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Uncountably many solutions of a third order nonlinear difference equation with neutral delay

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Abstract

In this paper, by using the Schauder fixed point theorem, Krasnoselskii fixed point theorem and some new techniques, we obtain the existence of uncountably many solutions for a third order nonlinear difference equation with neutral delay of the form

$$\begin{aligned}&\Delta\big(a(n, x_{a_{1n}}, x_{a_{2n}}, \ldots, x_{a_{rn}})\Delta^2(x_n + b_n x_{n-\tau})\big) + \Delta h\big(n, x_{h_{1n}}, x_{h_{2n}}, \ldots, x_{h_{kn}}\big)\\ &\quad + f\big(n, x_{f_{1n}}, x_{f_{2n}}, \ldots, x_{f_{kn}}\big) = c_n, \quad n \geq n_0.\end{aligned}$$

The results presented improve and generalize some results in literatures. Seven examples are given to illustrate the results presented in this paper.

Keywords: Third order nonlinear difference equation with neutral delay, uncountably many solutions, Schauder fixed point theorem, Krasnoselskii fixed point theorem.
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1. Introduction

In the past few years, many authors studied the oscillation, nonoscillation and existence of solutions for various linear and nonlinear difference equations with delays, see, for example, [1]-[16] and the references cited therein.

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Cheng [3] considered the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad \forall n \geq n_0, \tag{1.1}$$

where $p \in \mathbb{R} \setminus \{-1\}$. He obtained the existence of a bounded nonoscillatory solution by using the Banach fixed point theorem. Liu et al.[13] studied solvability of a second order nonlinear neutral delay difference equation

$$\Delta[a_n\Delta(x_n + b_n x_{n-\tau}) + f(n, x_{f_{1n}}, \dots, x_{f_{kn}})] + g(n, x_{g_{1n}}, \dots, x_{g_{kn}}) = c_n, \quad n \geq n_0. \tag{1.2}$$

Taking full advantage of the Schauder fixed point theorem, Yan and Liu [16] proved the existence of a bounded nonoscillatory solution for the third order difference equation

$$\Delta^3 x_n + f(n, x_n, x_{n-\tau}) = 0, \quad n \geq n_0. \tag{1.3}$$

In 2012, by applying the Leray-Schauder nonlinear alternative theorem, Liu et al.[8] established the existence results of bounded positive solutions for the third order difference equation

$$\Delta(a_n\Delta^2(x_n + p_n x_{n-\tau})) + f(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) = g_n, \quad n \geq n_0. \tag{1.4}$$

Inspired and motivated by the results in [1]-[16], in this paper, we are concerned with the third order nonlinear difference equation with neutral delay

$$\begin{aligned}\Delta(a(n, x_{a_{1n}}, x_{a_{2n}}, \dots, x_{a_{rn}})\Delta^2(x_n + b_n x_{n-\tau})) + \Delta h(n, x_{h_{1n}}, x_{h_{2n}}, \dots, x_{h_{kn}})\\ + f(n, x_{f_{1n}}, x_{f_{2n}}, \dots, x_{f_{kn}}) = c_n, \quad n \geq n_0,\end{aligned} \tag{1.5}$$

where $\tau, r, k \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{b_n\}_{n\in\mathbb{N}_{n_0}} \bigcup \{c_n\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, $a \in C(\mathbb{N}_{n_0} \times \mathbb{R}^r, \mathbb{R} \setminus \{0\})$, $h, f \in C(\mathbb{N}_{n_0} \times \mathbb{R}^k, \mathbb{R})$, a_{dn}, h_{ln} and $f_{ln} : \mathbb{N}_{n_0} \to \mathbb{N}$ with

$$\lim_{n\to\infty} a_{dn} = \lim_{n\to\infty} h_{ln} = \lim_{n\to\infty} f_{ln} = +\infty, \quad d \in \Lambda_r,\, l \in \Lambda_k.$$

By virtue of the Schauder fixed point theorem, Krasnoselskii fixed point theorem and some new techniques, we establish sufficient conditions for the existence results of uncountably many nonoscillatory, positive and negative solutions of Eq.(1.5), and construct seven nontrivial examples.

2. Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^3 x_n = \Delta(\Delta^2 x_n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of all positive integers and nonnegative integers, respectively,

$$\begin{aligned}\mathbb{N}_{n_0} &= \{n : n \in \mathbb{N}_0 \text{ with } n \geq n_0\}, \quad n_0 \in \mathbb{N}_0,\\ \Lambda_t &= \{1, 2, \dots, t\}, \quad t \in \mathbb{N},\\ \alpha &= \inf\{a_{dn}, h_{ln}, f_{ln} : d \in \Lambda_r, l \in \Lambda_k, n \in \mathbb{N}_{n_0}\},\\ \beta &= \min\{n_0 - \tau, \alpha\} \in \mathbb{N}.\end{aligned}$$

l_β^∞ represents the Banach space of all real sequences on $\mathbb{N}_\beta$ with norm

$$\|x\| = \sup_{n\in\mathbb{N}_\beta} \left|\frac{x_n}{n}\right| < +\infty \quad \text{for each } x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty.$$

For $d \in \mathbb{R}$ and $D > 0$, put

$$d_n = dn, \quad \forall n \in \mathbb{N}_\beta,$$

$$A(\{d_n\}_{n\in\mathbb{N}_\beta}, D) = \{\{x_n\}_{n\in\mathbb{N}_\beta} : \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty \text{and } |\frac{x_n}{n} - \frac{d_n}{n}| \leq D, \forall n \in \mathbb{N}_\beta\}.$$

It is easy to see that $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ is a closed and convex subset in l_β^∞.

By a solution of Eq.(1.5), we mean a sequence $\{x_n\}_{n\in\mathbb{N}_\beta}$ with a positive integer $T \geq n_0+\tau+\beta$ such that Eq.(1.5) is satisfied for all $n \geq T$. As is customary, a solution of Eq.(1.5) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

The following lemmas play important roles in this paper.

Lemma 2.1 ([4])**.** *A bounded, uniformly Cauchy subset Y of l_β^∞ is relatively compact.*

Lemma 2.2 (Krasnoselskii Fixed Point Theorem [5])**.** *Let Y be a nonempty bounded closed convex subset of a Banach space X and $S, G : Y \to X$ be mappings such that $Sx + Gy \in Y$ for every pair $x, y \in Y$. If S is a contraction and G is completely continuous, then the equation*

$$Sx + Gx = x$$

has a solution in Y.

Lemma 2.3 (Schauder Fixed Point Theorem [5])**.** *Let Y be a nonempty closed convex subset of a Banach space X and $S : Y \to Y$ be a continuous mapping such that $S(Y)$ is a relatively compact subset of X. Then S has a fixed point in Y.*

3. Existence of Uncountably Many Solutions

Now we show the existence of uncountably many nonoscillatory solutions for Eq.(1.5) by using the Schauder fixed point theorem.

Theorem 3.1. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}, d \in \mathbb{R}, D \in \mathbb{R}^+\backslash\{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$|d| > D, \quad b_n = -1, \quad \forall n \geq n_1; \tag{3.1}$$

$$\begin{aligned} &|a(n,u_1,u_2\dots,u_r)| \geq a_n, \quad \forall (n,u_d) \in \mathbb{N}_{n_0}\times(\mathbb{R}\setminus\{0\}),\, d\in\Lambda_r;\\ &|f(n,u_1,u_2\dots,u_k)| \leq P_n, \quad |h(n,u_1,u_2\dots,u_k)| \leq Q_n, \quad \forall (n,u_l)\in\mathbb{N}_{n_0}\times(\mathbb{R}\setminus\{0\}),\, l\in\Lambda_k; \end{aligned} \tag{3.2}$$

$$\sum_{j=1}^{\infty}\sum_{i=n_0+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\max\left\{Q_s, \sum_{t=s}^{\infty}\max\{P_t,|c_t|\}\right\} < +\infty. \tag{3.3}$$

Then for every $L \in (d-D, d+D)$, Eq.(1.5) has a nonoscillatory solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{n\to\infty}\frac{w_n}{n} = L$. Furthermore, Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Proof. Let $L \in (d-D, d+D)$. On account of Eq.(3.3), we infer that there exists $T \geq 1+n_0+n_1+\tau+\beta$ satisfying

$$\frac{1}{T}\sum_{j=1}^{\infty}\sum_{i=T+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)) < D-|L-d|. \tag{3.4}$$

Define a mapping $S_L : A(\{d_n\}_{n\in\mathbb{N}_\beta}, D) \to l_\beta^\infty$ by

$$S_Lx_n = \begin{cases} nL + \sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\big(h\big(s,x_{h_{1s}},\dots,x_{h_{ks}}\big) \\ \quad -\sum_{t=s}^{\infty}\big(f(t,x_{f_{1t}},\dots,x_{f_{kt}})-c_t\big)\big), \quad n\geq T,\\ \frac{n}{T}S_Lx_T, \quad \beta\leq n<T \end{cases} \tag{3.5}$$

for any $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Now we prove that

$$S_Lx \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D) \text{ and } \|S_Lx\| < |L| + D, \quad \forall x \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D). \tag{3.6}$$

Let $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$. It follows from (3.2), (3.4) and (3.5) that

$$\begin{aligned}\left|\frac{S_Lx_n}{n} - d\right| &= \left|L - d + \frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s, x_{a_{1s}}, \dots, x_{a_{rs}}\big)}\bigg(h\big(s, x_{h_{1s}}, \dots, x_{h_{ks}}\big)\right.\\ &\qquad \left. - \sum_{t=s}^{\infty}\big(f\big(t, x_{f_{1t}}, \dots, x_{f_{kt}}\big) - c_t\big)\bigg)\right|\\ &\le |L - d| + \frac{1}{T}\sum_{j=1}^{\infty}\sum_{i=T+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\bigg(Q_s + \sum_{t=s}^{\infty}\big(P_t + |c_t|\big)\bigg)\\ &< |L - d| + D - |L - d|\\ &= D, \quad n \ge T,\end{aligned}$$

$$\left|\frac{S_Lx_n}{n} - d\right| = \left|\frac{n}{T}\cdot\frac{S_Lx_T}{n} - d\right| = \left|\frac{S_Lx_T}{T} - d\right| < D, \quad \beta \le n < T,$$

$$\begin{aligned}\left|\frac{S_Lx_n}{n}\right| &= \left|L + \frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s, x_{a_{1s}}, \dots, x_{a_{rs}}\big)}\bigg(h\big(s, x_{h_{1s}}, \dots, x_{h_{ks}}\big)\right.\\ &\qquad \left. - \sum_{t=s}^{\infty}\big(f\big(t, x_{f_{1t}}, \dots, x_{f_{kt}}\big) - c_t\big)\bigg)\right|\\ &\le |L| + \frac{1}{T}\sum_{j=1}^{\infty}\sum_{i=T+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\bigg(Q_s + \sum_{t=s}^{\infty}\big(P_t + |c_t|\big)\bigg)\\ &< |L| + D - |L - d|\\ &< |L| + D, \quad n \ge T\end{aligned}$$

and

$$\left|\frac{S_Lx_n}{n}\right| = \left|\frac{n}{T}\cdot\frac{S_Lx_T}{n}\right| = \left|\frac{S_Lx_T}{T}\right| < |L| + D, \quad \beta \le n < T,$$

which imply (3.6).

