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A modified iterative algorithm for nonexpansive mappings

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Abstract

A modified iterative algorithm is presented based on the semi-implicit midpoint rule. Strong convergence analysis is demonstrated. Our method gives a unified framework related to the implicit midpoint rule. Our results improve and extend the corresponding results in the literature.

Keywords: Nonexpansive mapping, implicit midpoint rule, fixed point.
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1. Introduction

The purpose of the paper is to construct iterative methods for finding the fixed points of nonexpansive mappings. Fixed point methods for nonexpansive mappings have been studied extensively by many researchers due to its applications in engineering and natural science. Especially, the following fixed point methods have attracted so much attention.

Browder's method ([3]): for fixed $u \in C$,

$$x_t = tu + (1-t)Tx_t,$$

where $t \in (0,1)$. Mann's method ([14]):

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

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where $\alpha_n \in (0,1)$. Ishikawa's method ([12]):

$$y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,$$
$$x_{n+1} = \beta_n x_n + (1-\beta_n)Ty_n, n \geq 0,$$

where α_n and β_n are in $(0,1)$. Halpern's method ([11]):

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

where $u \in C$ is a fixed point and $\alpha_n \in (0,1)$. Moudafi's viscosity method ([16]):

$$x_{n+1} = \alpha_n Q(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

where $Q : C \to C$ is a contraction and $\alpha_n \in (0,1)$. Modified Mann's method ([13]):

$$y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,$$
$$x_{n+1} = \beta_n u + (1-\beta_n)y_n, n \geq 0.$$

Many researchers demonstrated the convergence results of the above methods and their variant forms. Related references, please refer to [2, 4–9, 15, 17, 18, 21, 23–30, 32, 33]. Very recently, in [1] and [22], the authors presented the following semi-implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \geq 0, \tag{1.1}$$

and

$$x_{n+1} = \alpha_n Q(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), n \geq 0, \tag{1.2}$$

where $\alpha_n \in (0,1)$ and Q is a contraction. Further, Yao, Shahzad and Liou [31] introduced the following semi-implicit midpoint method:

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right), n \geq 0. \tag{1.3}$$

Motivated and inspired by the above works, the purpose of the paper is to construct the following unified iterative algorithm for finding the fixed points of nonexpansive mappings

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n T(\delta_n x_n + (1-\delta_n)x_{n+1}), n \geq 0.$$

We prove that the above algorithm converges strongly to a fixed point of nonexpansive mappings T.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We use $Fix(T)$ to denote the set of fixed points of T.

A mapping $Q : C \to C$ is said to be contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|Q(x) - Q(y)\| \leq \alpha \|x - y\|$$

for all $x, y \in C$. In this case, Q is called α-contraction.

Lemma 2.1 ([10]). *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that $\{y_n\}$ is a sequence in C such that $y_n \rightharpoonup x^\dagger$ and $(I - T)y_n \to 0$. Then $x^\dagger \in Fix(T)$.*

Lemma 2.2 ([19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3 ([20]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \delta_n, \quad n \geq 0,$$

where

(i) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Throughout, we assume that H is a real Hilbert space and $C \subset H$ is a nonempty closed convex set. Let $T : C \to C$ be a nonexpansive mapping with its fixed points set being nonempty, that is, $Fix(T) \neq \emptyset$. Let $Q : C \to C$ be an α-contraction.

Now, we firstly present the following unified iterative algorithm.

Algorithm 3.1. *For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by the manner*

$$x_{n+1} = \alpha_n Q(x_n) + \beta_n x_n + \gamma_n T(\delta_n x_n + (1 - \delta_n)x_{n+1}), n \geq 0, \tag{3.1}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$, $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset [\kappa_1, \kappa_2] \subset (0, 1)$ are four sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$.

