Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5196–5201
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A study on a class of q-Euler polynomials under the symmetric group of degree n

Serkan Araci[a,*], Ugur Duran[b], Mehmet Acikgoz[b]

[a] *Department of Economics, Faculty of Economics, Administrative and Social Science, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey.*

[b] *Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, TR-27310 Gaziantep, Turkey.*

Communicated by S.-H. Rim

Abstract

Motivated by the paper of Kim et al. [T. Kim, D. S. Kim, H. I. Kwon, J. J. Seo, D. V. Dolgy, J. Nonlinear Sci. Appl., **9** (2016), 1077–1082], we study a class of q-Euler polynomials earlier given by Kim et al. in [T. Kim, Y. H. Kim, K. W. Hwang, Proc. Jangjeon Math. Soc., **12** (2009), 77–92]. We derive some new symmetric identities for q-extension of $\lambda-$Euler polynomials, using fermionic p-adic invariant integral over the p-adic number field originally introduced by Kim in [T. Kim, Russ. J. Math. Phys., **16** (2009), 484–491], under symmetric group of degree n denoted by S_n.

Keywords: Symmetric identities, q-extension of λ-Euler polynomials, fermionic p-adic invariant integral on $\mathbb{Z}_p$, invariant under S_n.
2010 MSC: 11B68, 05A19, 11S80, 05A30.

1. Introduction

Throughout the paper, we make use of the following notations: $\mathbb{Z}_p$ denotes topological closure of $\mathbb{Z}$, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes topological closure of $\mathbb{Q}$, and $\mathbb{C}_p$ indicates the field of p-adic completion of an algebraic closure of $\mathbb{Q}_p$, in which p be a fixed odd prime number. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$. For d an odd positive number with $(p, d) = 1$, let

$$X := X_d = \lim_{\overleftarrow{n}} \mathbb{Z}/dp^N\mathbb{Z} \text{ and } X_1 = \mathbb{Z}_p,$$

*Corresponding author
Email addresses: `mtsrkn@hotmail.com` (Serkan Araci), `duran.ugur@yahoo.com` (Ugur Duran), `acikgoz@gantep.edu.tr` (Mehmet Acikgoz)

Received 2016-06-09

and

$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\bmod dp^N \right) \right\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$ *cf.* [1, 2, 5–10].

The normalized absolute value according to the theory of p-adic analysis is given by $|p|_p = p^{-1}$. The notation q can be considered as an indeterminate a complex number $q \in \mathbb{C}$ with $|q| < 1$, or a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. It is always clear in the content of the paper. For any x, the q-analogue of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$ *cf.* [1, 2, 5–10].

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, Kim [6] originally constructed fermionic p-adic invariant integral on $\mathbb{Z}_p$, as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{n \to \infty} \sum_{x=0}^{p^n-1} f(x)(-1)^x, \tag{1.1}$$

and also

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

where $f_1(x) = f(x+1)$.

The λ-Euler polynomials $E_n(\lambda, x)$ (usually called Apostol-Euler polynomials) for $\lambda \in \mathbb{C}$ were defined by the power series expansion at $t = 0$:

$$\sum_{n=0}^{\infty} E_n(\lambda, x) \frac{t^n}{n!} = \frac{2}{\lambda e^t + 1} e^{xt}, \tag{1.2}$$
$$\left(|t| < \pi \text{ when } \lambda = 1; \ \ |t| < \log(-\lambda) \text{ when } \lambda \neq 1; 1^{\alpha} := 1 \right).$$

Taking $x = 0$ in the Eq. (1.2), we have $E_n(\lambda, 0) := E_n(\lambda)$ that is known as n-th λ-Euler numbers (see for details, [3, 4, 11]).

In [9], Kim et al. defined q-extension of λ-Euler polynomials by the following fermionic p-adic integral on $\mathbb{Z}_p$:

$$E_{n,q}(\lambda, x) = \frac{[2]_q}{2} \int_{\mathbb{Z}_p} \lambda^y \left[x + y \right]_q^n d\mu_{-1}(y). \tag{1.3}$$

Letting $x = 0$ into the Eq. (1.3) yields $E_{n,q}(\lambda, 0) := E_{n,q}(\lambda)$ called n-th q-extension of $\lambda-$Euler numbers, *cf.* [9].

By taking $q \to 1^-$ in the Eq. (1.3), we have

$$\lim_{q \to 1^-} E_{n,q}(\lambda, x) := E_n(\lambda, x) = \int_{\mathbb{Z}_p} \lambda^y (x+y)^n d\mu_{-1}(y).$$

Motivated by the paper of Kim et al. [10], we study q-extension of λ-Euler polynomials earlier given by Kim et al [9]. We derive some new symmetric identities for these polynomials, using fermionic p-adic invariant integral over the p-adic number field introduced by Kim [6], under symmetric group of degree n denoted by S_n.

We now give some interesting identities derived from the fermionic p-adic invariant integral on $\mathbb{Z}_p$ in the next section.

