks are not stable and in order to facilitate the utilization of neural networks in a wider scope of application, it is necessary to consider the stabilization control method to stabilize the unstable neural networks. Recently, stabilization control has been paid more attention due to its important impacts on the dynamic behavior of neural networks. As we know, about the stabilization control of neural networks, there are many existing references ([13, 15, 20, 22–26, 29, 37]). In [15], the output feedback stabilization was explored on the type of delayed nonlinear interconnected systems. In [25, 29], the state vector was chosen as the component of the controller to stabilize the neural network. Whereas, the models of the neural networks concerned above are either about constant delays, or about time-varying delays, or about distributed delays. That is, the deviation of the parameters are always lagging behind. In order to promote deeper understanding of neural networks, it is essential to consider more general types of deviations, i.e., generalized type neural networks with piecewise constant argument, in which the parameter can change its deviation type (delayed or advanced) during the motion. To the best of the authors' knowledge, although there are some excellent results on this type of neural networks, stabilization topic on this type of neural networks has not been investigated. Stabilization control is desirable as it can guarantee the dynamical behavior of the designed neural networks to some degree.

Motivated by the above discussion, in this paper, our aim is to investigate the stabilization control of the generalized type neural networks with piecewise constant argument. By estimating the state with argument, and meanwhile, based on the Lyapunov functions, three stabilization control rules, i.e., single state stabilization control rule, multiple state stabilization control rule, and output stabilization control rule, are proposed, and lots of stabilization results are obtained. The criteria acquired in this paper improve and extend some existing ones.

2. Preliminaries and model description

Throughout this paper, we denote N as the set of natural numbers, R^n stands for the n-dimensional Euclidean space, and $C([t_0-\zeta,t_0];R^n)$ represents the set of continuous function φ from $[t_0-\zeta,t_0]$ to R^n. For $x\in R^n$, its norm is defined as $\|x\|=\sum_{i=1}^n |x_i|$. Choose two real valued sequences $\theta_i,\eta_i, i\in N$, satisfying $\theta_i<\theta_{i+1}, \theta_i\le\eta_i\le\theta_{i+1}$, for all $i\in N$, and $\theta_i\to+\infty, \eta_i\to+\infty$ as $i\to+\infty$. Then we consider the generalized type neural networks with piecewise constant argument described by the following equations:

$$\dot{x}_i(t) = -\,a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(\gamma(t))) + I_i(t) \tag{2.1}$$

for $i=1,2,...,n$, $t>t_0$, where $x_i(t)$ denotes the state variable of the ith unit at time t, $a_i>0$ stands for the self-inhibition, b_{ij}, c_{ij} indicate the strength of the jth unit on the ith unit at time t and $\gamma(t)$, respectively; $f_j(\cdot), g_j(\cdot)$ signify the activation functions, and $\gamma(t)$ is a piecewise constant argument, satisfying $\gamma(t)=\eta_k$, $\theta_k\le\eta_k\le\theta_{k+1}$, if $\theta_k\le t<\theta_{k+1}$, $I_i(t)$ is the external input.

It is easy to see that neural network (2.1) is of mixed type. The argument is deviated when it is advanced or delayed. In fact, fix $k\in N$ and consider the neural network in the interval $[\theta_k,\theta_{k+1})$, the identification function $\gamma(t)$ is equal to η_k. If it is satisfied with $t \in [\theta_k, \eta_k)$, then $\gamma(t) > t$ and (2.1) is a system with advanced argument. Similarly, if it is satisfied with $t \in [\eta_k, \theta_{k+1})$, then $\gamma(t) < t$ and (2.1) is a system with delayed argument. Hence, neural network (2.1) varies the type of deviation of the argument as time t elapses.

Throughout the paper, the following hypotheses are needed.

(H1) For the activation functions $f_i(\cdot)$, $g_i(\cdot) \in C(R, R)$ satisfying $f_i(0) = 0$, $g_i(0) = 0$, there exist Lipschitz constants $L_i^1, L_i^2 > 0$ such that

$$|f_i(\omega) - f_i(\varrho)| \leq L_i^1 |\omega - \varrho|,$$
$$|g_i(\omega) - g_i(\varrho)| \leq L_i^2 |\omega - \varrho|,$$

for any $\omega, \varrho \in R$, $i = 1, 2, ..., n$.

(H2) For any $i \in N$, there exists a positive constant θ that satisfies

$$\theta_{i+1} - \theta_i \leq \theta, i = 1, 2, ..., n.$$

Remark 2.1. (H1) concerns with the property of the feedback functions, (H2) limits the maximum interval length of the variable sequence.

In what follows we introduce some definitions which are needed later.

Definition 2.2 ([5])**.** A solution of neural network (2.1) is a continuous function such that:

(i) the derivative $\dot{x}(t)$ exists at each point $t \in [0, +\infty)$ with the possible exception of the points $\theta_k, k \in N$, where a one-sided derivative exists;

(ii) neural network (2.1) is satisfied by $x(t)$ at each interval $(\theta_k, \theta_{k+1}), k \in N$.

Definition 2.3 ([29])**.** The zero solution of neural network (2.1), where $I_i(t) = 0, i = 1, 2, ..., n$, is called exponentially stable if there exist constants $\varrho > 0$, and $\beta > 0$ such that

$$\|x(t, t_0, x_0)\| \leq \varrho \|x_0\| e^{-\beta(t - t_0)}, t \geq t_0. \tag{2.2}$$

Definition 2.4 ([29])**.** Neural network (2.1) is called exponentially stablilizable if there exists an appropriate control rule such that the zero solution of the derived closed-loop system (2.1) satisfies (2.2).

3. Main results

In this section, we design three kinds of stabilization rules, i.e., single state stabilization control rule, multiple state stabilization control rule, and output stabilization control rule, and then derive the corresponding stabilization criteria to ensure system (2.1) to be globally exponentially stabilizable.

3.1. Single state stabilization

Suppose that the state variables of (2.1) are measurable and consider the single state stabilization control rule defined by:

$$\mathbf{I(t)} = \begin{pmatrix} k_1 x_1(t) \\ k_2 x_2(t) \\ \vdots \\ k_i x_i(t) \\ \vdots \\ k_n x_n(t) \end{pmatrix} = K x(t), \tag{3.1}$$

where $K = diag(k_1, k_2, \cdots, k_n)$ is the control gain.

From single state stabilization control rule (3.1), system (2.1) can be rewritten as follows:

$$\dot{x}_i(t) = -(a_i - k_i)x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(\gamma(t))). \tag{3.2}$$

Before giving the main result of the state stabilization control rule, we present a useful lemma.

In this subsection, we need the following assumption.

(H3) There exist positive constants θ, μ, and ν such that

$$1 - \theta\big[\nu + \mu(1+\nu\theta)e^{\mu\theta}\big] > 0,$$

where $\mu = \max_{1\le i\le n}\big(|a_i - k_i| + L_i^1 \sum_{j=1}^n |b_{ji}|\big)$, $\nu = \max_{1\le i\le n}\big(\sum_{j=1}^n L_i^2 |c_{ji}|\big)$.