Remark 3.2. Equation (3.1) is well-defined. As a matter of fact, for fixed $u \in C$, we can define a mapping

$$x \mapsto T_u x := \alpha Q(u) + \beta u + \gamma T(\delta u + (1 - \delta)x), \forall x \in C.$$

Then, we have

$$\begin{aligned} \|T_u x - T_u y\| &= \gamma\|T(\delta u + (1 - \delta)x) - T(\delta u + (1 - \delta)y)\| \\ &\leq (1 - \delta)\gamma\|x - y\|. \end{aligned}$$

This means T_u is a contraction with coefficient $(1 - \delta)\gamma \in (0, 1)$. Hence, Algorithm 3.1 is well-defined.

Next, we show the boundedness of the sequence $\{x_n\}$.

Proposition 3.3. *The sequence $\{x_n\}$ generated by (3.1) is bounded.*

Proof. Let $z^\sharp \in Fix(T)$. From (3.1), we get

$$\begin{aligned} \|x_{n+1} - z^\sharp\| &= \|\alpha_n(Q(x_n) - Q(z^\sharp)) + \alpha_n(Q(z^\sharp) - z^\sharp) + \beta_n(x_n - z^\sharp) \\ &\quad + \gamma_n(T(\delta_n x_n + (1 - \delta_n)x_{n+1}) - z^\sharp)\| \\ &\leq \alpha_n\|Q(x_n) - Q(z^\sharp)\| + \alpha_n\|Q(z^\sharp) - z^\sharp\| + \beta_n\|x_n - z^\sharp\| \\ &\quad + \gamma_n\|T(\delta_n x_n + (1 - \delta_n)x_{n+1}) - z^\sharp\| \\ &\leq \alpha_n\alpha\|x_n - z^\sharp\| + \alpha_n\|Q(z^\sharp) - z^\sharp\| + \beta_n\|x_n - z^\sharp\| \\ &\quad + \gamma_n\delta_n\|x_n - z^\sharp\| + \gamma_n(1 - \delta_n)\|x_{n+1} - z^\sharp\|. \end{aligned}$$

From the last inequality, we obtain

$$\begin{aligned}\|x_{n+1}-z^\sharp\| &\le \frac{\alpha_n\alpha+\beta_n+\gamma_n\delta_n}{1-\gamma_n(1-\delta_n)}\|x_n-z^\sharp\|+\frac{\alpha_n}{1-\gamma_n(1-\delta_n)}\|Q(z^\sharp)-z^\sharp\|\\ &=\left[1-\frac{(1-\alpha)\alpha_n}{1-\gamma_n(1-\delta_n)}\right]\|x_n-z^\sharp\|+\frac{(1-\alpha)\alpha_n}{1-\gamma_n(1-\delta_n)}\frac{1}{1-\alpha}\|Q(z^\sharp)-z^\sharp\|\\ &\le \max\{\|x_n-z^\sharp\|,\frac{1}{1-\alpha}\|Q(z^\sharp)-z^\sharp\|\}.\end{aligned}$$

By induction, we deduce

$$\|x_n-z^\sharp\|\le \max\{\|x_0-z^\sharp\|,\frac{1}{1-\alpha}\|Q(z^\sharp)-z^\sharp\|\}.$$

This indicates that $\{x_n\}$ is bounded. This completes the proof. □

Next, we state the following theorem

Theorem 3.4. *Assume $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions*

$(C1)$: $\lim_{n\to\infty}\alpha_n=0$;
$(C2)$: $\sum_{n=0}^\infty \alpha_n=\infty$;
$(C3)$: $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n^2}=0$;
$(C4)$: $\lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n^2}=0$;
$(C5)$: $\liminf_{n\to\infty}\gamma_n>0$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $q=P_{Fix(T)}Q(q)$.