Next we prove that S_L is continuous and $S_L(A(\{d_n\}_{n\in\mathbb{N}_\beta}, D))$ is uniformly Cauchy. Let $x^\nu = \{x_n^\nu\}_{n\in\mathbb{N}_\beta}$ and $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{\nu\to\infty} x^\nu = x$. Put $\varepsilon > 0$. Using (3.3) and the continuity of a, h and f, we deduce that there exist five positive integers T_1, T_2, T_3, T_4 and T_5 with $T_4 > T$, $T_2 > T_4 + T_1\tau$ and $T_3 > T_2 - 1$ satisfying

$$\begin{aligned}&\max\frac{1}{T}\Bigg\{\sum_{j=1}^{\infty}\sum_{i=T_4+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\bigg(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\bigg), \sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}(P_t + |c_t|),\\ &T_1(T_4 - T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\bigg(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\bigg), \sum_{j=T_1}^{\infty}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a_s}\bigg(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\bigg)\Bigg\} < \frac{\varepsilon}{25};\end{aligned} \tag{3.7}$$

$$\max\frac{1}{T}\Bigg\{\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\bigg(\big|h(s, x_{h_{1s}}^\nu, \dots, x_{h_{ks}}^\nu) - h(s, x_{h_{1s}}, \dots, x_{h_{ks}})\big|$$

$$
\begin{aligned}
&+\sum_{t=s}^{T_3-1}\left|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg),\\
&\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})-a(s,x_{a_{1s}},\dots,x_{a_{rs}})|}{a_s^2}\left(Q_s+\sum_{t=s}^{T_3-1}(P_t+|c_t|)\right)\Bigg\}<\frac{\varepsilon}{25}, \qquad (3.8)\\
&\forall\nu\ge T_5.
\end{aligned}
$$

By virtue of (3.2), (3.5), (3.7) and (3.8), we obtain that for each $\nu \ge T_5$

$$
\begin{aligned}
&\|S_Lx^{\nu}-S_Lx\|\\
&=\max\left\{\sup_{\beta\le n<T}\left|\frac{S_Lx^{\nu}_n}{n}-\frac{S_Lx_n}{n}\right|,\ \sup_{n\ge T}\left|\frac{S_Lx^{\nu}_n}{n}-\frac{S_Lx_n}{n}\right|\right\}\\
&=\sup_{n\ge T}\frac{1}{n}\Bigg|\sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})}\\
&\quad\times\left(h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})-\sum_{t=s}^{\infty}\left(f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})-c_t\right)\right)\\
&\quad-\sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\\
&\quad\times\left(h(s,x_{h_{1s}},\dots,x_{h_{ks}})-\sum_{t=s}^{\infty}\left(f(t,x_{f_{1t}},\dots,x_{f_{kt}})-c_t\right)\right)\Bigg|\\
&\le\sup_{n\ge T}\frac{1}{n}\Bigg[\sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})|}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})-h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right|\\
&\quad+\sum_{t=s}^{\infty}\left|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg)\\
&\quad+\sup_{n\ge T}\sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\left|\frac{1}{a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})}-\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\right|\\
&\quad\times\left(|h(s,x_{h_{1s}},\dots,x_{h_{ks}})|+\sum_{t=s}^{\infty}\left(|f(t,x_{f_{1t}},\dots,x_{f_{kt}})|+|c_t|\right)\right)\Bigg]\\
&\le\frac{1}{T}\Bigg[\sum_{j=1}^{\infty}\sum_{i=T+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})-h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right|\\
&\quad+\sum_{t=s}^{\infty}\left|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg)\\
&\quad+\sum_{j=1}^{\infty}\sum_{i=T+j\tau}^{\infty}\sum_{s=i}^{\infty}\left|\frac{1}{a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})}-\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\right|\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
&\le\frac{1}{T}\Bigg[\sum_{j=1}^{\infty}\sum_{i=T_4+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})\right|+\left|h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right|\\
&\quad+\sum_{t=s}^{\infty}\left(\left|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})\right|+\left|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\bigg(\big|h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})-h(s,x_{h_{1s}},\dots,x_{h_{ks}})\big|\\
&+\sum_{t=s}^{T_3-1}\big|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\dots,x_{f_{kt}})\big|\bigg)\\
&+\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}\big(\big|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})\big|+\big|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\big|\big)\\
&+\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=T_2}^{\infty}\frac{1}{a_s}\bigg(\big|h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})\big|+\big|h(s,x_{h_{1s}},\dots,x_{h_{ks}})\big|\\
&+\sum_{t=s}^{\infty}\big(\big|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})\big|+\big|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\big|\big)\bigg)\\
&+\sum_{j=T_1}^{\infty}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a_s}\bigg(\big|h(s,x^{\nu}_{h_{1s}},\dots,x^{\nu}_{h_{ks}})\big|+\big|h(s,x_{h_{1s}},\dots,x_{h_{ks}})\big|\\
&+\sum_{t=s}^{\infty}\big(\big|f(t,x^{\nu}_{f_{1t}},\dots,x^{\nu}_{f_{kt}})\big|+\big|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\big|\big)\bigg)\\
&+\sum_{j=1}^{\infty}\sum_{i=T_4+j\tau}^{\infty}\sum_{s=i}^{\infty}\left(\frac{1}{\big|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})\big|}+\frac{1}{\big|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\big|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&+\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{\big|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})-a(s,x_{a_{1s}},\dots,x_{a_{rs}})\big|}{a_s^2}\left(Q_s+\sum_{t=s}^{T_3-1}(P_t+|c_t|)\right)\\
&+\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\left(\frac{1}{\big|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})\big|}+\frac{1}{\big|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\big|}\right)\sum_{t=T_3}^{\infty}(P_t+|c_t|)\\
&+\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=T_2}^{\infty}\left(\frac{1}{\big|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})\big|}+\frac{1}{\big|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\big|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&+\sum_{j=T_1}^{\infty}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{\infty}\left(\frac{1}{\big|a(s,x^{\nu}_{a_{1s}},\dots,x^{\nu}_{a_{rs}})\big|}+\frac{1}{\big|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\big|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
\le{}&\frac{1}{T}\Bigg[2\sum_{j=1}^{\infty}\sum_{i=T_4+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)+\frac{\varepsilon}{25}+2\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}P_t\\
&+2T_1(T_4-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)+2\sum_{j=T_1}^{\infty}\sum_{i=T+j\tau}^{T_4+i\tau-1}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)\\
&+2\sum_{j=1}^{\infty}\sum_{i=T_4+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)+\frac{\varepsilon}{25}+2\sum_{j=1}^{T_1-1}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}(P_t+|c_t|)\\
&+2T_1(T_4-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)+2\sum_{j=T_1}^{\infty}\sum_{i=T+j\tau}^{T_4+j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
<{}&\varepsilon,
\end{aligned}
$$

which implies that $\lim_{\nu\to\infty}S_Lx^{\nu}=S_Lx$, that is, S_L is continuous in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.

Let $\varepsilon > 0$. It follows from (3.3) that there exists $T^* > T$ satisfying

$$\frac{1}{T^*}\sum_{j=1}^{\infty}\sum_{i=T^*+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)<\frac{\varepsilon}{2}, \tag{3.9}$$

which together with (3.2) and (3.5) yields that for all $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ and $t_1 > t_2 > T^*$

$$\begin{aligned}
&\left|\frac{S_L x_{t_1}}{t_1}-\frac{S_L x_{t_2}}{t_2}\right| \\
&=\left|\frac{1}{t_1}\sum_{j=1}^{\infty}\sum_{i=t_1+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\left(h(s,x_{h_{1s}},\dots,x_{h_{ks}})-\sum_{t=s}^{\infty}(f(t,x_{f_{1t}},\dots,x_{f_{kt}})-c_t)\right)\right.\\
&\quad\left.-\frac{1}{t_2}\sum_{j=1}^{\infty}\sum_{i=t_2+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\left(h(s,x_{h_{1s}},\dots,x_{h_{ks}})-\sum_{t=s}^{\infty}(f(t,x_{f_{1t}},\dots,x_{f_{kt}})-c_t)\right)\right|\\
&\le\frac{1}{t_1}\sum_{j=1}^{\infty}\sum_{i=t_1+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,x_{a_{1s}},\dots,x_{a_{rs}})|}\left(|h(s,x_{h_{1s}},\dots,x_{h_{ks}})|+\sum_{t=s}^{\infty}(|f(t,x_{f_{1t}},\dots,x_{f_{kt}})|+|c_t|)\right)\\
&\quad+\frac{1}{t_2}\sum_{j=1}^{\infty}\sum_{i=t_2+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,x_{a_{1s}},\dots,x_{a_{rs}})|}\left(|h(s,x_{h_{1s}},\dots,x_{h_{ks}})|+\sum_{t=s}^{\infty}(|f(t,x_{f_{1t}},\dots,x_{f_{kt}})|+|c_t|)\right)\\
&\le\frac{2}{T^*}\sum_{j=1}^{\infty}\sum_{i=T^*+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&<\varepsilon,
\end{aligned}$$

which means that $S_L(A(\{d_n\}_{n\in\mathbb{N}_\beta}, D))$ is uniformly Cauchy. Lemma 2.1 implies that $S_L(A(\{d_n\}_{n\in\mathbb{N}_\beta}, D))$ is relatively compact. It follows from Lemma 2.3 that the mapping S_L possesses a fixed point $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$, that is,

$$\begin{aligned}
w_n = nL &+ \sum_{j=1}^{\infty}\sum_{i=n+j\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,w_{a_{1s}},\dots,w_{a_{rs}})}\left(h(s,w_{h_{1s}},\dots,w_{h_{ks}})\right.\\
&\left.-\sum_{t=s}^{\infty}(f(t,w_{f_{1t}},\dots,w_{f_{kt}})-c_t)\right),\quad \forall n\ge T
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
w_{n-\tau} = (n-\tau)L &+ \sum_{j=1}^{\infty}\sum_{i=n+(j-1)\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,w_{a_{1s}},\dots,w_{a_{rs}})}\left(h(s,w_{h_{1s}},\dots,w_{h_{ks}})\right.\\
&\left.-\sum_{t=s}^{\infty}(f(t,w_{f_{1t}},\dots,w_{f_{kt}})-c_t)\right),\quad \forall n\ge T+\tau.
\end{aligned}\tag{3.11}$$

(3.10) and (3.11) lead to

$$\begin{aligned}
w_n - w_{n-\tau} = \tau L &- \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,w_{a_{1s}},\dots,w_{a_{rs}})}\left(h(s,w_{h_{1s}},\dots,w_{h_{ks}})\right.\\
&\left.-\sum_{t=s}^{\infty}(f(t,w_{f_{1t}},\dots,w_{f_{kt}})-c_t)\right),\quad \forall n\ge T+\tau,
\end{aligned}$$

$$\Delta(w_n - w_{n-\tau}) = \sum_{s=n}^{\infty} \frac{1}{a(s, w_{a_{1s}}, \ldots, w_{a_{rs}})} \Bigg(h(s, w_{h_{1s}}, \ldots, w_{h_{ks}}) - \sum_{t=s}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T + \tau,$$

$$\Delta^2(w_n - w_{n-\tau}) = -\frac{1}{a(n, w_{a_{1n}}, \ldots, w_{a_{rn}})} \Bigg(h(n, w_{h_{1n}}, \ldots, w_{h_{kn}}) - \sum_{t=n}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T + \tau,$$

which together with (3.1) yields that

$$\Delta(a(n, w_{a_{1n}}, \ldots, w_{a_{rn}})\Delta^2(w_n + b_n w_{n-\tau})) + \Delta h(n, w_{h_{1n}}, \ldots, w_{h_{kn}}) + f(n, w_{f_{1n}}, \ldots, w_{f_{kn}}) = c_n, \quad \forall n \geq T + \tau.$$

That is, $w = \{w_n\}_{n \in \mathbb{N}_\beta}$ is a nonoscillatory solution of Eq.(1.5) in $A(\{d_n\}_{n \in \mathbb{N}_\beta}, D)$.

In view of (3.3) and (3.10), we obtain that

$$\begin{aligned} \left| \frac{w_n}{n} - L \right| &\leq \frac{1}{n} \sum_{j=1}^{\infty} \sum_{i=n+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{|a(s, w_{a_{1s}}, \ldots, w_{a_{rs}})|} \Bigg(\left| h(s, w_{h_{1s}}, \ldots, w_{h_{ks}}) \right| \\ &\quad + \sum_{t=s}^{\infty} \left(\left| f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) \right| + |c_t| \right) \Bigg) \\ &< \frac{1}{n} \sum_{j=1}^{\infty} \sum_{i=n+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \Bigg(Q_s + \sum_{t=s}^{\infty} (P_t + |c_t|) \Bigg) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{3.12}$$

that is, $\lim_{n\to\infty} \frac{w_n}{n} = L$.