Lemma 3.1. *Under (H1), (H2), (H3), for* (3.2), *the following inequality holds*

$$\|x(\gamma(t))\| \le \alpha \|x(t)\|,$$

for any $t \ge t_0$, where

$$\alpha = 1\Big/\Big(1 - \theta\big[\nu + \mu(1+\nu\theta)e^{\mu\theta}\big]\Big), \mu = \max_{1\le i\le n}\big(|a_i - k_i| + L_i^1 \sum_{j=1}^n |b_{ji}|\big), \nu = \max_{1\le i\le n}\big(\sum_{j=1}^n L_i^2 |c_{ji}|\big).$$

Proof. For any $t \ge t_0$, by the property of $\gamma(t)$ and the sequences $\{\theta_k\}$ and $\{\eta_k\}$, there exists only one $k \in N$, which satisfies that

$$\gamma(t) = \eta_k \in [\theta_k, \theta_{k+1}), t \in [\theta_k, \theta_{k+1}),$$

and we get if $t \ge \eta_k$

$$x_i(t) = x_i(\eta_k) + \int_{\eta_k}^{t} \Big[-(a_i - k_i)x_i(s) + \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(\eta_k))\Big] ds, \tag{3.3}$$

for $i = 1, 2, ..., n$, then

$$\begin{aligned}
\|x(t)\| \le& \|x(\eta_k)\| + \sum_{i=1}^{n} \int_{\eta_k}^{t} \Big[|a_i - k_i||x_i(s)| + \sum_{j=1}^{n} L_j^1 |b_{ij}||x_j(s)| + \sum_{j=1}^{n} L_j^2 |c_{ij}||x_j(\eta_k)| \Big] ds \\
=& \|x(\eta_k)\| + \int_{\eta_k}^{t} \Big[\sum_{i=1}^{n} \Big(|a_i - k_i| + L_i^1 \sum_{j=1}^{n} |b_{ji}| \Big)|x_i(s)| + \sum_{i=1}^{n} \Big(L_i^2 \sum_{j=1}^{n} |c_{ji}| \Big)|x_i(\eta_k)| \Big] ds \\
\le& \|x(\eta_k)\| + \int_{\eta_k}^{t} \Big(\mu \sum_{i=1}^{n} |x_i(s)| + \nu \sum_{i=1}^{n} |x_i(\eta_k)| \Big) ds \\
\le& (1+\nu\theta)\|x(\eta_k)\| + \mu \int_{\eta_k}^{t} \|x(s)\| ds.
\end{aligned}$$

Based on the Gronwall-Bellman inequality, we obtain

$$\|x(t)\| \le (1+\nu\theta)\|x(\eta_k)\| e^{\mu\theta}. \tag{3.4}$$

Exchanging the location of $x_i(t)$ and $x_i(\eta_k)$ in (3.3), it derives that

$$\begin{aligned}
\|x(\eta_k)\| \le& \|x(t)\| + \int_{\eta_k}^{t} \Big[\sum_{i=1}^{n} \Big(|a_i - k_i| + L_i^1 \sum_{j=1}^{n} |b_{ji}| \Big)|x_i(s)| + \sum_{i=1}^{n} \Big(L_i^2 \sum_{j=1}^{n} |c_{ji}| \Big)|x_i(\eta_k)| \Big] ds \\
\le& \|x(t)\| + \int_{\eta_k}^{t} \Big(\mu \sum_{i=1}^{n} |x_i(s)| + \nu \sum_{i=1}^{n} |x_i(\eta_k)| \Big) ds \\
\le& \|x(t)\| + \nu\theta\|x(\eta_k)\| + \mu \int_{\eta_k}^{t} \|x(s)\| ds,
\end{aligned} \tag{3.5}$$

substituting (3.4) into (3.5)

$$\begin{aligned}\|x(\eta_k)\| \leq & \|x(t)\| + \nu\theta\|x(\eta_k)\| + \mu \int_{\eta_k}^{t} \big[(1+\nu\theta)\|x(\eta_k)\|e^{\mu\theta}\big] ds \\ \leq & \|x(t)\| + \theta\big[\nu + \mu(1+\nu\theta)e^{\mu\theta}\big]\|x(\eta_k)\|,\end{aligned}$$

it follows that

$$\|x(\gamma(t))\| \leq \alpha\|x(t)\|.$$

For the other case of $t < \eta_k$, the same conclusion can be drawn with the method above. And hence, the lemma is proved. □

Remark 3.2. Different from the conventional neural networks, the generalized type neural networks with piecewise constant argument can change its deviation type during the process, which is the difficulty of the study on this type of neural networks. Lemma 3.1 estimates the norm of the deviation term $x(\gamma(t))$ via the norm of the corresponding state vector $x(t)$ and establishes the link between deviation term and the state vector of the system itself.

Theorem 3.3. *Assume (H1), (H2), (H3) hold, system* (3.2) *is globally exponentially stable, which implies system* (2.1) *is globally exponentially stabilizable under the single state stabilization control rule* (3.1) *if there exist constants k_i, $i = 1, 2, ..., n$ such that*

$$A - \nu\alpha > 0, \tag{3.6}$$

where

$$\begin{aligned}\alpha &= 1\Big/\Big(1 - \theta\big[\nu + \mu(1+\nu\theta)e^{\mu\theta}\big]\Big), \\ \mu &= \max_{1\leq i\leq n} \big(|a_i - k_i| + L_i^1 \sum_{j=1}^{n} |b_{ji}|\big), \\ \nu &= \max_{1\leq i\leq n} \big(\sum_{j=1}^{n} L_i^2 |c_{ji}|\big), \\ A &= \min_{1\leq i\leq n} \Big[(a_i - k_i) - \sum_{j=1}^{n} |b_{ji}| L_i^1\Big].\end{aligned}$$

Proof. Consider the following Lyapunov function

$$V(x(t)) = \sum_{i=1}^{n} e^{\beta t}|x_i(t)|,$$

where $\beta > 0$, is a sufficiently small and positive constant.

Along trajectory (3.2), the upper right Dini derivative of V can be calculated as follows:

$$\begin{aligned}D^+V(x(t)) =& D^+\Big[\sum_{i=1}^{n} sgn(x_i(t))e^{\beta t}x_i(t)\Big] \\ =& \sum_{i=1}^{n} sgn(x_i(t))\Big[\beta e^{\beta t}x_i(t) + e^{\beta t}\dot{x}_i(t)\Big] \\ =& \beta e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))x_i(t) + e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))\dot{x}_i(t)\end{aligned}$$

$$
\begin{aligned}
&=\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}sgn(x_i(t))\Big(-(a_i-k_i)x_i(t)+\sum_{j=1}^{n}b_{ij}f_j(x_j(t))\\
&\quad+\sum_{j=1}^{n}c_{ij}g_j(x_j(\gamma(t)))\Big)\\
&=\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|-e^{\beta t}\sum_{i=1}^{n}(a_i-k_i)sgn(x_i(t))x_i(t)+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}sgn(x_i(t))f_j(x_j(t))\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}sgn(x_i(t))g_j(x_j(\gamma(t)))\\
&=\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|-e^{\beta t}\sum_{i=1}^{n}(a_i-k_i)|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}sgn(x_i(t))f_j(x_j(t))\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}sgn(x_i(t))g_j(x_j(\gamma(t)))\\
&\le\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|-e^{\beta t}\sum_{i=1}^{n}(a_i-k_i)|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}f_j(x_j(t))|\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}g_j(x_j(\gamma(t)))|\\
&\le\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|-e^{\beta t}\sum_{i=1}^{n}(a_i-k_i)|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}L_j^1x_j(t)|\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}L_j^2x_j(\gamma(t))|\\
&=\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|-e^{\beta t}\sum_{i=1}^{n}(a_i-k_i)|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ji}L_i^1|\right)|x_i(t)|\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ji}L_i^2|\right)|x_i(\gamma(t))|.
\end{aligned}
$$

And hence

$$
\begin{aligned}
D^+V(x(t))&\le\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|c_{ji}L_i^2|\right)|x_i(\gamma(t))|-e^{\beta t}\sum_{i=1}^{n}\left[(a_i-k_i)-\left(\sum_{j=1}^{n}|b_{ji}L_i^1|\right)\right]|x_i(t)|\\
&\le\beta e^{\beta t}\|x(t)\|-Ae^{\beta t}\sum_{i=1}^{n}|x_i(t)|+\nu e^{\beta t}\sum_{i=1}^{n}|x_i(\gamma(t))|\\
&=\beta e^{\beta t}\|x(t)\|-Ae^{\beta t}\|x(t)\|+\nu e^{\beta t}\|x(\gamma(t))\|.
\end{aligned}
$$