Proof. Set $z_n=\frac{\alpha_n}{1-\beta_n}Q(z_n)+(1-\frac{\alpha_n}{1-\beta_n})Tz_n$ for all n. By [21], we have that the sequence $\{z_n\}$ converges strongly to $q=P_{Fix(T)}Q(q)$ provided $\lim_{n\to\infty}\alpha_n=0$. Note that the sequences $\{x_n\}$ and $\{z_n\}$ are all bounded. We can rewrite y_n as

$$z_n=\alpha_nQ(z_n)+\beta_nz_n+\gamma_nTz_n, n\ge 0.$$

First, we note that

$$\begin{aligned}\|x_{n+1}-z_n\| &= \|\alpha_n(Q(x_n)-Q(z_n))+\beta_n(x_n-z_n)+\gamma_n(T(\delta_nx_n+(1-\delta)x_{n+1})-Tz_n)\|\\ &\le \alpha_n\alpha\|x_n-z_n\|+\beta_n\|x_n-z_n\|+\gamma_n\delta_n\|x_n-z_n\|+\gamma_n(1-\delta_n)\|x_{n+1}-z_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-z_n\| &\le \left[1-\frac{(1-\alpha)\alpha_n}{1-\gamma_n(1-\delta_n)}\right]\|x_n-z_n\|\\ &\le \left[1-\frac{(1-\alpha)\alpha_n}{1-\gamma_n(1-\delta_n)}\right]\|x_n-z_{n-1}\|+\|z_n-z_{n-1}\|.\end{aligned}$$

It is easily seen that if $\sum_{n=0}^\infty\alpha_n=\infty$ and $\lim_{n\to\infty}\frac{\|z_n-z_{n-1}\|}{\alpha_n}=0$, then we get $\lim_{n\to\infty}\|x_{n+1}-z_n\|=0$ by Lemma 2.3. Consequently, $x_n\to q=P_{Fix(T)}Q(q)$ provided $\lim_{n\to\infty}\alpha_n=0$.

Next, we estimate $\frac{\|z_n-z_{n-1}\|}{\alpha_n}$. As a matter of fact, we have

$$\begin{aligned}\|z_n-z_{n-1}\| &= \|\alpha_n(Q(z_n)-Q(z_{n-1}))+(\alpha_n-\alpha_{n-1})Q(z_{n-1})+\beta_n(z_n-z_{n-1})\\ &\quad+(\beta_n-\beta_{n-1})z_{n-1}+\gamma_n(Tz_n-Tz_{n-1})+(\gamma_n-\gamma_{n-1})Tz_{n-1}\|\\ &\le (\alpha\alpha_n+\beta_n+\gamma_n)\|z_n-z_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Q(z_{n-1})\|\\ &\quad+|\beta_n-\beta_{n-1}|\|z_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|Tz_{n-1}\|.\end{aligned}$$

Hence,

$$\frac{\|z_n - z_{n-1}\|}{\alpha_n} \leq \frac{|\alpha_n - \alpha_{n-1}|}{(1-\alpha)\alpha_n^2}(\|Q(z_{n-1})\| + \|Tz_{n-1}\|) + \frac{|\beta_n - \beta_{n-1}|}{(1-\alpha)\alpha_n^2}(\|z_{n-1}\| + \|Tz_{n-1}\|).$$

Since $\lim_{n\to\infty} \frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n^2} = \lim_{n\to\infty} \frac{|\beta_n-\beta_{n-1}|}{\alpha_n^2} = 0$, we derive that $\lim_{n\to\infty} \frac{\|z_n-z_{n-1}\|}{\alpha_n} = 0$. Therefore, $\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0$ and thus $x_n \to q = P_{Fix(T)}Q(q)$. This completes the proof. □

Theorem 3.5. *Assume* $\{\alpha_n\}$ *satisfies* $(C1)$ *and* $(C2)$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *satisfies*

$(C1):\ \lim_{n\to\infty} \alpha_n = 0$;
$(C2):\ \sum_{n=0}^{\infty} \alpha_n = \infty$;
$(C6):\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
$(C7):\ \lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;
$(C8):\ \lim_{n\to\infty}(\delta_{n+1} - \delta_n) = 0$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $q = P_{Fix(T)}Q(q)$.