Next we show that Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n \in \mathbb{N}_\beta}, D)$. Let $L_1, L_2 \in (d - D, d + D)$ and $L_1 \neq L_2$. Similarly we infer that for each $l \in \Lambda_2$, there exist a constant $T_l^* \geqq 1 + n_0 + n_1 + \tau + \beta$ and a mapping S_{L_l} satisfying (3.5), where L, T and S_L are replaced by L_l, T_l^* and S_{L_l}, respectively, the mapping S_{L_l} has a fixed point $w^l = \{w_n^l\}_{n \in \mathbb{N}_\beta} \in A(\{d_n\}_{n \in \mathbb{N}_\beta}, D)$, which is a nonoscillatory solution of Eq.(1.5), that is,

$$\begin{aligned} w_n^l = nL_l + \sum_{j=1}^{\infty} \sum_{i=n+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a(s, w_{a_{1s}}^l, \ldots, w_{a_{rs}}^l)} \Bigg(h\Big(s, w_{h_{1s}}^l, \ldots, w_{h_{ks}}^l\Big) \\ - \sum_{t=s}^{\infty} \Big(f\Big(t, w_{f_{1t}}^l, \ldots, w_{f_{kt}}^l\Big) - c_t \Big) \Bigg), \quad \forall n \geq T_l^*, \quad l \in \Lambda_2. \end{aligned} \tag{3.13}$$

Note that (3.3) implies that there exists $T_3^* > \max\{T_1^*, T_2^*\}$ satisfying

$$\frac{1}{T_3^*} \sum_{j=1}^{\infty} \sum_{i=T_3^*+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \Bigg(Q_s + \sum_{t=s}^{\infty} (P_t + |c_t|) \Bigg) < \frac{|L_1 - L_2|}{4}, \tag{3.14}$$

which together with (3.2), (3.13) gives that

$$\left| \frac{w_n^1}{n} - \frac{w_n^2}{n} \right| = \Bigg| L_1 + \frac{1}{n} \sum_{j=1}^{\infty} \sum_{i=n+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a\big(s, w_{a_{1s}}^1, \ldots, w_{a_{rs}}^1\big)}$$

$$
\begin{aligned}
&\times \left(h\Big(s, w^1_{h_{1s}}, \dots, w^1_{h_{ks}}\Big) - \sum_{t=s}^{\infty} \Big(f\Big(t, w^1_{f_{1t}}, \dots, w^1_{f_{kt}}\Big) - c_t \Big) \right) \\
&- L_2 - \frac{1}{n} \sum_{j=1}^{\infty} \sum_{i=n+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a(s, w^2_{a_{1s}}, \dots, w^2_{a_{rs}})} \\
&\times \left(h\Big(s, w^2_{h_{1s}}, \dots, w^2_{h_{ks}}\Big) - \sum_{t=s}^{\infty} \Big(f\Big(t, w^2_{f_{1t}}, \dots, w^2_{f_{kt}}\Big) - c_t \Big) \right) \Bigg| \\
\geq{}& |L_1 - L_2| - \frac{1}{n} \sum_{j=1}^{\infty} \sum_{i=n+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \left(\Big| h\Big(s, w^1_{h_{1s}}, \dots, w^1_{h_{ks}}\Big) \Big| + \Big| h\Big(s, w^2_{h_{1s}}, \dots, w^2_{h_{ks}}\Big) \Big| \right. \\
&\left. + \sum_{t=s}^{\infty} \Big(\Big| f\Big(t, w^1_{f_{1t}}, \dots, w^1_{f_{kt}}\Big) \Big| + \Big| c_t \Big| + \Big| f\Big(t, w^2_{f_{1t}}, \dots, w^2_{f_{kt}}\Big) \Big| + \Big| c_t \Big| \Big) \right) \\
\geq{}& |L_1 - L_2| - \frac{2}{T_3^*} \sum_{j=1}^{\infty} \sum_{i=T_3^*+j\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \left(Q_s + \sum_{t=s}^{\infty} (P_t + |c_t|) \right) \\
>{}& \frac{1}{2} |L_1 - L_2| > 0, \quad \forall n \geq T_3^*,
\end{aligned}
$$

that is, $w^1 \neq w^2$. Therefore, Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$. This completes the proof. □

Theorem 3.2. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}, d \in \mathbb{R}, D \in \mathbb{R}^+ \backslash \{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2) *and*

$$
|d| > D, \quad b_n = 1, \quad \forall n \geq n_1; \tag{3.15}
$$

$$
\sum_{i=n_0}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \max \left\{ Q_s, \sum_{t=s}^{\infty} \max \{P_t, |c_t|\} \right\} < +\infty. \tag{3.16}
$$

Then for every $L \in (d-D, d+D)$, Eq.(1.5) *has a nonoscillatory solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{n\to\infty} \frac{w_n}{n} = L$. Furthermore, Eq.*(1.5) *has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.*

Proof. Let $L \in (d-D, d+D)$. It follows from (3.16) that there exists $T \geq 1 + n_0 + n_1 + \tau + \beta$ satisfying

$$
\frac{1}{T} \sum_{i=T}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \left(Q_s + \sum_{t=s}^{\infty} \big(P_t + |c_t|\big) \right) < D - |L - d|. \tag{3.17}
$$

Define a mapping $S_L : A(\{d_n\}_{n\in\mathbb{N}_\beta}, D) \to l^\infty_\beta$ by

$$
S_L x_n = \begin{cases} nL - \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)\tau}^{n+2j\tau-1} \sum_{s=i}^{\infty} \frac{1}{a(s, x_{a_{1s}}, \dots, x_{a_{rs}})} \big(h\big(s, x_{h_{1s}}, \dots, x_{h_{ks}}\big) \\ \quad - \sum_{t=s}^{\infty} \big(f\big(t, x_{f_{1t}}, \dots, x_{f_{kt}}\big) - c_t \big) \big), \quad n \geq T, \\ \frac{n}{T} S_L x_T, \quad \beta \leq n < T \end{cases} \tag{3.18}
$$

for any $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$. Combining (3.2), (3.17) and (3.18), we get that for any $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$

$$\begin{aligned}\left|\frac{S_L x_n}{n}-d\right| &= \left|L-d-\frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a\big(s,x_{a_{1s}},\ldots,x_{a_{rs}}\big)}\Bigg(h\big(s,x_{h_{1s}},\ldots,x_{h_{ks}}\big)\right.\\ &\qquad \left.-\sum_{t=s}^{\infty}\big(f\big(t,x_{f_{1t}},\ldots,x_{f_{kt}}\big)-c_t\big)\Bigg)\right|\\ &\le |L-d|+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}\big(P_t+|c_t|\big)\right)\\ &< |L-d|+D-|L-d|\\ &= D, \quad n\ge T,\end{aligned}$$

$$\left|\frac{S_L x_n}{n}-d\right|=\left|\frac{n}{T}\cdot\frac{S_L x_T}{n}-d\right|=\left|\frac{S_L x_T}{T}-d\right|<D, \quad \beta\le n<T,$$

$$\begin{aligned}\left|\frac{S_L x_n}{n}\right| &= \left|L-\frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a\big(s,x_{a_{1s}},\ldots,x_{a_{rs}}\big)}\Bigg(h\big(s,x_{h_{1s}},\ldots,x_{h_{ks}}\big)\right.\\ &\qquad \left.-\sum_{t=s}^{\infty}\big(f\big(t,x_{f_{1t}},\ldots,x_{f_{kt}}\big)-c_t\big)\Bigg)\right|\\ &\le |L|+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}\big(P_t+|c_t|\big)\right)\\ &< |L|+D-|L-d|\\ &< |L|+D, \quad n\ge T\end{aligned}$$

and

$$\left|\frac{S_L x_n}{n}\right|=\left|\frac{n}{T}\cdot\frac{S_L x_T}{n}\right|=\left|\frac{S_L x_T}{T}\right|<|L|+D, \quad \beta\le n<T,$$

which yield (3.6).

Now we assert that S_L is continuous and $S_L(A(\{d_n\}_{n\in\mathbb{N}_\beta},D))$ is uniformly Cauchy. Let $x^\nu=\{x_n^\nu\}_{n\in\mathbb{N}_\beta}$ and $x=\{x_n\}_{n\in\mathbb{N}_\beta}\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$ with $\lim_{\nu\to\infty}x^\nu=x$. Let $\varepsilon>0$. It follows from (3.16) and the continuity of a, h and f that there exist T_1, T_2, T_3 and $T_4\in\mathbb{N}$ with $T_1>T$, $T_2>T_1+\tau-1$ and $T_3>T_2-1$ satisfying

$$\begin{aligned}&\max\frac{1}{T}\Bigg\{\sum_{i=T_1+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right),\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}(P_t+|c_t|),\\ &\quad (T_1-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg\}<\frac{\varepsilon}{16};\end{aligned}\tag{3.19}$$

$$\begin{aligned}&\max\frac{1}{T}\Bigg\{\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\Bigg(\big|h(s,x^\nu_{h_{1s}},\ldots,x^\nu_{h_{ks}})-h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\big|\\ &\quad+\sum_{t=s}^{T_3-1}\big|f(t,x^\nu_{f_{1t}},\ldots,x^\nu_{f_{kt}})-f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\big|\Bigg),\\ &\quad\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{|a(s,x^\nu_{a_{1s}},\ldots,x^\nu_{a_{rs}})-a(s,x_{a_{1s}},\ldots,x_{a_{rs}})|}{a_s^2}\left(Q_s+\sum_{t=s}^{T_3-1}(P_t+|c_t|)\right)\Bigg\}<\frac{\varepsilon}{16},\\ &\quad\forall\nu\ge T_4.\end{aligned}\tag{3.20}$$