Applying Lemma 3.1, it derives that

$$
\begin{aligned}
D^+V(x(t))&\le\beta e^{\beta t}\|x(t)\|-Ae^{\beta t}\|x(t)\|+\nu\alpha e^{\beta t}\|x(t)\|\\
&=-(A-\nu\alpha-\beta)\|x(t)\|e^{\beta t}.
\end{aligned}
$$

According to (3.6) and utilizing the continuity of the parameters, there exists some sufficiently small and positive $\beta>0$ such that

$$\delta_0=A-\nu\alpha-\beta>0.$$

Then

$$D^+V(x(t)) \le -\delta_0 e^{\beta t}\|x(t)\| < 0.$$

Therefore

$$\|x(t)\|e^{\beta t} = V(x(t)) \le V(x(t_0)) = \|x(t_0)\|e^{\beta t_0}.$$

That is

$$\|x(t)\| = V(x(t))e^{-\beta t} \le V(x(t_0))e^{-\beta t} = \|x(t_0)\|e^{-\beta(t-t_0)}.$$

It implies that system (2.1) is of globally exponential stabilization under the single state stabilization control rule (3.1). □

3.2. Multiple state stabilization

We propose the following multiple state stabilization control rule:

$$\mathbf{I(t)} = \begin{pmatrix} k_{11}x_1(t) + k_{12}x_1(\gamma(t)) \\ k_{21}x_2(t) + k_{22}x_2(\gamma(t)) \\ \vdots \\ k_{i1}x_i(t) + k_{i2}x_i(\gamma(t)) \\ \vdots \\ k_{n1}x_n(t) + k_{n2}x_n(\gamma(t)) \end{pmatrix} = K_1x(t) + K_2x(\gamma(t)), \tag{3.7}$$

where $K_1 = diag(k_{11}, k_{21}, \cdots, k_{i1}, \cdots, k_{n1})$, $K_2 = diag(k_{12}, k_{22}, \cdots, k_{i2}, \cdots, k_{n2})$ are the control gain of state vector and state vector with piecewise constant argument, respectively.

With multiple state stabilization control rule (3.7), system (2.1) can be rewritten as follows:

$$\dot{x}_i(t) = -(a_i - k_{i1})x_i(t) + \sum_{j=1}^{n} b_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(\gamma(t))) + k_{i2}x_i(\gamma(t)). \tag{3.8}$$

In order to verify Theorem 3.5 more expediently, we give Lemma 3.4. The following hypothesis is needed in this subsection.

(H4) There exist positive constants θ, τ, and ς such that

$$1 - \theta\big[\tau + \varsigma(1+\tau\theta)e^{\varsigma\theta}\big] > 0,$$

where $\varsigma = \max_{1\le i\le n}\big(|a_i - k_{i1}| + L_i^1\sum_{j=1}^{n}|b_{ji}|\big)$, $\tau = \max_{1\le i\le n}\big(|k_{i2}| + \sum_{j=1}^{n}L_i^2|c_{ji}|\big)$.

Lemma 3.4. *Under (H1), (H2), (H4), for* (3.8)*, the following inequality holds*

$$\|x(\gamma(t))\| \le \xi\|x(t)\|$$

for all $t \ge t_0$*, where* $\xi = 1\Big/\Big(1 - \theta\big[\tau + \varsigma(1+\tau\theta)e^{\varsigma\theta}\big]\Big)$, $\varsigma = \max_{1\le i\le n}\big(|a_i - k_{i1}| + L_i^1\sum_{j=1}^{n}|b_{ji}|\big)$, $\tau = \max_{1\le i\le n}\big(|k_{i2}| + \sum_{j=1}^{n}L_i^2|c_{ji}|\big)$.

Proof. For any $t \ge t_0$, by the property of $\gamma(t)$ and the sequences $\{\theta_k\}$ and $\{\eta_k\}$, there exists only one $k \in N$, which satisfies that

$$\gamma(t) = \eta_k \in [\theta_k, \theta_{k+1}), t \in [\theta_k, \theta_{k+1}),$$

and we get if $t \ge \eta_k$

$$x_i(t) = x_i(\eta_k) + \int_{\eta_k}^{t}\Big[-(a_i - k_{i1})x_i(s) + \sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(\eta_k)) + k_{i2}x_i(\eta_k)\Big]ds, \tag{3.9}$$

for $i = 1, 2, ..., n$, then

$$
\begin{aligned}
\|x(t)\| \leq & \|x(\eta_k)\| + \sum_{i=1}^{n} \int_{\eta_k}^{t} \Big[|a_i - k_{i1}||x_i(s)| + \sum_{j=1}^{n} L_j^1 |b_{ij}||x_j(s)| + \sum_{j=1}^{n} L_j^2 |c_{ij}||x_j(\eta_k)| + |k_{i2} x_i(\eta_k)| \Big] ds \\
= & \|x(\eta_k)\| + \int_{\eta_k}^{t} \Big[\sum_{i=1}^{n} \Big(|a_i - k_{i1}| + L_i^1 \sum_{j=1}^{n} |b_{ji}| \Big) |x_i(s)| + \sum_{i=1}^{n} \Big(|k_{i2}| + L_i^2 \sum_{j=1}^{n} |c_{ji}| \Big) |x_i(\eta_k)| \Big] ds \\
\leq & \|x(\eta_k)\| + \int_{\eta_k}^{t} \Big(\varsigma \sum_{i=1}^{n} |x_i(s)| + \tau \sum_{i=1}^{n} |x_i(\eta_k)| \Big) ds \\
\leq & (1 + \tau\theta) \|x(\eta_k)\| + \varsigma \int_{\eta_k}^{t} \|x(s)\| ds,
\end{aligned}
$$

Based on the Gronwall-Bellman inequality

$$
\|x(t)\| \leq (1 + \tau\theta) \|x(\eta_k)\| e^{\varsigma\theta}. \tag{3.10}
$$

Exchanging the location of $x_i(t)$ and $x_i(\eta_k)$ in (3.9), it derives

$$
\begin{aligned}
\|x(\eta_k)\| \leq & \|x(t)\| + \sum_{i=1}^{n} \int_{\eta_k}^{t} \Big[|a_i - k_{i1}||x_i(s)| + \sum_{j=1}^{n} L_j^1 |b_{ij}||x_j(s)| + \sum_{j=1}^{n} L_j^2 |c_{ij}||x_j(\eta_k)| + |k_{i2} x_i(\eta_k)| \Big] ds \\
= & \|x(t)\| + \int_{\eta_k}^{t} \Big[\sum_{i=1}^{n} \Big(|a_i - k_{i1}| + L_i^1 \sum_{j=1}^{n} |b_{ji}| \Big) |x_i(s)| + \sum_{i=1}^{n} \Big(|k_{i2}| + L_i^2 \sum_{j=1}^{n} |c_{ji}| \Big) |x_i(\eta_k)| \Big] ds \\
\leq & \|x(t)\| + \int_{\eta_k}^{t} \Big(\varsigma \sum_{i=1}^{n} |x_i(s)| + \tau \sum_{i=1}^{n} |x_i(\eta_k)| \Big) ds \\
\leq & \|x(t)\| + \tau\theta \|x(\eta_k)\| + \varsigma \int_{\eta_k}^{t} \|x(s)\| ds,
\end{aligned} \tag{3.11}
$$

substituting (3.10) into (3.11)

$$
\begin{aligned}
\|x(\eta_k)\| \leq & \|x(t)\| + \tau\theta \|x(\eta_k)\| + \varsigma \int_{\eta_k}^{t} \big[(1 + \tau\theta) \|x(\eta_k)\| e^{\varsigma\theta} \big] ds \\
\leq & \|x(t)\| + \theta \big[\tau + \varsigma (1 + \tau\theta) e^{\varsigma\theta} \big] \|x(\eta_k)\|,
\end{aligned}
$$

it follows that

$$
\|x(\gamma(t))\| \leq \xi \|x(t)\|.
$$

For the other case of $t < \eta_k$, the same conclusion can be drawn with the method above. And hence, the lemma is proved. □