Proof. From Proposition 3.3, we can choose a constant M such that

$$\sup_n \left\{ \left(\frac{3}{1-\beta_n} + \frac{3}{1-\gamma_n} \right) \Big(\|Q(x_n)\| + \|T(\delta_n x_n + (1-\delta_n)x_{n+1})\| + \|x_n\| \Big) \right\} \leq M.$$

Set $y_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ for all $n \geq 0$. Thus, we have

$$\begin{aligned} y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}Q(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})T(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2})}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n Q(x_n) + (1-\alpha_n-\beta_n)T(\delta_n x_n + (1-\delta_n)x_{n+1})}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(Q(x_{n+1}) - Q(x_n)) \\ &\quad + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}(T(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}) - T(\delta_n x_n + (1-\delta_n)x_{n+1})) \\ &\quad + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right)(Q(x_n) - T(\delta_n x_n + (1-\delta_n)x_{n+1})). \end{aligned}$$

It follows that

$$\begin{aligned} \|y_{n+1} - y_n\| \leq &\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right)\|Q(x_n) - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\ &+ \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}(\delta_{n+1}\|x_{n+1} - x_n\| + (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\|) \\ &+ \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}|\delta_{n+1} - \delta_n|(\|x_n\| + \|x_{n+1}\|) \\ &+ \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\|. \end{aligned} \tag{3.2}$$

From (3.1), we have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| = &\|\alpha_{n+1}(Q(x_{n+1}) - Q(x_n)) + (\alpha_{n+1} - \alpha_n)Q(x_n) \\ &+ \beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)(x_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})) \end{aligned}$$

$$\begin{aligned}
&+ \gamma_{n+1}(T(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})x_{n+2}) - T(\delta_n x_n + (1-\delta_n)x_{n+1})) \\
&+ (\alpha_n - \alpha_{n+1})T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
\leq\ & \alpha\alpha_{n+1}\|x_{n+1} - x_n\| + (\alpha_{n+1} + \alpha_n)\|Q(x_n)\| + \beta_{n+1}\|x_{n+1} - x_n\| \\
&+ \gamma_{n+1}(\delta_{n+1}\|x_{n+1} - x_n\| + (1-\delta_{n+1})\|x_{n+2} - x_{n+1}\|) \\
&+ \gamma_{n+1}|\delta_{n+1} - \delta_n|(\|x_n\| + \|x_{n+1}\|) + (\alpha_n + \alpha_{n+1})\|T(\delta_n x_n + (1-\delta_n)x_{n+1})\| \\
&+ |\beta_{n+1} - \beta_n|\|x_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| \leq\ & \left[1 - \frac{(1-\alpha)\alpha_{n+1}}{1 - \gamma_{n+1}(1-\delta_{n+1})}\right]\|x_{n+1} - x_n\| \\
&+ \frac{\alpha_{n+1} + \alpha_n}{1 - \gamma_{n+1}(1-\delta_{n+1})}(\|Q(x_n)\| + \|T(\delta_n x_n + (1-\delta_n)x_{n+1})\|) \\
&+ \frac{|\beta_{n+1} - \beta_n| + |\delta_{n+1} - \delta_n|}{1 - \gamma_{n+1}(1-\delta_{n+1})}(\|x_n - T(\delta_n x_n + (1-\delta_n)x_{n+1})\| + \|x_n\| + \|x_{n+1}\|) \\
\leq\ & \|x_{n+1} - x_n\| + M(\alpha_n + \alpha_{n+1} + |\beta_{n+1} - \beta_n| + |\delta_{n+1} - \delta_n|).
\end{aligned} \tag{3.3}$$

Substitute (3.3) into (3.2) to get

$$\|y_{n+1} - y_n\| \leq \left[1 - \frac{(1-\alpha)\alpha_{n+1}}{1 - \beta_{n+1}}\right]\|x_{n+1} - x_n\| + 3M(\alpha_{n+1} + \alpha_n + |\beta_{n+1} - \beta_n| + |\delta_{n+1} - \delta_n|).$$