It follows from (3.2) and (3.18)-(3.20) that for any $\nu \geq T_4$

$$
\begin{aligned}
&\left\|S_L x^\nu - S_L x\right\| \\
&= \max\left\{\sup_{\beta\leq n<T}\left|\frac{S_L x_n^\nu}{n} - \frac{S_L x_n}{n}\right|, \sup_{n\geq T}\left|\frac{S_L x_n^\nu}{n} - \frac{S_L x_n}{n}\right|\right\} \\
&= \sup_{n\geq T}\frac{1}{n}\left|\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}}^\nu,\dots,x_{a_{rs}}^\nu)}\Bigg(h(s,x_{h_{1s}}^\nu,\dots,x_{h_{ks}}^\nu) - h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right. \\
&\quad - \sum_{t=s}^{\infty}\left(f(t,x_{f_{1t}}^\nu,\dots,x_{f_{kt}}^\nu) - f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right)\Bigg) \\
&\quad + \sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\left(\frac{1}{a(s,x_{a_{1s}}^\nu,\dots,x_{a_{rs}}^\nu)} - \frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\right) \\
&\quad \left.\times\left(h(s,x_{h_{1s}},\dots,x_{h_{ks}}) - \sum_{t=s}^{\infty}\left(f(t,x_{f_{1t}},\dots,x_{f_{kt}}) - c_t\right)\right)\right| \\
&\leq \frac{1}{T}\Bigg[\sup_{n\geq T}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{|a(s,x_{a_{1s}}^\nu,\dots,x_{a_{rs}}^\nu)|}\Bigg(\left|h(s,x_{h_{1s}}^\nu,\dots,x_{h_{ks}}^\nu) - h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right| \\
&\quad + \sum_{t=s}^{\infty}\left|f(t,x_{f_{1t}}^\nu,\dots,x_{f_{kt}}^\nu) - f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg) \\
&\quad + \sup_{n\geq T}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\left|\frac{1}{a(s,x_{a_{1s}}^\nu,\dots,x_{a_{rs}}^\nu)} - \frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\right| \\
&\quad \times\left(|h(s,x_{h_{1s}},\dots,x_{h_{ks}})| + \sum_{t=s}^{\infty}\left(|f(t,x_{f_{1t}},\dots,x_{f_{kt}})| + |c_t|\right)\right)\Bigg] \\
&\leq \frac{1}{T}\Bigg[\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x_{h_{1s}}^\nu,\dots,x_{h_{ks}}^\nu) - h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right| \\
&\quad + \sum_{t=s}^{\infty}\left|f(t,x_{f_{1t}}^\nu,\dots,x_{f_{kt}}^\nu) - f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg) \\
&\quad + \sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}\left|\frac{1}{a(s,x_{a_{1s}}^\nu,\dots,x_{a_{rs}}^\nu)} - \frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\right|\left(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\right)\Bigg] \\
&\leq \frac{1}{T}\Bigg[\sum_{i=T_1+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x_{h_{1s}}^\nu,\dots,x_{h_{ks}}^\nu)\right| + \left|h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right| \\
&\quad + \sum_{t=s}^{\infty}\left(\left|f(t,x_{f_{1t}}^\nu,\dots,x_{f_{kt}}^\nu)\right| + \left|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\right)\Bigg) \\
&\quad + \sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\Bigg(\left|h(s,x_{h_{1s}}^\nu,\dots,x_{h_{ks}}^\nu) - h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right| \\
&\quad + \sum_{t=s}^{T_3-1}\left|f(t,x_{f_{1t}}^\nu,\dots,x_{f_{kt}}^\nu) - f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}\left(\left|f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})\right|+\left|f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right|\right)\\
&+\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=T_2}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\ldots,x^{\nu}_{h_{ks}})\right|+\left|h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\right|\\
&+\sum_{t=s}^{\infty}\left(\left|f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})\right|+\left|f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right|\right)\Bigg)\\
&+\sum_{i=T_1+\tau}^{\infty}\sum_{s=i}^{\infty}\left(\frac{1}{\left|a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})\right|}+\frac{1}{\left|a(s,x_{a_{1s}},\ldots,x_{a_{rs}})\right|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&+\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{\left|a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})-a(s,x_{a_{1s}},\ldots,x_{a_{rs}})\right|}{a_s^2}\left(Q_s+\sum_{t=s}^{T_3-1}(P_t+|c_t|)\right)\\
&+\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\left(\frac{1}{\left|a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})\right|}+\frac{1}{\left|a(s,x_{a_{1s}},\ldots,x_{a_{rs}})\right|}\right)\sum_{t=T_3}^{\infty}(P_t+|c_t|)\\
&+\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=T_2}^{\infty}\left(\frac{1}{\left|a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})\right|}+\frac{1}{\left|a(s,x_{a_{1s}},\ldots,x_{a_{rs}})\right|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
&\le\frac{1}{T}\Bigg[2\sum_{i=T_1+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)+\frac{\varepsilon}{16}+2\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}P_t\\
&+2(T_1-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)+2\sum_{i=T_1+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&+\frac{\varepsilon}{16}+2\sum_{i=T+\tau}^{T_1+\tau-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}(P_t+|c_t|)+2(T_1-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
&<\varepsilon,
\end{aligned}
$$

which gives that $\lim_{\nu\to\infty} S_L x^{\nu} = S_L x$, that is, S_L is continuous in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Given $\varepsilon > 0$. Using (3.16), we infer that there exists $T^* > T$ satisfying

$$
\frac{1}{T^*}\sum_{i=T^*+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)<\frac{\varepsilon}{2}, \tag{3.21}
$$

which together with (3.2) and (3.18) implies that for all $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ and $t_1 > t_2 > T^*$

$$
\begin{aligned}
&\left|\frac{S_L x_{t_1}}{t_1}-\frac{S_L x_{t_2}}{t_2}\right|\\
&=\Bigg|\frac{1}{t_1}\sum_{j=1}^{\infty}\sum_{i=t_1+(2j-1)\tau}^{t_1+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\ldots,x_{a_{rs}})}\left(h(s,x_{h_{1s}},\ldots,x_{h_{ks}})-\sum_{t=s}^{\infty}(f(t,x_{f_{1t}},\ldots,x_{f_{kt}})-c_t)\right)\\
&-\frac{1}{t_2}\sum_{j=1}^{\infty}\sum_{i=t_2+(2j-1)\tau}^{t_2+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\ldots,x_{a_{rs}})}\left(h(s,x_{h_{1s}},\ldots,x_{h_{ks}})-\sum_{t=s}^{\infty}(f(t,x_{f_{1t}},\ldots,x_{f_{kt}})-c_t)\right)\Bigg|\\
&\le\frac{1}{t_1}\sum_{j=1}^{\infty}\sum_{i=t_1+(2j-1)\tau}^{t_1+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{\left|a(s,x_{a_{1s}},\ldots,x_{a_{rs}})\right|}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(|h(s, x_{h_{1s}}, \ldots, x_{h_{ks}})| + \sum_{t=s}^{\infty} (|f(t, x_{f_{1t}}, \ldots, x_{f_{kt}})| + |c_t|) \right) \\
&+ \frac{1}{t_2} \sum_{j=1}^{\infty} \sum_{i=t_2+(2j-1)\tau}^{t_2+2j\tau-1} \sum_{s=i}^{\infty} \frac{1}{|a(s, x_{a_{1s}}, \ldots, x_{a_{rs}})|} \\
&\times \left(|h(s, x_{h_{1s}}, \ldots, x_{h_{ks}})| + \sum_{t=s}^{\infty} (|f(t, x_{f_{1t}}, \ldots, x_{f_{kt}})| + |c_t|) \right) \\
\leq\ & \frac{2}{T^*} \sum_{i=T^*+\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \left(Q_s + \sum_{t=s}^{\infty} (P_t + |c_t|) \right) \\
<\ & \varepsilon,
\end{aligned}
$$

which yields that $S_L(A(\{d_n\}_{n\in\mathbb{N}_\beta}, D))$ is uniformly Cauchy. Lemma 2.1 means that $S_L(A(\{d_n\}_{n\in\mathbb{N}_\beta}, D))$ is relatively compact. Lemma 2.3 ensures that the mapping S_L has a fixed point $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$, that is,

$$
\begin{aligned}
w_n = nL - \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)\tau}^{n+2j\tau-1} \sum_{s=i}^{\infty} \frac{1}{a(s, w_{a_{1s}}, \ldots, w_{a_{rs}})} \Bigg(h(s, w_{h_{1s}}, \ldots, w_{h_{ks}}) \\
- \sum_{t=s}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
w_{n-\tau} = (n-\tau)L - \sum_{j=1}^{\infty} \sum_{i=n+(2j-2)\tau}^{n+(2j-1)\tau-1} \sum_{s=i}^{\infty} \frac{1}{a(s, w_{a_{1s}}, \ldots, w_{a_{rs}})} \Bigg(h(s, w_{h_{1s}}, \ldots, w_{h_{ks}}) \\
- \sum_{t=s}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T+\tau,
\end{aligned}
$$

which lead to

$$
\begin{aligned}
w_n + w_{n-\tau} = (2n-\tau)L - \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a(s, w_{a_{1s}}, \ldots, w_{a_{rs}})} \Bigg(h(s, w_{h_{1s}}, \ldots, w_{h_{ks}}) \\
- \sum_{t=s}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T+\tau,
\end{aligned}
$$

$$
\begin{aligned}
\Delta(w_n + w_{n-\tau}) = 2L + \sum_{s=n}^{\infty} \frac{1}{a(s, w_{a_{1s}}, \ldots, w_{a_{rs}})} \Bigg(h(s, w_{h_{1s}}, \ldots, w_{h_{ks}}) \\
- \sum_{t=s}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T+\tau,
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2(w_n + w_{n-\tau}) = -\frac{1}{a(n, w_{a_{1n}}, \ldots, w_{a_{rn}})} \Bigg(h(n, w_{h_{1n}}, \ldots, w_{h_{kn}}) \\
- \sum_{t=n}^{\infty} (f(t, w_{f_{1t}}, \ldots, w_{f_{kt}}) - c_t) \Bigg), \quad \forall n \geq T+\tau,
\end{aligned}
$$

which together with (3.15) yields that

$$\Delta(a(n, w_{a_{1n}}, \dots, w_{a_{rn}})\Delta^2(w_n + b_n w_{n-\tau})) + \Delta h(n, w_{h_{1n}}, \dots, w_{h_{kn}}) \\ + f(n, w_{f_{1n}}, \dots, w_{f_{kn}}) = c_n, \quad \forall n \geq T + \tau.$$

That is, $w = \{w_n\}_{n\in\mathbb{N}_\beta}$ is a nonoscillatory solution of Eq.(1.5) in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

It follows from (3.16) and (3.22) that

$$\begin{aligned}\left|\frac{w_n}{n} - L\right| &\leq \frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{|a(s, w_{a_{1s}}, \dots, w_{a_{rs}})|}\Bigg(|h(s, w_{h_{1s}}, \dots, w_{h_{ks}})| \\ &\quad + \sum_{t=s}^{\infty}\left(|f(t, w_{f_{1t}}, \dots, w_{f_{kt}})| + |c_t|\right)\Bigg) \\ &< \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\right) \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that $\lim_{n\to\infty}\frac{w_n}{n} = L$.

Next we claim that Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$. Let $L_1, L_2 \in (d - D, d + D)$ and $L_1 \neq L_2$. We deduce similarly that for every $l \in \Lambda_2$, there exist a constant $T_l^* \geqq 1 + n_0 + n_1 + \tau + \beta$ and a mapping S_{L_l} satisfying (3.18), where L, T and S_L are replaced by L_l, T_l^* and S_{L_l}, respectively, the mapping S_{L_l} has a fixed point $w^l = \{w_n^l\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$, which is a nonoscillatory solution of Eq.(1.5), that is,

$$\begin{aligned}w_n^l = nL_l - \sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a\big(s, w_{a_{1s}}^l, \dots, w_{a_{rs}}^l\big)}\Bigg(h\Big(s, w_{h_{1s}}^l, \dots, w_{h_{ks}}^l\Big) \\ - \sum_{t=s}^{\infty}\Big(f\Big(t, w_{f_{1t}}^l, \dots, w_{f_{kt}}^l\Big) - c_t\Big)\Bigg), \quad \forall n \geq T_l^*, \quad l \in \Lambda_2.\end{aligned} \tag{3.23}$$

Obviously (3.16) yields that there exists $T_3^* > \max\{T_1^*, T_2^*\}$ satisfying

$$\frac{1}{T_3^*}\sum_{i=T_3^*+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\right) < \frac{|L_1 - L_2|}{4}, \tag{3.24}$$

which together with (3.2), (3.23) means that

$$\begin{aligned}\left|\frac{w_n^1}{n} - \frac{w_n^2}{n}\right| &= \Bigg|L_1 - \frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a\big(s, w_{a_{1s}}^1, \dots, w_{a_{rs}}^1\big)} \\ &\quad \times \Bigg(h\Big(s, w_{h_{1s}}^1, \dots, w_{h_{ks}}^1\Big) - \sum_{t=s}^{\infty}\Big(f\Big(t, w_{f_{1t}}^1, \dots, w_{f_{kt}}^1\Big) - c_t\Big)\Bigg) \\ &\quad - L_2 + \frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a\big(s, w_{a_{1s}}^2, \dots, w_{a_{rs}}^2\big)} \\ &\quad \times \Bigg(h\Big(s, w_{h_{1s}}^2, \dots, w_{h_{ks}}^2\Big) - \sum_{t=s}^{\infty}\Big(f\Big(t, w_{f_{1t}}^2, \dots, w_{f_{kt}}^2\Big) - c_t\Big)\Bigg)\Bigg| \\ &\geq |L_1 - L_2| - \frac{1}{n}\sum_{j=1}^{\infty}\sum_{i=n+(2j-1)\tau}^{n+2j\tau-1}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\Big|h\Big(s, w_{h_{1s}}^1, \dots, w_{h_{ks}}^1\Big)\Big| + \Big|h\Big(s, w_{h_{1s}}^2, \dots, w_{h_{ks}}^2\Big)\Big|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=s}^{\infty}\left(\left|f\left(t,w^1_{f_{1t}},\ldots,w^1_{f_{kt}}\right)\right|+\left|c_t\right|+\left|f\left(t,w^2_{f_{1t}},\ldots,w^2_{f_{kt}}\right)\right|+\left|c_t\right|\right)\Bigg)\\
&\geq |L_1-L_2|-\frac{2}{T_3^*}\sum_{i=T_3^*+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&>\frac{1}{2}|L_1-L_2|>0,\quad \forall n\geq T_3^*,
\end{aligned}
$$

that is, $w^1\neq w^2$. Therefore, Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$. This completes the proof. □

Now we use the Krasnoselskii fixed point theorem to prove the existence of uncountably many nonoscillatory, positive and negative solutions of Eq.(1.5).