Theorem 3.5. *Assume (H1), (H2) and (H4) hold, system* (3.8) *is globally exponentially stable, which implies system* (2.1) *is globally exponentially stabilizable under the multiple state stabilization control rule* (3.7) *if there exist constants k_{i1}, k_{i2}, $i = 1, 2, ..., n$, such that*

$$
\hat{A} - \tau\xi > 0, \tag{3.12}
$$

where

$$
\begin{aligned}
\xi = & 1 \Big/ \Big(1 - \theta \big[\tau + \varsigma (1 + \tau\theta) e^{\varsigma\theta} \big] \Big), \\
\varsigma = & \max_{1 \leq i \leq n} \Big(|a_i - k_{i1}| + L_i^1 \sum_{j=1}^{n} |b_{ji}| \Big),
\end{aligned}
$$

$$\tau = \max_{1\le i\le n}\big(|k_{i2}| + \sum_{j=1}^{n} L_i^2 |c_{ji}|\big),$$
$$\hat{A} = \min_{1\le i\le n}\left[(a_i - k_{i1}) - \sum_{j=1}^{n} |b_{ji}| L_i^1\right].$$

Proof. Consider the following Lyapunov function

$$V(x(t)) = \sum_{i=1}^{n} e^{\beta t}|x_i(t)|,$$

where $\beta > 0$, is a sufficiently small positive constant.

Along trajectory (3.8), evaluating the upper right Dini derivative of V, we can have

$$\begin{aligned}
D^+V(x(t)) =& \sum_{i=1}^{n} sgn(x_i(t))\Big(\beta e^{\beta t}x_i(t) + e^{\beta t}\dot{x}_i(t)\Big)\\
=& \sum_{i=1}^{n} sgn(x_i(t))\beta e^{\beta t}x_i(t) + e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))\dot{x}_i(t)\\
=& \beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)| + e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))\Big[-(a_i - k_{i1})x_i(t) + \sum_{j=1}^{n} b_{ij}f_j(x_j(t))\\
&+ \sum_{j=1}^{n} c_{ij}g_j(x_j(\gamma(t))) + k_{i2}x_i(\gamma(t))\Big]\\
=& \beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)| - e^{\beta t}\sum_{i=1}^{n}(a_i - k_{i1})|x_i(t)| + e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}f_j(x_j(t))sgn(x_i(t))\\
&+ e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}g_j(x_j(\gamma(t)))sgn(x_i(t)) + e^{\beta t}\sum_{i=1}^{n} k_{i2}x_i(\gamma(t))sgn(x_i(t))\\
\le& \beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)| - e^{\beta t}\sum_{i=1}^{n}(a_i - k_{i1})|x_i(t)| + e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}L_j^1x_j(t)|\\
&+ e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}L_j^2x_j(\gamma(t))| + e^{\beta t}\sum_{i=1}^{n}|k_{i2}x_i(\gamma(t))|\\
\le& \beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)| - e^{\beta t}\sum_{i=1}^{n}(a_i - k_{i1})|x_i(t)| + e^{\beta t}\sum_{j=1}^{n}\sum_{i=1}^{n}|b_{ji}L_i^1x_i(t)|\\
&+ e^{\beta t}\sum_{j=1}^{n}\sum_{i=1}^{n}|c_{ji}L_i^2x_i(\gamma(t))| + e^{\beta t}\sum_{i=1}^{n}|k_{i2}x_i(\gamma(t))|\\
=& \beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)| - e^{\beta t}\sum_{i=1}^{n}(a_i - k_{i1})|x_i(t)| + e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ji}L_i^1||x_i(t)|\\
&+ e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}L_i^2x_i(\gamma(t))| + e^{\beta t}\sum_{i=1}^{n}|k_{i2}x_i(\gamma(t))|,
\end{aligned}$$

and hence, we can get

$$D^+V(x(t)) \le \beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)| - e^{\beta t}\sum_{i=1}^{n}\left[(a_i - k_{1i}) - \Big(\sum_{j=1}^{n}|b_{ji}L_i^1|\Big)\right]|x_i(t)| + e^{\beta t}\sum_{i=1}^{n}\left[\Big(\sum_{j=1}^{n}|c_{ji}L_i^2|\Big)\right.$$

$$
\begin{aligned}
&+|k_{i2}|\Big]\,|x_i(\gamma(t))| \\
\leq&\beta e^{\beta t}\sum_{i=1}^{n}|x_i(t)|-e^{\beta t}\hat{A}\sum_{i=1}^{n}|x_i(t)|+e^{\beta t}\tau\sum_{i=1}^{n}|x_i(\gamma(t))| \\
=&\beta e^{\beta t}\|x(t)\|-e^{\beta t}\hat{A}\|x(t)\|+e^{\beta t}\tau\|x(\gamma(t))\|.
\end{aligned}
$$

Applying Lemma 3.4, we have

$$
\begin{aligned}
D^{+}V(x(t))\leq&\beta e^{\beta t}\|x(t)\|-\hat{A}e^{\beta t}\|x(t)\|+\tau\xi e^{\beta t}\|x(t)\| \\
=&-(\hat{A}-\tau\xi-\beta)\|x(t)\|e^{\beta t}.
\end{aligned}
$$

According to (3.12) and utilizing the continuity of the parameters, there exists some sufficiently small and positive constant $\beta>0$ such that

$$\delta_1=\hat{A}-\tau\xi-\beta>0.$$

Then

$$D^{+}V(x(t))\leq-\delta_1 e^{\beta t}\|x(t)\|<0.$$

Therefore, we have

$$\|x(t)\|e^{\beta t}=V(x(t))\leq V(x(t_0))=\|x(t_0)\|e^{\beta t_0}.$$

That is

$$\|x(t)\|=V(x(t))e^{-\beta t}\leq V(x(t_0))e^{-\beta t}=\|x(t_0)\|e^{-\beta(t-t_0)}.$$

It implies that system (2.1) is of global exponential stabilization under the multiple state stabilization control rule (3.7). □

Remark 3.6. It is clear that Theorem 3.3 is a special case of Theorem 3.5. In fact, when the conditions in Theorem 3.5 are satisfied, let $k_{i1}=k_i$, and $k_{i2}=0$, it is an immediate consequence that Theorem 3.3 follows.

Remark 3.7. Compared with the control rule of Theorem 3.3, the control rule of Theorem 3.5 contains two adjustable parameters, which increase the flexibility of Theorem 3.5. The control rule of Theorem 3.3 is simpler, which is implemented more easily in practical applications.

3.3. Output stabilization

In many integrated systems, the states of some components are unable or inconvenient to be measured, but their outputs are easy to be measured. Hence, we can use the outputs as the elements of the control rule to stabilize the integrated systems.

Assume that the state variables $x_i(t), i=1,2,...,n$ of (2.1) are not measurable, but the corresponding outputs $f_i(x_i), g_i(\gamma(x_i)), i=1,2,...,n$ are measurable. Hence we can use the outputs as the entry of the stabilization control rule to stabilize the system. We propose the following output stabilization control rule:

$$
\mathbf{I(t)}=\begin{pmatrix}
k_{11}f_1(x_1(t))+k_{12}g_1(x_1(\gamma(t))) \\
k_{21}f_2(x_2(t))+k_{22}g_2(x_2(\gamma(t))) \\
\vdots \\
k_{i1}f_i(x_i(t))+k_{i2}g_i(x_i(\gamma(t))) \\
\vdots \\
k_{n1}f_n(x_n(t))+k_{n2}g_n(x_n(\gamma(t)))
\end{pmatrix}=K_1f(x(t))+K_2g(x(\gamma(t))), \tag{3.13}
$$

where $K_1 = diag(k_{11}, k_{21}, \cdots, k_{i1}, \cdots, k_{n1})$, $K_2 = diag(k_{12}, k_{22}, \cdots, k_{i2}, \cdots, k_{n2})$ are the control gain of outputs $f_i(x_i(t))$ and $g_i(x_i(\gamma(t)))$, respectively.