Hence,

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

This together with Lemma 2.2 implies that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Note that

$$y_n - x_n = \frac{x_{n+1} - x_n}{1 - \beta_n}.$$

So,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Again, from (3.1), we have

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|Q(x_n) - Tx_n\| + \beta_n\|x_n - Tx_n\| \\
&\quad + \gamma_n(1-\delta_n)\|x_n - x_{n+1}\|.
\end{aligned}$$

It follows that

$$\|x_n - Tx_n\| \leq \frac{\alpha_n}{1 - \beta_n}\|Q(x_n) - Tx_n\| + \frac{1 + \gamma_n(1-\delta_n)}{1 - \beta_n}\|x_{n+1} - x_n\|.$$

This together with $(C1)$ and (3.4) imply that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.5}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle q - Q(q), q - x_n\rangle \leq 0, \tag{3.6}$$

where $q \in Fix(T)$ is the unique fixed point of the contraction $P_{Fix(T)}Q$, that is, $q = P_{Fix(T)}Q(q)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point $\breve{x}$ and

$$\begin{aligned}&\limsup_{n\to\infty}\langle P_{Fix(T)}Q(q)-Q(q),P_{Fix(T)}Q(q)-x_n\rangle\\&=\lim_{i\to\infty}\langle P_{Fix(T)}Q(q)-Q(q),P_{Fix(T)}Q(q)-x_{n_i}\rangle.\end{aligned}\tag{3.7}$$

By Lemma 2.1 and (3.5), we deduce $\breve{x}\in Fix(T)$. Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\langle P_{Fix(T)}Q(q)-Q(q),P_{Fix(T)}Q(q)-x_n\rangle\\&=\lim_{i\to\infty}\langle P_{Fix(T)}Q(q)-Q(q),P_{Fix(T)}Q(q)-x_{n_i}\rangle\\&=\langle P_{Fix(T)}Q(q)-Q(q),P_{Fix(T)}Q(q)-\breve{x}\rangle\\&\le 0.\end{aligned}$$

Finally, we prove that $x_n\to q$. From (3.1), we have

$$\begin{aligned}\|x_{n+1}-q\|^2&=\alpha_n\langle Q(x_n)-Q(q),x_{n+1}-q\rangle+\alpha_n\langle Q(q)-q,x_{n+1}-q\rangle\\&\quad+\gamma_n\langle T(\delta_n x_n+(1-\delta_n)x_{n+1})-q,x_{n+1}-q\rangle\\&\quad+\beta_n\langle x_n-q,x_{n+1}-q\rangle\\&\le\alpha_n\alpha\|x_n-q\|\|x_{n+1}-q\|+\alpha_n\langle Q(q)-q,x_{n+1}-q\rangle\\&\quad+\gamma_n(\delta_n\|x_n-q\|+(1-\delta_n)\|x_{n+1}-q\|)\|x_{n+1}-q\|\\&\quad+\beta_n\|x_n-q\|\|x_{n+1}-q\|\\&\le\frac{1-\gamma_n(1-\delta_n)-(1-\alpha)\alpha_n}{2}\|x_n-q\|^2+\frac{1+\gamma_n(1-\delta_n)-(1-\alpha)\alpha_n}{2}\|x_{n+1}-q\|^2\\&\quad+\alpha_n\langle Q(q)-q,x_{n+1}-q\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2\le&\left[1-\frac{2(1-\alpha)\alpha_n}{1-\gamma_n(1-\delta_n)+(1-\alpha)\alpha_n}\right]\|x_n-q\|^2\\&+\frac{2\alpha_n}{1-\gamma_n(1-\delta_n)+(1-\alpha)\alpha_n}\langle Q(q)-q,x_{n+1}-q\rangle.\end{aligned}\tag{3.8}$$

Applying Lemma 2.3 and (3.6) to (3.8) to deduce that $x_n\to q$. This completes the proof. □

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