Theorem 3.3. *Assume that there exist $n_1\in\mathbb{N}_{n_0}, d\in\mathbb{R}, D,b\in\mathbb{R}^+\setminus\{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2), (3.16) *and*

$$1<\frac{|d|}{D}<\frac{1-b}{b},\quad |b_n|\leq b,\quad \forall n\geq n_1. \tag{3.25}$$

Then for every $L\in\big(d-(1-b)D+b|d|,d+(1-b)D-b|d|\big)$, Eq.(1.5) *has a nonoscillatory solution $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$ with $\lim_{n\to\infty}\left(\frac{w_n}{n}+\frac{b_n}{n}w_{n-\tau}\right)=L$. Furthermore, Eq.*(1.5) *has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.*

Proof. Let $L\in\big(d-(1-b)D+b|d|,d+(1-b)D-b|d|\big)$. In view of (3.16), we deduce that there exists $T\geq 1+n_0+n_1+\tau+\beta$ satisfying

$$\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}\big(P_t+|c_t|\big)\right)<(1-b)D-b|d|-|L-d|. \tag{3.26}$$

Define two mappings S_L and $G_L:A(\{d_n\}_{n\in\mathbb{N}_\beta},D)\to l^\infty_\beta$ by

$$S_Lx_n=\begin{cases}nL-b_nx_{n-\tau}, & n\geq T,\\ \frac{n}{T}S_Lx_T, & \beta\leq n<T\end{cases} \tag{3.27}$$

and

$$G_Lx_n=\begin{cases}-\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\ldots,x_{a_{rs}})}\big(h\big(s,x_{h_{1s}},\ldots,x_{h_{ks}}\big)\\ \quad-\sum_{t=s}^{\infty}\big(f\big(t,x_{f_{1t}},\ldots,x_{f_{kt}}\big)-c_t\big)\big), \quad n\geq T,\\ \frac{n}{T}G_Lx_T,\quad \beta\leq n<T\end{cases} \tag{3.28}$$

for any $x=\{x_n\}_{n\in\mathbb{N}_\beta}\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.

Now we assert that

$$S_Lx+G_Ly\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D),\quad \forall x,y\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D), \tag{3.29}$$

$$\|S_Lx-S_Ly\|\leq b\|x-y\|,\quad \forall x,y\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D) \tag{3.30}$$

and

$$\|G_Lx\|<D,\quad \forall x\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D). \tag{3.31}$$

In light of (3.2) and (3.25)-(3.28), we get that for any $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ and $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$

$$\begin{aligned}\left|\frac{S_Lx_n}{n}+\frac{G_Ly_n}{n}-d\right| &= \left|L-d-b_n\frac{x_{n-\tau}}{n}-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,y_{a_{1s}},\ldots,y_{a_{rs}}\big)}\Bigg(h\big(s,y_{h_{1s}},\ldots,y_{h_{ks}}\big)\right.\\ &\qquad \left.-\sum_{t=s}^{\infty}\big(f\big(t,y_{f_{1t}},\ldots,y_{f_{kt}}\big)-c_t\big)\Bigg)\right|\\ &\le |L-d|+b(|d|+D)+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(Q_s+\sum_{t=s}^{\infty}\big(P_t+|c_t|\big)\Bigg)\\ &< |L-d|+b(|d|+D)+(1-b)D-b|d|-|L-d|\\ &= D,\quad n\ge T,\end{aligned}$$

$$\left|\frac{S_Lx_n}{n}+\frac{G_Ly_n}{n}-d\right| = \left|\frac{n}{T}\cdot\frac{S_Lx_T}{n}+\frac{n}{T}\cdot\frac{G_Ly_T}{n}-d\right| = \left|\frac{S_Lx_T}{T}+\frac{G_Ly_T}{T}-d\right| < D,\quad \beta\le n<T,$$

$$\left|\frac{S_Lx_n}{n}-\frac{S_Ly_n}{n}\right| = \left|\frac{(n-\tau)b_n}{n}\left(\frac{x_{n-\tau}}{n-\tau}-\frac{y_{n-\tau}}{n-\tau}\right)\right| \le b\|x-y\|,\quad n\ge T,$$

$$\left|\frac{S_Lx_n}{n}-\frac{S_Ly_n}{n}\right| = \left|\frac{n}{T}\cdot\frac{S_Lx_T}{n}-\frac{n}{T}\cdot\frac{S_Ly_T}{n}\right| = \left|\frac{S_Lx_T}{T}-\frac{S_Ly_T}{T}\right| \le b\|x-y\|,\quad \beta\le n<T,$$

$$\begin{aligned}\left|\frac{G_Lx_n}{n}\right| &= \left|\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,x_{a_{1s}},\ldots,x_{a_{rs}}\big)}\Bigg(h\big(s,x_{h_{1s}},\ldots,x_{h_{ks}}\big)-\sum_{t=s}^{\infty}\big(f\big(t,x_{f_{1t}},\ldots,x_{f_{kt}}\big)-c_t\big)\Bigg)\right|\\ &\le \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(Q_s+\sum_{t=s}^{\infty}\big(P_t+|c_t|\big)\Bigg)\\ &< (1-b)D-b|d|-|L-d|\\ &< D,\quad n\ge T\end{aligned}$$

and

$$\left|\frac{G_Lx_n}{n}\right| = \left|\frac{n}{T}\cdot\frac{G_Lx_T}{n}\right| = \left|\frac{G_Lx_T}{T}\right| < D,\quad \beta\le n<T,$$

which imply (3.29)-(3.31).

Next we show that G_L is continuous and $G_L(A(\{d_n\}_{n\in\mathbb{N}_\beta}, D))$ is uniformly Cauchy. Let $x^\nu = \{x_n^\nu\}_{n\in\mathbb{N}_\beta}$ and $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{\nu\to\infty} x^\nu = x$. By virtue of (3.16) and the continuity of a, h and f, we infer that for given $\varepsilon > 0$ there exist T_1, T_2, T_3 and $T_4 \in \mathbb{N}$ with $T_1 > T$, $T_2 > T_1 - 1$ and $T_3 > T_2 - 1$ satisfying

$$\begin{aligned}\max\frac{1}{T}\Bigg\{&\sum_{i=T_1}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\Bigg),\ \sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}(P_t+|c_t|),\\ &(T_1-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\Bigg(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\Bigg)\Bigg\} < \frac{\varepsilon}{16};\end{aligned}\tag{3.32}$$

$$\max\frac{1}{T}\Bigg\{\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\Bigg(\big|h(s,x^\nu_{h_{1s}},\ldots,x^\nu_{h_{ks}})-h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\big|$$

$$
\begin{aligned}
&+\sum_{t=s}^{T_3-1}\left|f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right|\Bigg),\\
&\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{|a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})-a(s,x_{a_{1s}},\ldots,x_{a_{rs}})|}{a_s^2}\left(Q_s+\sum_{t=s}^{T_3-1}(P_t+|c_t|)\right)\Bigg\}<\frac{\varepsilon}{16},\\
&\forall\nu\ge T_4.
\end{aligned}\tag{3.33}
$$

In view of (3.2), (3.28), (3.32) and (3.33), we deduce that for any $\nu \ge T_4$

$$
\begin{aligned}
&\left\|G_Lx^{\nu}-G_Lx\right\|\\
&=\max\left\{\sup_{\beta\le n<T}\left|\frac{G_Lx^{\nu}_n}{n}-\frac{G_Lx_n}{n}\right|,\sup_{n\ge T}\left|\frac{G_Lx^{\nu}_n}{n}-\frac{G_Lx_n}{n}\right|\right\}\\
&=\sup_{n\ge T}\frac{1}{n}\Bigg|\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})}\Bigg(h(s,x^{\nu}_{h_{1s}},\ldots,x^{\nu}_{h_{ks}})-h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\\
&\quad-\sum_{t=s}^{\infty}\left(f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right)\Bigg)\\
&\quad+\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\left(\frac{1}{a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})}-\frac{1}{a(s,x_{a_{1s}},\ldots,x_{a_{rs}})}\right)\\
&\quad\times\left(h(s,x_{h_{1s}},\ldots,x_{h_{ks}})-\sum_{t=s}^{\infty}\left(f(t,x_{f_{1t}},\ldots,x_{f_{kt}})-c_t\right)\right)\Bigg|\\
&\le\frac{1}{T}\Bigg[\sup_{n\ge T}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})|}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\ldots,x^{\nu}_{h_{ks}})-h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\right|\\
&\quad+\sum_{t=s}^{\infty}\left|f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right|\Bigg)\\
&\quad+\sup_{n\ge T}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\left|\frac{1}{a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})}-\frac{1}{a(s,x_{a_{1s}},\ldots,x_{a_{rs}})}\right|\\
&\quad\times\left(|h(s,x_{h_{1s}},\ldots,x_{h_{ks}})|+\sum_{t=s}^{\infty}\left(|f(t,x_{f_{1t}},\ldots,x_{f_{kt}})|+|c_t|\right)\right)\Bigg]\\
&\le\frac{1}{T}\Bigg[\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\ldots,x^{\nu}_{h_{ks}})-h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\right|\\
&\quad+\sum_{t=s}^{\infty}\left|f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})-f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right|\Bigg)\\
&\quad+\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\left|\frac{1}{a(s,x^{\nu}_{a_{1s}},\ldots,x^{\nu}_{a_{rs}})}-\frac{1}{a(s,x_{a_{1s}},\ldots,x_{a_{rs}})}\right|\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
&\le\frac{1}{T}\Bigg[\sum_{i=T_1}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x^{\nu}_{h_{1s}},\ldots,x^{\nu}_{h_{ks}})\right|+\left|h(s,x_{h_{1s}},\ldots,x_{h_{ks}})\right|\\
&\quad+\sum_{t=s}^{\infty}\left(\left|f(t,x^{\nu}_{f_{1t}},\ldots,x^{\nu}_{f_{kt}})\right|+\left|f(t,x_{f_{1t}},\ldots,x_{f_{kt}})\right|\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\Bigg(\left|h(s,x^\nu_{h_{1s}},\dots,x^\nu_{h_{ks}})-h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right|\\
&+\sum_{t=s}^{T_3-1}\left|f(t,x^\nu_{f_{1t}},\dots,x^\nu_{f_{kt}})-f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\Bigg)\\
&+\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}\left(\left|f(t,x^\nu_{f_{1t}},\dots,x^\nu_{f_{kt}})\right|+\left|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\right)\\
&+\sum_{i=T}^{T_1-1}\sum_{s=T_2}^{\infty}\frac{1}{a_s}\Bigg(\left|h(s,x^\nu_{h_{1s}},\dots,x^\nu_{h_{ks}})\right|+\left|h(s,x_{h_{1s}},\dots,x_{h_{ks}})\right|\\
&+\sum_{t=s}^{\infty}\left(\left|f(t,x^\nu_{f_{1t}},\dots,x^\nu_{f_{kt}})\right|+\left|f(t,x_{f_{1t}},\dots,x_{f_{kt}})\right|\right)\Bigg)\\
&+\sum_{i=T_1}^{\infty}\sum_{s=i}^{\infty}\left(\frac{1}{\left|a(s,x^\nu_{a_{1s}},\dots,x^\nu_{a_{rs}})\right|}+\frac{1}{\left|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\right|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&+\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{\left|a(s,x^\nu_{a_{1s}},\dots,x^\nu_{a_{rs}})-a(s,x_{a_{1s}},\dots,x_{a_{rs}})\right|}{a_s^2}\left(Q_s+\sum_{t=s}^{T_3-1}(P_t+|c_t|)\right)\\
&+\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\left(\frac{1}{\left|a(s,x^\nu_{a_{1s}},\dots,x^\nu_{a_{rs}})\right|}+\frac{1}{\left|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\right|}\right)\sum_{t=T_3}^{\infty}(P_t+|c_t|)\\
&+\sum_{i=T}^{T_1-1}\sum_{s=T_2}^{\infty}\left(\frac{1}{\left|a(s,x^\nu_{a_{1s}},\dots,x^\nu_{a_{rs}})\right|}+\frac{1}{\left|a(s,x_{a_{1s}},\dots,x_{a_{rs}})\right|}\right)\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
\le&\frac{1}{T}\Bigg[2\sum_{i=T_1}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)+\frac{\varepsilon}{16}+2\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}P_t\\
&+2(T_1-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}P_t\right)+2\sum_{i=T_1}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&+\frac{\varepsilon}{16}+2\sum_{i=T}^{T_1-1}\sum_{s=i}^{T_2-1}\frac{1}{a_s}\sum_{t=T_3}^{\infty}(P_t+|c_t|)+2(T_1-T)\sum_{s=T_2}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\Bigg]\\
<&\ \varepsilon,
\end{aligned}
$$

which yields that $\lim_{\nu\to\infty}\|G_Lx^\nu-G_Lx\|=0$, that is, G_L is continuous in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.