With the output stabilization control rule (3.13), the system (2.1) can be rewritten as follows

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} \tilde{b}_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} \tilde{c}_{ij} g_j(x_j(\gamma(t))), \tag{3.14}$$

where

$$\tilde{b}_{ij} = \begin{cases} b_{ij} + k_{i1}, & \text{if } i = j, \\ b_{ij}, & \text{if } i \neq j, \end{cases}$$

$$\tilde{c}_{ij} = \begin{cases} c_{ij} + k_{i2}, & \text{if } i = j, \\ c_{ij}, & \text{if } i \neq j. \end{cases}$$

In order to formulate Theorem 3.9 more conveniently, we introduce the following assumption and Lemma 3.8.

(H5) There exist positive constants θ, ζ, and κ such that

$$1 - \theta\big[\kappa + \zeta(1 + \kappa\theta)e^{\zeta\theta}\big] > 0,$$

where $\zeta = \max_{1 \le i \le n} \big(a_i + L_i^1 \sum_{j=1}^{n} |\tilde{b}_{ji}|\big)$, $\kappa = \max_{1 \le i \le n} \big(\sum_{j=1}^{n} L_i^2 |\tilde{c}_{ji}|\big)$.

Lemma 3.8. *Under (H1), (H2), (H5), for* (3.14)*, the following inequality holds*

$$\|x(\gamma(t))\| \le \rho \|x(t)\|,$$

for all $t \ge t_0$, where

$$\rho = 1 \Big/ \Big(1 - \theta\big[\kappa + \zeta(1 + \kappa\theta)e^{\zeta\theta}\big]\Big), \zeta = \max_{1 \le i \le n} \big(a_i + L_i^1 \sum_{j=1}^{n} |\tilde{b}_{ji}|\big), \kappa = \max_{1 \le i \le n} \big(\sum_{j=1}^{n} L_i^2 |\tilde{c}_{ji}|\big).$$

Proof. For any $t \ge t_0$, by the property of $\gamma(t)$ and the sequences $\{\theta_k\}$ and $\{\eta_k\}$, there exists only one $k \in N$, which satisfies that

$$\gamma(t) = \eta_k \in [\theta_k, \theta_{k+1}), t \in [\theta_k, \theta_{k+1}),$$

and we get if $t \ge \eta_k$

$$x_i(t) = x_i(\eta_k) + \int_{\eta_k}^{t} \Big[- a_i x_i(s) + \sum_{j=1}^{n} \tilde{b}_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} \tilde{c}_{ij} g_j(x_j(\eta_k)) \Big] ds. \tag{3.15}$$

for $i = 1, 2, ..., n$, then

$$\begin{aligned}
\|x(t)\| \le & \|x(\eta_k)\| + \sum_{i=1}^{n} \int_{\eta_k}^{t} \Big[a_i |x_i(s)| + \sum_{j=1}^{n} L_j^1 |\tilde{b}_{ij}||x_j(s)| + \sum_{j=1}^{n} L_j^2 |\tilde{c}_{ij}||x_j(\eta_k)| \Big] ds \\
= & \|x(\eta_k)\| + \int_{\eta_k}^{t} \Big[\sum_{i=1}^{n} \Big(a_i + L_i^1 \sum_{j=1}^{n} |\tilde{b}_{ji}|\Big)|x_i(s)| + \sum_{i=1}^{n} \Big(L_i^2 \sum_{j=1}^{n} |\tilde{c}_{ji}|\Big)|x_i(\eta_k)| \Big] ds \\
\le & \|x(\eta_k)\| + \int_{\eta_k}^{t} \Big(\zeta \sum_{i=1}^{n} |x_i(s)| + \kappa \sum_{i=1}^{n} |x_i(\eta_k)|\Big) ds \\
\le & (1 + \kappa\theta)\|x(\eta_k)\| + \zeta \int_{\eta_k}^{t} \|x(s)\| ds.
\end{aligned}$$

Based on the Gronwall-Bellman inequality, we obtain

$$\|x(t)\| \leq (1+\kappa\theta)\|x(\eta_k)\|e^{\zeta\theta}. \tag{3.16}$$

Exchanging the location of $x_i(t)$ and $x_i(\eta_k)$ in (3.15), it follows that

$$\begin{aligned}\|x(\eta_k)\| \leq &\|x(t)\| + \int_{\eta_k}^{t}\Big[\sum_{i=1}^{n}\Big(a_i + L_i^1\sum_{j=1}^{n}|\tilde{b}_{ji}|\Big)|x_i(s)| + \sum_{i=1}^{n}\Big(L_i^2\sum_{j=1}^{n}|\tilde{c}_{ji}|\Big)|x_i(\eta_k)|\Big]ds \\ \leq &\|x(t)\| + \int_{\eta_k}^{t}\Big(\zeta\sum_{i=1}^{n}|x_i(s)| + \kappa\sum_{i=1}^{n}|x_i(\eta_k)|\Big)ds \\ \leq &\|x(t)\| + \kappa\theta\|x(\eta_k)\| + \zeta\int_{\eta_k}^{t}\|x(s)\|ds.\end{aligned} \tag{3.17}$$

substituting (3.16) into (3.17)

$$\begin{aligned}\|x(\eta_k)\| \leq &\|x(t)\| + \kappa\theta\|x(\eta_k)\| + \zeta\int_{\eta_k}^{t}\big[(1+\kappa\theta)\|x(\eta_k)\|e^{\zeta\theta}\big]ds \\ \leq &\|x(t)\| + \theta\big[\kappa + \zeta(1+\kappa\theta)e^{\zeta\theta}\big]\|x(\eta_k)\|,\end{aligned}$$

it follows that

$$\|x(\gamma(t))\| \leq \rho\|x(t)\|,$$

For the other case of $t < \eta_k$, the same conclusion can be drawn with the method above. And hence, the lemma is proved. □

Theorem 3.9. *Assume H(1), (H2) and H(5) hold, system* (3.14) *is globally exponentially stable, which implies system* (2.1) *is globally exponentially stabilizable under stabilization control rule* (3.13) *if there exist constants* $k_{i1}, k_{i2}, i = 1, 2, ..., n$, *such that*

$$\tilde{A} - \kappa\rho > 0, \tag{3.18}$$

where

$$\rho = 1\Big/\Big(1 - \theta\big[\kappa + \zeta(1+\kappa\theta)e^{\zeta\theta}\big]\Big), \qquad \zeta = \max_{1\leq i\leq n}\big(a_i + L_i^1\sum_{j=1}^{n}|\tilde{b}_{ji}|\big),$$

$$\kappa = \max_{1\leq i\leq n}\big(\sum_{j=1}^{n}L_i^2|\tilde{c}_{ji}|\big), \qquad \tilde{A} = \min_{1\leq i\leq n}\Big(a_i - \sum_{j=1}^{n}|\tilde{b}_{ji}|L_i^1\Big).$$

Proof. Consider the following Lyapunov function

$$V(x(t)) = \sum_{i=1}^{n}e^{\beta t}|x_i(t)|,$$

where $\beta > 0$, is a sufficiently small and positive constant.

Evaluating the upper right Dini derivative of V along the trajectory of (3.14), we can have

$$\begin{aligned}D^+V(x(t)) = &D^+\Big[\sum_{i=1}^{n}sgn(x_i(t))e^{\beta t}x_i(t)\Big] \\ = &\sum_{i=1}^{n}sgn(x_i(t))\Big[\beta e^{\beta t}x_i(t) + e^{\beta t}\dot{x}_i(t)\Big]\end{aligned}$$

$$
\begin{aligned}
&=\beta e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))x_i(t)+e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))\dot{x}_i(t)\\
&=\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|+e^{\beta t}\sum_{i=1}^{n} sgn(x_i(t))\Big(-a_ix_i(t)+\sum_{j=1}^{n}\tilde{b}_{ij}f_j(x_j(t))\\
&\quad+\sum_{j=1}^{n}\tilde{c}_{ij}g_j(x_j(\gamma(t)))\Big)\\
&=\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|-e^{\beta t}\sum_{i=1}^{n} a_i sgn(x_i(t))x_i(t)+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{b}_{ij}sgn(x_i(t))f_j(x_j(t))\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ij}sgn(x_i(t))g_j(x_j(\gamma(t)))\\
&=\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|-e^{\beta t}\sum_{i=1}^{n} a_i|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{b}_{ij}sgn(x_i(t))f_j(x_j(t))\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ij}sgn(x_i(t))g_j(x_j(\gamma(t)))\\
&\leq\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|-e^{\beta t}\sum_{i=1}^{n} a_i|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|\tilde{b}_{ij}f_j(x_j(t))|\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|\tilde{c}_{ij}g_j(x_j(\gamma(t)))|\\
&\leq\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|-e^{\beta t}\sum_{i=1}^{n} a_i|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|\tilde{b}_{ij}L_j^1x_j(t)|\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\sum_{j=1}^{n}|\tilde{c}_{ij}L_j^2x_j(\gamma(t))|\\
&=\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|-e^{\beta t}\sum_{i=1}^{n} a_i|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\tilde{b}_{ji}L_i^1|\right)|x_i(t)|\\
&\quad+e^{\beta t}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\tilde{c}_{ji}L_i^2|\right)|x_i(\gamma(t))|\\
&=\beta e^{\beta t}\sum_{i=1}^{n} |x_i(t)|-e^{\beta t}\sum_{i=1}^{n}\left[a_i-\left(\sum_{j=1}^{n}|\tilde{b}_{ji}L_i^1|\right)\right]|x_i(t)|+e^{\beta t}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\tilde{c}_{ji}L_i^2|\right)|x_i(\gamma(t))|\\
&\leq\beta e^{\beta t}\|x(t)\|-\tilde{A}e^{\beta t}\sum_{i=1}^{n}|x_i(t)|+\kappa e^{\beta t}\sum_{i=1}^{n}|x_i(\gamma(t))|\\
&=\beta e^{\beta t}\|x(t)\|-\tilde{A}e^{\beta t}\|x(t)\|+\kappa e^{\beta t}\|x(\gamma(t))\|,
\end{aligned}
$$

Applying Lemma 3.8, it derives that

$$
\begin{aligned}
D^+V(x(t)) &\leq \beta e^{\beta t}\|x(t)\|-\tilde{A}e^{\beta t}\|x(t)\|+\kappa\rho e^{\beta t}\|x(t)\|\\
&=-(\tilde{A}-\kappa\rho-\beta)\|x(t)\|e^{\beta t}.
\end{aligned}
$$

According to (3.18) and utilizing the continuity of the parameters, there exists some sufficiently small and positive $\beta > 0$ such that

$$\delta_2 = \tilde{A} - \kappa\rho - \beta > 0.$$

Then

$$D^+V(x(t)) \leq -\delta_2 e^{\beta t}\|x(t)\| < 0.$$

Therefore

$$\|x(t)\|e^{\beta t} = V(x(t)) \leq V(x(t_0)) = \|x(t_0)\|e^{\beta t_0}.$$

That is

$$\|x(t)\| = V(x(t))e^{-\beta t} \leq V(x(t_0))e^{-\beta t} = \|x(t_0)\|e^{-\beta(t-t_0)}.$$

It implies that system (2.1) is of global exponential stabilization under the output stabilization control rule (3.13). □

4. Numerical examples

In this section, we introduce two illustrative examples to demonstrate the effectiveness of the obtained criteria.

Example 4.1. We consider the generalized type neural network with piecewise constant argument as follows, where $\theta_k = k/9$, $\gamma(t) = \eta_k = (2k+1)/18$, $k \in N$.

$$\begin{cases} \dot{x}_1(t) = & -0.8x_1(t) + 0.02\tanh(x_1(t)) + 0.03\tanh(x_2(t)) \\ & +0.08\tanh(x_1(\gamma(t))/7) + 0.1\tanh(x_2(\gamma(t))) + I_1(t), \\ \dot{x}_2(t) = & -x_2(t) + 0.01\tanh(x_1(t)) + \tanh(x_2(t)) \\ & +0.01\tanh(x_1(\gamma(t))/7) + 0.1\tanh(x_2(\gamma(t))) + I_2(t). \end{cases} \tag{4.1}$$

The state trajectory of (4.1) is depicted in Figure 1. Clearly, its states converge to different equilibrium points and even unstable.

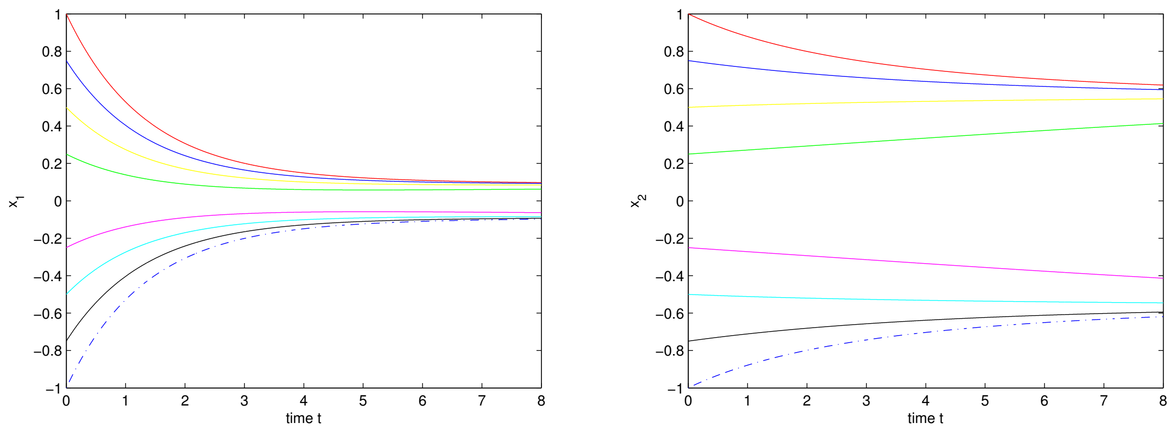

Figure 1: Transient behavior of $x_1(t), x_2(t)$ for (4.1) without external controller.

Choose the single state stabilization control rule as following

$$I(t) = \begin{cases} I_1(t) = k_1x_1(t) = -2.2x_1(t), \\ I_2(t) = k_2x_2(t) = -1.8x_2(t). \end{cases} \tag{4.2}$$

We can easily calculate

$$\begin{aligned}
&|a_1 - k_1| + L_1^1 \sum_{j=1}^{2} b_{j1} = |0.8 + 2.2| + 1 \times (0.02 + 0.01) = 3.03, \\
&|a_2 - k_2| + L_2^1 \sum_{j=1}^{2} b_{j2} = |1 + 1.8| + 1 \times (0.03 + 1) = 3.83, \\
&\sum_{j=1}^{2} L_1^2 |c_{j1}| = 1/7 \times (0.08 + 0.01) = 9/700, \\
&\sum_{j=1}^{2} L_2^2 |c_{j2}| = 1 \times (0.1 + 0.1) = 0.2.
\end{aligned}$$