Let $\varepsilon>0$. It follows from (3.16) that there exists $T^*>T$ satisfying

$$
\frac{1}{T^*}\sum_{i=T^*}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)<\frac{\varepsilon}{2}, \tag{3.34}
$$

which together with (3.2) and (3.28) implies that for all $x=\{x_n\}_{n\in\mathbb{N}_\beta}\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$ and $t_1>t_2>T^*$

$$
\begin{aligned}
&\left|\frac{G_Lx_{t_1}}{t_1}-\frac{G_Lx_{t_2}}{t_2}\right|\\
&=\Bigg|\frac{1}{t_1}\sum_{i=t_1}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\left(h(s,x_{h_{1s}},\dots,x_{h_{ks}})-\sum_{t=s}^{\infty}(f(t,x_{f_{1t}},\dots,x_{f_{kt}})-c_t)\right)\\
&\quad-\frac{1}{t_2}\sum_{i=t_2}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\left(h(s,x_{h_{1s}},\dots,x_{h_{ks}})-\sum_{t=s}^{\infty}(f(t,x_{f_{1t}},\dots,x_{f_{kt}})-c_t)\right)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{t_1}\sum_{i=t_1}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,x_{a_{1s}},\ldots,x_{a_{rs}})|}\left(|h(s,x_{h_{1s}},\ldots,x_{h_{ks}})|+\sum_{t=s}^{\infty}(|f(t,x_{f_{1t}},\ldots,x_{f_{kt}})|+|c_t|)\right)\\
&\quad+\frac{1}{t_2}\sum_{i=t_2}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,x_{a_{1s}},\ldots,x_{a_{rs}})|}\left(|h(s,x_{h_{1s}},\ldots,x_{h_{ks}})|+\sum_{t=s}^{\infty}(|f(t,x_{f_{1t}},\ldots,x_{f_{kt}})|+|c_t|)\right)\\
&\le \frac{2}{T^*}\sum_{i=T^*}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&<\varepsilon,
\end{aligned}
$$

which yields that $G_L(A(\{d_n\}_{n\in\mathbb{N}_\beta},D))$ is uniformly Cauchy, which together with (3.31) and Lemma 2.1 means that $G_L(A(\{d_n\}_{n\in\mathbb{N}_\beta},D))$ is relatively compact. Consequently G_L is completely continuous in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$. Thus (3.29), (3.30) and Lemma 2.2 ensure that the mapping S_L+G_L has a fixed point $w=\{w_n\}_{n\in\mathbb{N}_\beta}\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$, that is,

$$
\begin{aligned}
w_n = nL - b_n w_{n-\tau} - \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,w_{a_{1s}},\ldots,w_{a_{rs}}\big)}\Bigg(h\big(s,w_{h_{1s}},\ldots,w_{h_{ks}}\big)\\
-\sum_{t=s}^{\infty}\big(f\big(t,w_{f_{1t}},\ldots,w_{f_{kt}}\big)-c_t\big)\Bigg),\quad \forall n\ge T+\tau,
\end{aligned}
\tag{3.35}
$$

which yields that

$$
\begin{aligned}
&\Delta(a(n,w_{a_{1n}},\ldots,w_{a_{rn}})\Delta^2(w_n+b_nw_{n-\tau}))+\Delta h(n,w_{h_{1n}},\ldots,w_{h_{kn}})\\
&\quad+f(n,w_{f_{1n}},\ldots,w_{f_{kn}})=c_n,\quad \forall n\ge T+\tau.
\end{aligned}
$$

That is, $w=\{w_n\}_{n\in\mathbb{N}_\beta}$ is a nonoscillatory solution of Eq.(1.5) in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.

In view of (3.16) and (3.35), we get that

$$
\begin{aligned}
\left|\frac{w_n}{n}+\frac{b_n}{n}w_{n-\tau}-L\right| &\le \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s,w_{a_{1s}},\ldots,w_{a_{rs}})|}\Bigg(\big|h(s,w_{h_{1s}},\ldots,w_{h_{ks}})\big|\\
&\quad+\sum_{t=s}^{\infty}\big(\big|f(t,w_{f_{1t}},\ldots,w_{f_{kt}})\big|+|c_t|\big)\Bigg)\\
&<\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\to 0\quad\text{as}\quad n\to\infty,
\end{aligned}
$$

which implies that $\lim_{n\to\infty}\left(\frac{w_n}{n}+\frac{b_n}{n}w_{n-\tau}\right)=L$.

Next we show that Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$. Let $L_1,L_2\in\big(d-(1-b)D+b|d|,d+(1-b)D-b|d|\big)$ and $L_1\ne L_2$. Similarly, we conclude that for every $l\in\Lambda_2$, there exist a constant $T_l^*\geqq 1+n_0+n_1+\tau+\beta$ and two mapping S_{L_l} and G_{L_l} satisfying (3.27) and (3.28), where L,T, S_L and G_L are replaced by L_l,T_l^*, S_{L_l} and G_{L_l}, respectively, the mapping $S_{L_l}+G_{L_l}$ has a fixed point $w^l=\{w_n^l\}_{n\in\mathbb{N}_\beta}\in A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$, which is a nonoscillatory solution of Eq.(1.5), that is,

$$
\begin{aligned}
w_n^l = nL_l - b_n w_{n-\tau}^l - \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,w_{a_{1s}}^l,\ldots,w_{a_{rs}}^l\big)}\Bigg(h\Big(s,w_{h_{1s}}^l,\ldots,w_{h_{ks}}^l\Big)\\
-\sum_{t=s}^{\infty}\Big(f\Big(t,w_{f_{1t}}^l,\ldots,w_{f_{kt}}^l\Big)-c_t\Big)\Bigg),\quad \forall n\ge T_l^*,\quad l\in\Lambda_2.
\end{aligned}
\tag{3.36}
$$

Note that (3.16) implies that there exists $T_3^* > \max\{T_1^*, T_2^*\}$ satisfying

$$\frac{1}{T_3^*}\sum_{i=T_3^*}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)<\frac{|L_1-L_2|}{4}. \tag{3.37}$$

Using (3.2), (3.36) and (3.37), we get that

$$\begin{aligned}
&\left|\frac{w_n^1}{n}-\frac{w_n^2}{n}+b_n\left(\frac{w_{n-\tau}^1}{n}-\frac{w_{n-\tau}^2}{n}\right)\right|\\
&=\Bigg|L_1-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,w_{a_{1s}}^1,\dots,w_{a_{rs}}^1\big)}\left(h\Big(s,w_{h_{1s}}^1,\dots,w_{h_{ks}}^1\Big)-\sum_{t=s}^{\infty}\Big(f\Big(t,w_{f_{1t}}^1,\dots,w_{f_{kt}}^1\Big)-c_t\Big)\right)\\
&\quad-L_2+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,w_{a_{1s}}^2,\dots,w_{a_{rs}}^2\big)}\left(h\Big(s,w_{h_{1s}}^2,\dots,w_{h_{ks}}^2\Big)-\sum_{t=s}^{\infty}\Big(f\Big(t,w_{f_{1t}}^2,\dots,w_{f_{kt}}^2\Big)-c_t\Big)\right)\Bigg|\\
&\ge|L_1-L_2|-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\Bigg(\Big|h\Big(s,w_{h_{1s}}^1,\dots,w_{h_{ks}}^1\Big)\Big|+\Big|h\Big(s,w_{h_{1s}}^2,\dots,w_{h_{ks}}^2\Big)\Big|\\
&\quad+\sum_{t=s}^{\infty}\Big(\Big|f\Big(t,w_{f_{1t}}^1,\dots,w_{f_{kt}}^1\Big)\Big|+\Big|c_t\Big|+\Big|f\Big(t,w_{f_{1t}}^2,\dots,w_{f_{kt}}^2\Big)\Big|+\Big|c_t\Big|\Big)\Bigg)\\
&\ge|L_1-L_2|-\frac{2}{T_3^*}\sum_{i=T_3^*}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s+\sum_{t=s}^{\infty}(P_t+|c_t|)\right)\\
&>\frac{1}{2}|L_1-L_2|>0,\quad\forall n\ge T_3^*,
\end{aligned}$$

which yields that $w^1 \neq w^2$. Therefore, Eq.(1.5) has uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$. This completes the proof. □

Similar to the proofs of Theorems 3.3, we have the following results.

Theorem 3.4. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}, d, D, b \in \mathbb{R}^+ \setminus \{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.16) *and*

$$d>D,\quad \frac{d}{D}<\frac{1-b}{b},\quad |b_n|\le b,\quad \forall n\ge n_1; \tag{3.38}$$

$$\begin{aligned}
&|a(n,u_1,u_2\dots,u_r)|\geqq a_n,\quad \forall (n,u_d)\in\mathbb{N}_{n_0}\times(\mathbb{R}^+\setminus\{0\}),\ d\in\Lambda_r;\\
&|f(n,u_1,u_2\dots,u_k)|\le P_n,\quad |h(n,u_1,u_2\dots,u_k)|\le Q_n,\quad \forall (n,u_l)\in\mathbb{N}_{n_0}\times(\mathbb{R}^+\setminus\{0\}),\ l\in\Lambda_k.
\end{aligned} \tag{3.39}$$

Then for every $L \in \big(d-(1-b)D+bd, d+(1-b)D-bd\big)$, Eq.(1.5) *has a positive solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{n\to\infty}\left(\frac{w_n}{n}+\frac{b_n}{n}w_{n-\tau}\right)=L$. Furthermore, Eq.*(1.5) *has uncountably many positive solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.*

Theorem 3.5. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}, d \in \mathbb{R}_-, D, b \in \mathbb{R}^+ \setminus \{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.16) *and*

$$d+D<0,\quad \frac{d}{D}>\frac{b-1}{b},\quad |b_n|\le b,\quad \forall n\ge n_1; \tag{3.40}$$

$$\begin{aligned}
&|a(n,u_1,u_2\dots,u_r)|\geqq a_n,\quad \forall (n,u_d)\in\mathbb{N}_{n_0}\times\mathbb{R}_-,\ d\in\Lambda_r;\\
&|f(n,u_1,u_2\dots,u_k)|\le P_n,\quad |h(n,u_1,u_2\dots,u_k)|\le Q_n,\quad \forall (n,u_l)\in\mathbb{N}_{n_0}\times\mathbb{R}_-,\ l\in\Lambda_k.
\end{aligned} \tag{3.41}$$

Then for every $L \in \big(d-(1-b)D-bd, d+(1-b)D+bd\big)$, Eq.(1.5) *has a negative solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{n\to\infty}\left(\frac{w_n}{n}+\frac{b_n}{n}w_{n-\tau}\right)=L$. Furthermore, Eq.*(1.5) *has uncountably many negative solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.*

Theorem 3.6. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}, d, D, b \in \mathbb{R}^+ \setminus \{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2), (3.16) *and*

$$d < D, \quad \frac{d}{D} < \frac{1-b}{b}, \quad |b_n| \le b, \quad \forall n \ge n_1. \tag{3.42}$$