Hence

$$\begin{aligned}
\mu &= \max_{1\le i\le 2}(|a_i - k_i| + L_i^1 \sum_{j=1}^{2} b_{ji}) = 3.83, \\
\nu &= \max_{1\le i\le 2}(\sum_{j=1}^{2} L_i^2 |c_{ji}|) = 0.2, \\
1 - \theta[\nu + \mu(1 + \nu\theta)e^{\mu\theta}] &= 1 - 1/9 \times [0.2 + 3.83 \times (1 + 0.2 \times 1/9) \times e^{3.83\times 1/9}] \\
&= 0.3120 > 0,
\end{aligned}$$

and consequently, (H3) is satisfied.

$$\begin{aligned}
&\alpha = 1/(1 - \theta[\nu + \mu(1 + \nu\theta)e^{\mu\theta}]) = 1/0.3120 = 3.2051, \\
&\left[(a_1 - k_1) - \sum_{j=1}^{2} |b_{j1}| L_1^1\right] = (0.8 + 2.2) - 1 \times (0.02 + 0.01) = 2.97, \\
&\left[(a_2 - k_2) - \sum_{j=1}^{2} |b_{j2}| L_2^1\right] = (1 + 1.8) - 1 \times (0.03 + 1) = 1.77, \\
&A = \min_{1\le i\le 2}\left[(a_i - k_i) - \sum_{j=1}^{2} |b_{ji}| L_i^1\right] = 1.77,
\end{aligned}$$

it satisfies that

$$A - \nu\alpha = 1.77 - 0.2 \times 3.2051 = 1.1290 > 0.$$

The conditions in Theorem 3.3 are all satisfied. Consequently, (4.1) under the single state stabilization control rule (4.2) is of global exponential stabilization. The simulations in Figure 2 are well suited to the theoretical results.

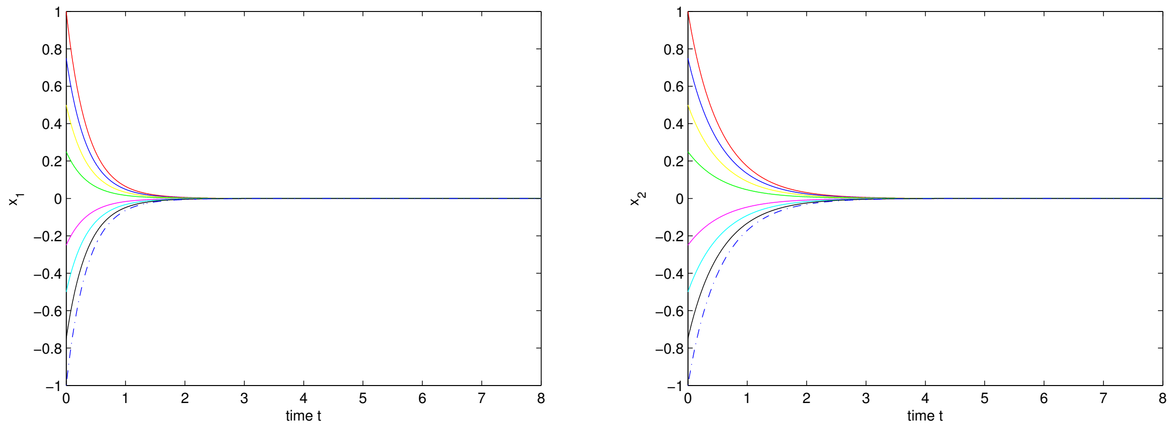

Figure 2: Transient behavior of $x_1(t), x_2(t)$ for (4.1) with the single state stabilization control rule $I_1(t) = -2.2x_1(t), I_2(t) = -1.8x_2(t)$.

In addition, select

$$I(t) = \begin{cases} I_1(t) = -2.2x_1(t) - 0.2x_1(\gamma(t)), \\ I_2(t) = -1.8x_2(t) - 0.2x_2(\gamma(t)), \end{cases} \tag{4.3}$$

as the multiple state stabilization control rule.

We can easily compute

$$|a_1 - k_{11}| + L_1^1 \sum_{j=1}^{2} |b_{j1}| = |0.8 + 2.2| + 1 \times (0.02 + 0.01) = 3.03,$$

$$|a_2 - k_{21}| + L_2^1 \sum_{j=1}^{2} |b_{j2}| = |1.0 + 1.8| + 1 \times (0.03 + 1) = 3.83,$$

$$|k_{12}| + \sum_{j=1}^{2} L_1^2 |c_{j1}| = 0.2 + 1/7 \times (0.08 + 0.01) = 149/700,$$

$$|k_{22}| + \sum_{j=1}^{2} L_2^2 |c_{j2}| = 0.2 + 1 \times (0.1 + 0.1) = 0.4,$$

$$\varsigma = \max_{1 \le i \le 2} \left(|a_i - k_{i1}| + L_i^1 \sum_{j=1}^{2} |b_{ji}| \right) = 3.83,$$

$$\tau = \max_{1 \le i \le 2} \left(|k_{i2}| + \sum_{j=1}^{2} L_i^2 |c_{ji}| \right) = 0.4,$$

$$1 - \theta\big[\tau + \varsigma(1 + \tau\theta)e^{\varsigma\theta}\big] = 1 - 1/9 \times [0.4 + 3.83 \times (1 + 0.4 \times 1/9) \times e^{3.83 \times 1/9}] = 0.2753 > 0,$$

and (H4) is fulfilled.

$$\xi = 1/0.2753 = 3.6324,$$

$$\left[(a_1 - k_{11}) - \sum_{j=1}^{1} |b_{j1}| L_1^1 \right] = (0.8 + 2.2) - 1 \times (0.02 + 0.01) = 2.97,$$

$$\left[(a_2 - k_{21}) - \sum_{j=1}^{2} |b_{j2}| L_2^1\right] = (1 + 1.8) - 1 \times (0.03 + 1) = 1.77,$$

$$\hat{A} = \min_{1\le i\le 2} \left[(a_i - k_{i1}) - \sum_{j=1}^{2} |b_{ji}| L_i^1\right] = 1.77,$$

$$\hat{A} - \tau\xi = 1.77 - 0.4 \times 3.6324 = 0.3170 > 0.$$

It is easy to see the conditions of Theorem 3.5 are satisfied. The simulations depicted in Figure 3 also demonstrate the feasibility of the multiple state stabilization control rule (4.3).

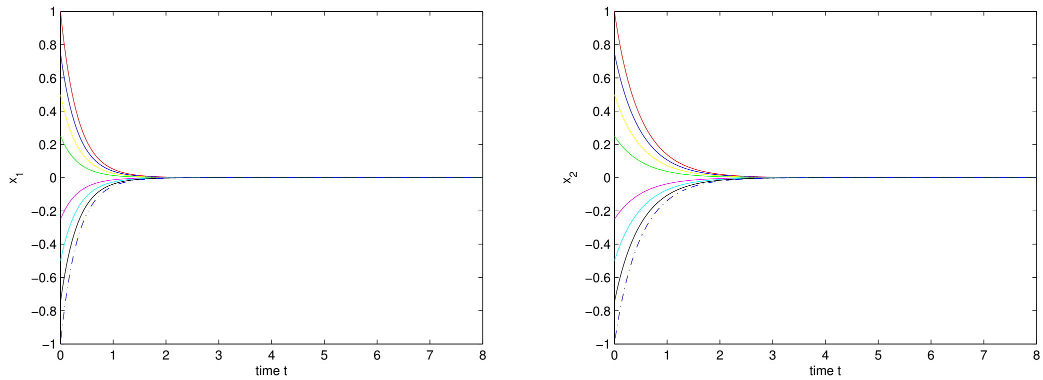

Figure 3: Transient behavior of $x_1(t), x_2(t)$ for (4.1) with the multiple state stabilization control rule $I_1(t) = -2.2x_1(t) - 0.2x_1(\gamma(t)), I_2(t) = -1.8x_2(t) - 0.2x_2(\gamma(t))$.