Then for every $L \in \big(d - (1-b)D + bd, d + (1-b)D - bd\big)$, Eq.(1.5) *has a solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{n\to\infty}\big(\frac{w_n}{n} + \frac{b_n}{n}w_{n-\tau}\big) = L$. Furthermore, Eq.*(1.5) *has uncountably many solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.*

Theorem 3.7. *Assume that there exist $n_1 \in \mathbb{N}_{n_0}, d \in \mathbb{R}, D, b_* \in \mathbb{R}^+ \setminus \{0\}$, two nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$ and a positive sequence $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.2), (3.16) *and*

$$|d| < D, \quad \frac{D+|d|}{D-|d|} < b_* \le |b_n|, \quad \forall n \ge n_1. \tag{3.43}$$

Then for every $L \in \big(-b_*(D-|d|) + D + |d|, b_*(D-|d|) - D - |d|\big)$, Eq.(1.5) *has a solution $w = \{w_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ with $\lim_{n\to\infty}\big(\frac{w_n}{n} + \frac{b_n}{n}w_{n-\tau}\big) = L$. Furthermore, Eq.*(1.5) *has uncountably many solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.*

Proof. Let $L \in \big(-b_*(D-|d|) + D + |d|, b_*(D-|d|) - D - |d|\big)$. Using (3.16), we infer that there exists $T \ge 1 + n_0 + n_1 + \tau + \beta$ satisfying

$$\frac{1}{b_*}\left(1 + \frac{\tau}{T}\right) < 1, \tag{3.44}$$

$$L \in \left(-b_*(D-|d|) + (D+|d|)\left(1+\frac{\tau}{T}\right), b_*(D-|d|) - (D+|d|)\left(1+\frac{\tau}{T}\right)\right), \tag{3.45}$$

$$\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s + \sum_{t=s}^{\infty}\big(P_t + |c_t|\big)\right) < b_*(D-|d|) - (D+|d|)\left(1+\frac{\tau}{T}\right) - |L|. \tag{3.46}$$

Define two mappings S_L and $G_L : A(\{d_n\}_{n\in\mathbb{N}_\beta}, D) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} \frac{nL}{b_{n+\tau}} - \frac{x_{n+\tau}}{b_{n+\tau}}, & n \ge T, \\ \frac{n}{T}S_L x_T, & \beta \le n < T \end{cases} \tag{3.47}$$

and

$$G_L x_n = \begin{cases} -\frac{1}{b_{n+\tau}}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a(s,x_{a_{1s}},\dots,x_{a_{rs}})}\big(h(s,x_{h_{1s}},\dots,x_{h_{ks}}) \\ \quad -\sum_{t=s}^{\infty}\big(f\big(t,x_{f_{1t}},\dots,x_{f_{kt}}\big) - c_t\big)\big), \quad n \ge T, \\ \frac{n}{T}G_L x_T, \quad \beta \le n < T \end{cases} \tag{3.48}$$

for any $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Now we assert that (3.29), (3.31) and

$$\|S_L x - S_L y\| \le \frac{1}{b_*}\left(1+\frac{\tau}{T}\right)\|x-y\|, \quad \forall x, y \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D) \tag{3.49}$$

hold. It follows from (3.2) and (3.43)-(3.48) that for any $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$ and $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$

$$\left|\frac{S_L x_n}{n} + \frac{G_L y_n}{n} - d\right| = \left|\frac{L}{b_{n+\tau}} - d - \frac{x_{n+\tau}}{nb_{n+\tau}} - \frac{1}{nb_{n+\tau}}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a\big(s,y_{a_{1s}},\dots,y_{a_{rs}}\big)}\right(h\big(s,y_{h_{1s}},\dots,y_{h_{ks}}\big)$$

$$
\begin{aligned}
&\quad - \sum_{t=s}^{\infty} \big(f\big(t, y_{f_{1t}}, \dots, y_{f_{kt}}\big) - c_t \big) \bigg) \bigg| \\
&\le \frac{|L|}{b_*} + |d| + \frac{D + |d|}{b_*} \left(1 + \frac{\tau}{T} \right) + \frac{1}{b_* T} \sum_{i=T}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \left(Q_s + \sum_{t=s}^{\infty} \big(P_t + |c_t| \big) \right) \\
&< \frac{|L|}{b_*} + |d| + \frac{D + |d|}{b_*} \left(1 + \frac{\tau}{T} \right) + \frac{1}{b_*} \left(b_*(D - |d|) - (D + |d|) \left(1 + \frac{\tau}{T} \right) - |L| \right) \\
&= D, \quad n \ge T,
\end{aligned}
$$

$$
\left| \frac{S_L x_n}{n} + \frac{G_L y_n}{n} - d \right| = \left| \frac{n}{T} \cdot \frac{S_L x_T}{n} + \frac{n}{T} \cdot \frac{G_L y_T}{n} - d \right| = \left| \frac{S_L x_T}{T} + \frac{G_L y_T}{T} - d \right| < D, \quad \beta \le n < T,
$$

$$
\left| \frac{S_L x_n}{n} - \frac{S_L y_n}{n} \right| = \left| \frac{n + \tau}{n b_{n+\tau}} \left(\frac{x_{n+\tau}}{n + \tau} - \frac{y_{n+\tau}}{n + \tau} \right) \right| \le \frac{1}{b_*} \left(1 + \frac{\tau}{T} \right) \|x - y\|, \quad n \ge T,
$$

$$
\begin{aligned}
\left| \frac{S_L x_n}{n} - \frac{S_L y_n}{n} \right| &= \left| \frac{n}{T} \cdot \frac{S_L x_T}{n} - \frac{n}{T} \cdot \frac{S_L y_T}{n} \right| = \left| \frac{S_L x_T}{T} - \frac{S_L y_T}{T} \right| \\
&\le \frac{1}{b_*} \left(1 + \frac{\tau}{T} \right) \|x - y\|, \quad \beta \le n < T,
\end{aligned}
$$

$$
\begin{aligned}
\left| \frac{G_L x_n}{n} \right| &= \bigg| \frac{1}{n b_{n+\tau}} \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a\big(s, x_{a_{1s}}, \dots, x_{a_{rs}}\big)} \\
&\quad \times \left(h\big(s, x_{h_{1s}}, \dots, x_{h_{ks}}\big) - \sum_{t=s}^{\infty} \big(f\big(t, x_{f_{1t}}, \dots, x_{f_{kt}}\big) - c_t \big) \right) \bigg| \\
&\le \frac{1}{b_* T} \sum_{i=T}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a_s} \left(Q_s + \sum_{t=s}^{\infty} \big(P_t + |c_t| \big) \right) \\
&< \frac{1}{b_*} \left(b_*(D - |d|) - (D + |d|) \left(1 + \frac{\tau}{T} \right) - |L| \right) \\
&< D, \quad n \ge T
\end{aligned}
$$

and

$$
\left| \frac{G_L x_n}{n} \right| = \left| \frac{n}{T} \cdot \frac{G_L x_T}{n} \right| = \left| \frac{G_L x_T}{T} \right| < D, \quad \beta \le n < T,
$$

which imply (3.29), (3.31) and (3.49).

As in the proof of Theorem 3.3, we prove similarly that G_L is continuous in $A(\{d_n\}_{n \in \mathbb{N}_\beta}, D)$ and $G_L(A(\{d_n\}_{n \in \mathbb{N}_\beta}, D))$ is relatively compact. Thus G_L is completely continuous, which together with (3.29), (3.49) and Lemma 2.2 ensures that the equation $S_L x + G_L x = x$ has a solution $w = \{w_n\}_{n \in \mathbb{N}_\beta} \in A(\{d_n\}_{n \in \mathbb{N}_\beta}, D)$, which together with (3.47) and (3.48) implies that

$$
\begin{aligned}
w_n = \frac{nL}{b_{n+\tau}} - \frac{w_{n+\tau}}{b_{n+\tau}} - \frac{1}{b_{n+\tau}} \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \frac{1}{a(s, w_{a_{1s}}, \dots, w_{a_{rs}})} \Big(h\big(s, w_{h_{1s}}, \dots, w_{h_{ks}}\big) \\
- \sum_{t=s}^{\infty} \big(f\big(t, w_{f_{1t}}, \dots, w_{f_{kt}}\big) - c_t \big) \Big), \quad \forall n \ge T + \tau,
\end{aligned}
\tag{3.50}
$$

which yields that

$$\begin{aligned}&\Delta(a(n, w_{a_{1n}}, \dots, w_{a_{rn}})\Delta^2(w_n + b_n w_{n-\tau})) + \Delta h(n, w_{h_{1n}}, \dots, w_{h_{kn}})\\&\quad + f(n, w_{f_{1n}}, \dots, w_{f_{kn}}) = c_n, \quad \forall n \geq T + \tau.\end{aligned}$$

That is, $w = \{w_n\}_{n\in\mathbb{N}_\beta}$ is a solution of Eq.(1.5) in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

In light of (3.16) and (3.50), we get that

$$\begin{aligned}\left|\frac{w_n}{n} + \frac{b_n}{n}w_{n-\tau} - L\right| &\leq \frac{1}{n}\left[\tau L + \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{|a(s, w_{a_{1s}}, \dots, w_{a_{rs}})|}\left(\left|h(s, w_{h_{1s}}, \dots, w_{h_{ks}})\right|\right.\right.\\&\quad\left.\left. + \sum_{t=s}^{\infty}\left(\left|f(t, w_{f_{1t}}, \dots, w_{f_{kt}})\right| + |c_t|\right)\right)\right]\\&< \frac{1}{n}\left[\tau L + \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\frac{1}{a_s}\left(Q_s + \sum_{t=s}^{\infty}(P_t + |c_t|)\right)\right] \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that $\lim_{n\to\infty}\left(\frac{w_n}{n} + \frac{b_n}{n}w_{n-\tau}\right) = L$. The rest of the proof is similar to that of Theorem 3.3 and is omitted. This completes the proof. □

4. Examples and Applications

In this section we suggest seven examples to explain the advantage and applications of the results presented in Section 3. Note that all known results are not applicable to the seven examples.

Example 4.1. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}&\Delta\big((n^8 + n^3 + 2)(1 + x_{n^2}^2)\Delta^2(x_n - x_{n-\tau})\big) + \Delta\frac{(-1)^2 n^3 x_{n^2-n}x_{2n}}{(n^7 + n^2 + 3)(1 + x_{n^2-n}^2 + x_{2n}^2)}\\&\quad + \frac{\sin^2 x_{3n-1}}{(n^6 + n + 1)(1 + x_{n-8}^2)} = \frac{(-1)^n(n^3 + 1)}{n^9 + \sin\sqrt{n^2 + 2}}, \quad n \geq 14,\end{aligned} \tag{4.1}$$

where $n_0 = 14$ and $\tau \in \mathbb{N}$ is fixed. Let $n_1 = 14,\ r = 1,\ k = 2,\ d = \pm 9,\ D = 5,\ \beta = \min\{14 - \tau, 6\} \in \mathbb{N}$ and

$$\begin{aligned}&a_{1n} = n^2, \quad f_{1n} = n - 8, \quad f_{2n} = 3n - 1, \quad h_{1n} = n^2 - n, \quad h_{2n} = 2n,\\&a(n, u) = (n^8 + n^3 + 2)(1 + u^2), \quad b_n = -1, \quad c_n = \frac{(-1)^n(n^3 + 1)}{n^9 + \sin\sqrt{n^2 + 2}},\\&f(n, u, v) = \frac{\sin^2 v}{(n^6 + n + 1)(1 + u^2)}, \quad h(n, u, v) = \frac{(-1)^2 n^3 uv}{(n^7 + n^2 + 3)(1 + u^2 + v^2)},\\&a_n = n^8 + n^3 + 2, \quad P_n = \frac{4}{n^6}, \quad Q_n = \frac{4}{n^4}, \quad (n, u, v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.\end{aligned}$$

It is easy to show that (3.1)-(3.3) hold. It follows from Theorem 3.1 that Eq.(4.1) possesses uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Example 4.2. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}&\Delta\big((-1)^{n^2-1}(n^2 + 3)(1 + 3x_{3n-5}^2 + |\sin x_{n-2}|)\Delta^2(x_n + x_{n-\tau})\big)\\&\quad + \Delta\left(\frac{\sin^4(n^5 x_{5n^2-9}^7 x_{n-5}^8)}{n^{10} - \ln n + 2} + \frac{n - (-1)^{n-1}}{(n^5 + 5)(x_{n-5}^2 + 9)}\right) + \frac{\cos^5 x_{3n^3-1}}{n^3(n^5 + 3)(2 + |x_{n^2-3}|)}\\&= \frac{n^2 - \sqrt{n} - 10}{n^{10} + n^3 + 8}, \quad n \geq 15,\end{aligned} \tag{4.2}$$

where $n_0 = 15$ and $\tau \in \mathbb{N}$ is fixed. Let $n_1 = 15,\ r = 2,\ k = 2,\ d = \pm 6,\ D = 3,\ \beta = \min\{15-\tau, 10\} \in \mathbb{N}$ and

$$\begin{aligned}
&a_{1n} = 3n-5, \quad a_{2n} = n-2, \quad f_{1n} = n^2-3, \quad f_{2n} = 3n^3-1, \quad h_{1n} = 5n^2-9, \quad h_{2n} = n-5,\\
&a(n,u,v) = (-1)^{n^2-1}(n^2+3)(1+3u^2+|\sin v|), \quad b_n = 1, \quad c_n = \frac{n^2-\sqrt{n}-10}{n^{10}+n^3+8},\\
&f(n,u,v) = \frac{\cos^5 v}{n^3(n^5+3)(2+|u|)}, \quad h(n,u,v) = \frac{\sin^4(n^5u^7v^8)}{n^{10}-\ln n+2} + \frac{n-(-1)^{n-1}}{(n^5+5)(v^2+9)},\\
&a_n = n^2+3, \quad P_n = \frac{5}{n^8}, \quad Q_n = \frac{9}{n^4}, \quad (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{aligned}$$

It is easy to verify that (3.2), (3.15) and (3.16) hold. Theorem 3.2 ensures that Eq.(4.2) possesses uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Example 4.3. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}
&\Delta\left(n^6(2+\sin x_{n-3})\Delta^2\left(x_n + \frac{(-1)^n(n-3)}{4n+2}x_{n-\tau}\right)\right) + \Delta\frac{(n^3+n+1)\cos^5 x_{n^3-8n}}{(n^6+2n^2+9)(1+x_{n-4}^4)}\\
&\quad + \frac{(-1)^n\sin^2 x_{3n-15}}{n^7} - \frac{1}{(n^{10}+6n^2)(1+x_{n^2-3n}^2)} = \frac{(-1)^n(n^2+2)}{n^{10}(1+\ln n)}, \quad n \ge 13,
\end{aligned} \tag{4.3}$$

where $n_0 = 13$ and $\tau \in \mathbb{N}$ is fixed. Let $n_1 = 13,\ r = 1,\ k = 2,\ d = \pm 8,\ D = 5,\ b = \frac{1}{4},\ \beta = \min\{13-\tau, 9\} \in \mathbb{N}$ and

$$\begin{aligned}
&a_{1n} = n+3, \quad f_{1n} = 3n-15, \quad f_{2n} = n^2-3n, \quad h_{1n} = n-4, \quad h_{2n} = n^3-8n,\\
&a(n,u) = n^6(2+\sin u), \quad b_n = \frac{(-1)^n(n-3)}{4n+2}, \quad c_n = \frac{(-1)^n(n^2+2)}{n^{10}(1+\ln n)},\\
&f(n,u,v) = \frac{(-1)^n\sin^2 u}{n^7} - \frac{1}{(n^{10}+6n^2)(1+v^2)}, \quad h(n,u,v) = \frac{(n^3+n+1)\cos^5 v}{(n^6+2n^2+9)(1+u^4)},\\
&a_n = n^6, \quad P_n = \frac{3}{n^7}, \quad Q_n = \frac{10}{n^3}, \quad (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{aligned}$$

It is easy to see that (3.2), (3.16) and (3.25) hold. It follows from Theorem 3.3 that Eq.(4.3) possesses uncountably many nonoscillatory solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Example 4.4. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}
&\Delta\left((n^3-n)(6+3\sin x_{n^2-6n}+2\cos x_{10n-6})\Delta^2\left(x_n + \frac{(-1)^{n^2-2}(n-4)}{6n+3}x_{n-\tau}\right)\right)\\
&\quad + \Delta\frac{(n-3)x_{n-4}x_{5n^2-7}}{(n^5+n+4)(x_{n-4}^2+x_{5n^2-7}^2)} + \frac{\sin^3 x_{n-5}}{n^7(1+x_{3n^3-9}^2)} = \frac{(-1)^n(n^2-2)}{7n^{11}}, \quad n \ge 24,
\end{aligned} \tag{4.4}$$

where $n_0 = 24$ and $\tau \in \mathbb{N}$ is fixed. Let $n_1 = 24,\ r = 2,\ k = 2,\ d = 9,\ D = 3,\ b = \frac{1}{5},\ \beta = \min\{24-\tau, 19\} \in \mathbb{N}$ and

$$\begin{aligned}
&a_{1n} = n^2-6n, \quad a_{2n} = 10n-6, \quad f_{1n} = n-5, \quad f_{2n} = 3n^3-9,\\
&h_{1n} = n-4, \quad h_{2n} = 5n^2-7, \quad a(n,u,v) = (n^3-n)(6+3\sin u+2\cos v),\\
&b_n = \frac{(-1)^{n^2-2}(n-4)}{6n+3}, \quad c_n = \frac{(-1)^n(n^2-2)}{7n^{11}}, \quad f(n,u,v) = \frac{\sin^3 u}{n^7(1+v^2)},\\
&h(n,u,v) = \frac{(n-3)uv}{(n^5+n+4)(u^2+v^2)}, \quad a_n = n^2, \quad P_n = \frac{3}{n^7},\\
&Q_n = \frac{5}{n^4}, \quad (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2.
\end{aligned}$$

It is easy to get that (3.16), (3.38) and (3.39) hold. Theorem 3.4 ensures that Eq.(4.4) possesses uncountably many positive solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

Example 4.5. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}\Delta\Big(n^3(9+n|x_{7n-4}|)\Delta^2\Big(x_n+\frac{2+e^n}{9+3e^n}x_{n-\tau}\Big)\Big)+\Delta\frac{\cos(3n-1)}{(n+2)^5\sqrt{2+3|x_{n^2-16}|}}\\ +\frac{\sin(n^3x_{n-2})}{n^8+3n^2}=\frac{n-\ln n}{n^{12}+3n^4+3},\quad n\geq 13,\end{aligned}\tag{4.5}$$

where $n_0=13$ and $\tau\in\mathbb{N}$ is fixed. Let $n_1=15,\ r=1,\ k=1,\ d=-7,\ D=4,\ b=\frac{1}{3},\ \beta=\min\{13-\tau,11\}\in\mathbb{N}$ and

$$\begin{aligned}&a_{1n}=7n-4,\quad f_{1n}=n-2,\quad h_{1n}=n^2-16,\quad a(n,u)=n^3(9+n|u|),\\ &b_n=\frac{2+e^n}{9+3e^n},\quad c_n=\frac{n-\ln n}{n^{12}+3n^4+3},\quad f(n,u)=\frac{\sin(n^3u)}{n^8+3n^2},\\ &h(n,u)=\frac{\cos(3n-1)}{(n+2)^5\sqrt{2+3|u|}},\quad a_n=n^3,\quad P_n=\frac{10}{n^8},\quad Q_n=\frac{16}{n^5},\quad (n,u)\in\mathbb{N}_{n_0}\times\mathbb{R}.\end{aligned}$$

It is easy to see that (3.16), (3.40) and (3.41) hold. It follows from Theorem 3.5 that Eq.(4.5) possesses uncountably many negative solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.

Example 4.6. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}\Delta\Big(n^2\big(2+x^2_{(2n+1)^2}-\sin x_{(2n+1)^2}\big)\Delta^2\Big(x_n+\frac{(-1)^{n+1}(2n^2-n-3)}{4n^2+9}x_{n-\tau}\Big)\Big)\\ +\Delta\frac{\sin(x^4_{7n}x^5_{n^2-n+3})}{n^5+4n^2+3}+\frac{x_{n-5}x_{10n-4}\cos(n^3+5)}{(n+2)^{11}(x^2_{n-5}+x^2_{10n-4})}=\frac{(-1)^nn^3+6}{n^{10}+n+1},\quad n\geq 12,\end{aligned}\tag{4.6}$$

where $n_0=12$ and $\tau\in\mathbb{N}$ is fixed. Let $n_1=17,\ r=1,\ k=2,\ d=2,\ D=9,\ b=\frac{3}{4},\ \beta=\min\{12-\tau,7\}\in\mathbb{N}$ and

$$\begin{aligned}&a_{1n}=(2n+1)^2,\quad f_{1n}=n-5,\quad f_{2n}=10n-4,\quad h_{1n}=7n,\quad h_{2n}=n^2-n+3,\\ &a(n,u)=n^2\big(2+u^2-\sin u\big),\quad b_n=\frac{(-1)^{n+1}(2n^2-n-3)}{4n^2+9},\quad c_n=\frac{(-1)^nn^3+6}{n^{10}+n+1},\\ &f(n,u,v)=\frac{uv\cos(n^3+5)}{(n+2)^{11}(u^2+v^2)},\quad h(n,u,v)=\frac{\sin(u^4v^5)}{n^5+4n^2+3},\\ &a_n=n^2,\quad P_n=\frac{3}{n^{11}},\quad Q_n=\frac{7}{n^5},\quad (n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.\end{aligned}$$

It is easy to show that (3.2), (3.16) and (3.42) hold. It follows from Theorem 3.6 that Eq.(4.6) possesses uncountably many solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta},D)$.

Example 4.7. Consider the third order nonlinear difference equation with neutral delay

$$\begin{aligned}\Delta\Big(\big(n^3-5n+3\big)\big(1+\sin^2x_{3n+2}\big)\Delta^2\Big(x_n+\frac{(-1)^n(8n^4-3n+5)}{n^4+n+7}x_{n-\tau}\Big)\Big)\\ +\Delta\frac{(-1)^{n+1}}{(n+1)^8(2+\cos x^2_{n-5})}+\frac{n^2\sin(n^3x^9_{n^2-1})}{n^{13}+4n^2+1}=\frac{7\sqrt{n^4-2}}{n^{12}(1+e^n)},\quad n\geq 20,\end{aligned}\tag{4.7}$$

where $n_0=20$ and $\tau\in\mathbb{N}$ is fixed. Let $n_1=20,\ r=1,\ k=1,\ d=\pm4,\ D=9,\ b_*=6,\ \beta=\min\{20-\tau,15\}\in\mathbb{N}$ and

$$\begin{aligned}&a_{1n}=3n+2,\quad f_{1n}=n^2-1,\quad h_{1n}=n-5,\quad a(n,u)=(n^3-5n+3)(1+\sin^2u),\\ &b_n=\frac{(-1)^n(8n^4-3n+5)}{n^4+n+7},\quad c_n=\frac{7\sqrt{n^4-2}}{n^{12}(1+e^n)},\quad f(n,u)=\frac{n^2\sin(n^3u^9)}{n^{13}+4n^2+1},\\ &h(n,u)=\frac{(-1)^{n+1}}{(n+1)^8(2+\cos u^2)},\quad a_n=n^2,\quad P_n=\frac{5}{n^{11}},\quad Q_n=\frac{10}{n^8},\quad (n,u)\in\mathbb{N}_{n_0}\times\mathbb{R}.\end{aligned}$$

It is clear that (3.2), (3.16) and (3.43) hold. Theorem 3.7 ensures that Eq.(4.7) possesses uncountably many solutions in $A(\{d_n\}_{n\in\mathbb{N}_\beta}, D)$.

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