Example 4.2. Consider the following generalized type neural network with piecewise constant argument, where $\theta_k = k/9$, $\gamma(t) = \eta_k = (2k+1)/18$, $k \in N$.

$$\begin{cases} \dot{x}_1(t) = & -2x_1(t) + 6.62\tanh(x_1(t)) + 0.03\tanh(x_2(t)) \\ & +5.08\tanh(x_1(\gamma(t))/7) + 0.1\tanh(x_2(\gamma(t))/6) + I_1(t), \\ \dot{x}_2(t) = & -1.5x_2(t) + 0.01\tanh(x_1(t)) + 8.3\tanh(x_2(t)) \\ & +0.01\tanh(x_1(\gamma(t))/7) + 8.1\tanh(x_2(\gamma(t))/6) + I_2(t). \end{cases} \tag{4.4}$$

The numerical simulations of (4.4) are shown in Figure 4. It is clear that the system (4.4) converges to different states for different initial values.

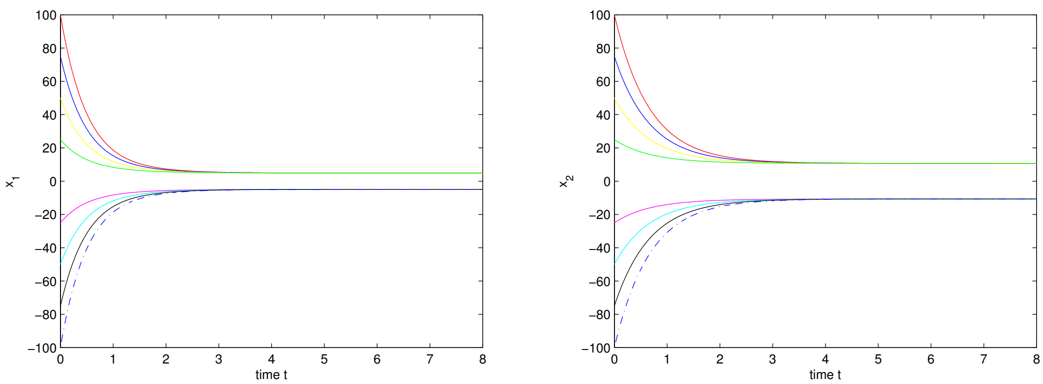

Figure 4: Transient behavior of $x_1(t), x_2(t)$ for (4.4) without external controller.

Select the output stabilization control rule as follows

$$I(t) = \begin{cases} I_1(t) = -6.6\tanh(x_1(t)) - 5\tanh(x_1(\gamma(t))/7), \\ I_2(t) = -7.3\tanh(x_2(t)) - 8\tanh(x_2(\gamma(t))/6). \end{cases} \tag{4.5}$$

Then

$$(\tilde{b}_{ij})_{2\times 2} = \begin{pmatrix} 0.02 & 0.03 \\ 0.01 & 1 \end{pmatrix},$$

$$(\tilde{c}_{ij})_{2\times 2} = \begin{pmatrix} 0.08 & 0.1 \\ 0.01 & 0.1 \end{pmatrix}.$$

We calculate as follows

$$a_1 + L_1^1 \sum_{j=1}^{2} |\tilde{b}_{j1}| = 2 + 1 \times (0.02 + 0.01) = 2.03,$$

$$a_2 + L_2^1 \sum_{j=1}^{2} |\tilde{b}_{j2}| = 1.5 + 1 \times (0.03 + 1) = 2.53,$$

$$\sum_{j=1}^{2} L_1^2 |\tilde{c}_{j1}| = 1/7 \times (0.08 + 0.01) = 9/700,$$

$$\sum_{j=1}^{2} L_2^2 |\tilde{c}_{j2}| = 1/6 \times (0.1 + 0.1) = 1/30, \zeta = \max_{1\le i\le 2} \big(a_i + L_i^1 \sum_{j=1}^{2} |\tilde{b}_{ji}|\big) = 2.53,$$

$$\kappa = \max_{1\le i\le 2} \big(\sum_{j=1}^{2} L_i^2 |\tilde{c}_{ji}|\big) = 1/30,$$

$$1 - \theta\big[\kappa + \zeta(1 + \kappa\theta)e^{\zeta\theta}\big] > 0 = 1 - 1/9 \times [1/30 + 2.53 \times (1 + 1/30 \times 1/9) \times e^{2.53\times 1/9}] = 0.6226 > 0,$$

$$\rho = 1/0.6226 = 1.6062,$$

$$a_1 - \sum_{j=1}^{2} |\tilde{b}_{j1}| L_1^1 = 2 - 1 \times (0.02 + 0.01) = 1.97,$$

$$a_2 - \sum_{j=1}^{2} |\tilde{b}_{j2}| L_2^1 = 1.5 - 1 \times (0.03 + 1) = 0.47,$$

and hence

$$\tilde{A} = \min_{1\le i\le 2} \left(a_i - \sum_{j=1}^{2} |\tilde{b}_{ji}| L_i^1\right) = 0.47.$$

It satisfies that

$$\tilde{A} - \kappa\rho = 0.47 - 1.6062 \times 1/30 = 0.4165 > 0.$$

Consequently, the system (4.4) is of global exponential stabilization with the output stabilization control rule (4.5), and the simulation results in Figure 5 agree well with the theoretical results.

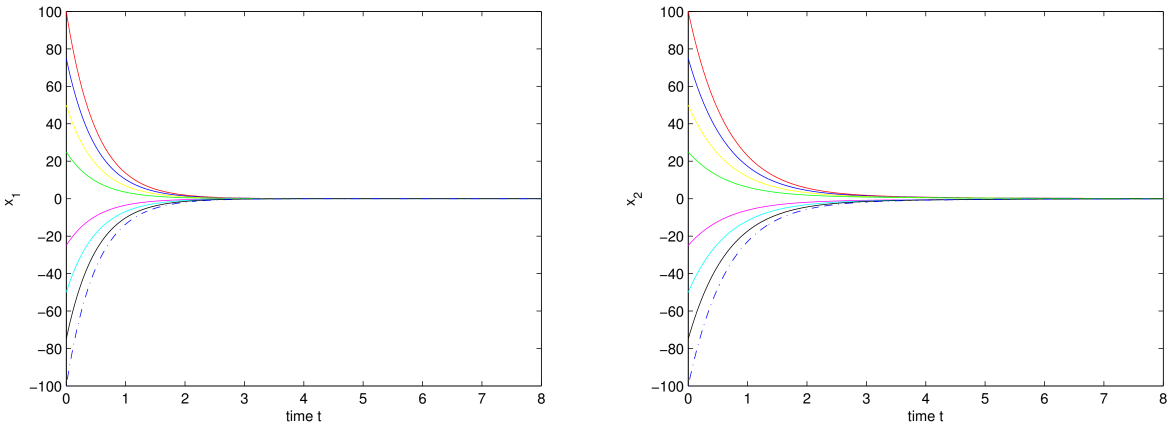

Figure 5: Transient behavior of $x_1(t), x_2(t)$ for (4.4) with the output stabilization control rule $I_1(t) = -6.6\tanh(x_1(t)) - 5\tanh(x_1(\gamma(t))/7), I_2(t) = -7.3\tanh(x_2(t)) - 8\tanh(x_2(\gamma(t))/6)$.

5. Concluding remarks

Generalized type neural networks with piecewise constant argument have attracted more attention over the past years. Different from the conventional neural networks with or without delays, the generalized type neural networks with piecewise constant argument can be both advanced and delayed during the motion. In this paper, the stabilization control for generalized type neural networks with piecewise constant argument is explored. Three kinds of different stabilization controllers are considered, and correspondingly sufficient conditions are established to guarantee the stabilization of the neural networks, which are not discussed in the existing literature. The obtained results in this paper are the preliminary research on the generalized type neural networks with piecewise constant argument, and further investigation may be focused on the synchronization, chaos and other dynamic behaviors of this type of neural networks.

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