2. New symmetric identities for $E_{n,q}(\lambda, x)$ under S_n

Let w_i be odd natural numbers where $i = \{1, 2, \cdots, n\}$. From the Eqs. (1.1) and (1.3), we consider

$$\int_{\mathbb{Z}_p} \lambda^{y \prod_{j=1}^{n-1} w_j} e^{\left[\left(\prod_{j=1}^{n-1} w_j \right) y + \left(\prod_{j=1}^{n} w_j \right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q t} d\mu_{-1}(y)$$

$$= \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (-1)^y \lambda^{y\prod_{j=1}^{n-1} w_j} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t}$$

$$= \lim_{N\to\infty} \sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1} (-1)^{m+y} \lambda^{(m+w_n y)\prod_{j=1}^{n-1} w_j}$$

$$\times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.$$

Therefore, we derive that

$$\begin{aligned} I &= \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)} \lambda^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j} \\ &\quad\times \int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-1}(y) \\ &= \lim_{N\to\infty}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{l=0}^{w_n-1}\sum_{y=0}^{p^N-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)+m+y} \lambda^{\left(\prod_{j=1}^{n-1} w_j m+\prod_{j=1}^{n} w_j y+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right)} \\ &\quad\times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q t}. \end{aligned} \tag{2.1}$$

Observe that the Eq. (2.1) is invariant under any permutation $\sigma \in S_n$. Hence, we state the following theorem.

Theorem 2.1. *Let w_i be odd natural numbers where $i = \{1, 2, \cdots, n\}$. Then the following*

$$\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1} (-1)^{\left(\sum_{s=1}^{n-1} k_s\right)} \lambda^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j}$$

$$\times \int_{\mathbb{Z}_p} \lambda^{y\left(\prod_{j=1}^{n-1} w_{\sigma(j)}\right)} e^{\left[\left(\prod_{j=1}^{n-1} w_{\sigma(j)}\right)y+\left(\prod_{j=1}^{n} w_{\sigma(j)}\right)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j\right]_q t} d\mu_{-1}(y)$$

holds true for any $\sigma \in S_n$.

By using the definition of $[x]_q$, we have

$$\begin{aligned} &\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j\right]_q \\ &\quad= \left[\prod_{j=1}^{n-1} w_j\right]_q \left[y+w_n x+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1 w_2\cdots w_{n-1}}} \\ &\quad= \left[\prod_{j=1}^{n-1} w_j\right]_q \left[y+w_n x+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1 w_2\cdots w_{n-1}}}. \end{aligned} \tag{2.2}$$

It follows from the Eq. (2.2) that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-1}(y) \\
&= \sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1} w_j\right]_q^m \left(\int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j}\left[y+w_n x+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m d\mu_{-1}(y)\right)\frac{t^m}{m!} \\
&= \frac{2}{[2]_q}\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1} w_j\right]_q^m E_{m,q^{w_1w_2\cdots w_{n-1}}}\left(\lambda^{w_1w_2\dots w_{n-1}}, w_n x+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right)\frac{t^m}{m!}.
\end{aligned} \tag{2.3}$$

From Eq. (2.3), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j}\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q^m d\mu_{-1}(y) \\
&= \frac{2}{[2]_q}\left[\prod_{j=1}^{n-1} w_j\right]_q^m E_{m,q^{w_1w_2\cdots w_{n-1}}}\left(\lambda^{w_1w_2\dots w_{n-1}}, w_n x+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right) \quad (m\geq 0).
\end{aligned} \tag{2.4}$$

Thus, by Theorem 2.1 and Eq. (2.4), we derive the following theorem.

Theorem 2.2. *Let w_i be odd natural numbers where $i=\{1,2,\cdots,n\}$. For $m\geq 0$, the following*

$$\begin{aligned}
&\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^m \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\left(\sum_{s=1}^{n-1}k_s\right)}\lambda^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j} \\
&\qquad\times\frac{2}{[2]_q}E_{m,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}\left(\lambda^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}, w_{\sigma(n)}x+\sum_{j=1}^{n-1}\frac{w_{\sigma(n)}}{w_{\sigma(j)}}k_j\right)
\end{aligned}$$

holds true for any $\sigma\in S_n$.

By using the definitions of $[x]_q$ and binomial theorem, we can write :

$$\begin{aligned}
\left[y+w_n x+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m &= \sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_{q^{w_n}}^{m-l} \\
&\quad\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}[y+w_n x]_{q^{w_1w_2\cdots w_{n-1}}}^l .
\end{aligned}$$

Taking $\int_{\mathbb{Z}_p}\lambda^{y\prod_{j=1}^{n-1}w_j}d\mu_{-1}(y)$ on the both sides of the above equation gives

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j}\left[y+w_n x+\sum_{j=1}^{n-1}\frac{w_n}{w_j}k_j\right]_{q^{w_1w_2\cdots w_{n-1}}}^m d\mu_{-1}(y) \\
&= \sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1}w_j\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i=j}}^{n-1} w_i\right)k_j\right]_{q^{w_n}}^{m-l}
\end{aligned}$$

$$\times q^{lw_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j} \int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j} [y + w_n x]^l_{q^{w_1 w_2 \cdots w_{n-1}}} \, d\mu_{-1}(y)$$

$$= \frac{2}{[2]_q} \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q} \right)^{m-l} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i \right) k_j \right]^{m-l}_{q^{w_n}} \tag{2.5}$$

$$\times q^{lw_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j} E_{l,q^{w_1 w_2 \cdots w_{n-1}}} \left(\lambda^{w_1 w_2 \ldots w_{n-1}}, w_n x\right).$$

As a result of the Eq. (2.5), we obtain

$$\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]^m_q \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)} \lambda^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}$$

$$\times \int_{\mathbb{Z}_p} \lambda^{y\prod_{j=1}^{n-1} w_j} \left[y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right]_{q^{w_1 w_2 \cdots w_{n-1}}} d\mu_{-1}(y)$$

$$= \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_j\right]^l_q [w_n]^{m-l}_q \frac{2}{[2]_q} E_{l,q^{w_1 w_2 \cdots w_{n-1}}} \left(\lambda^{w_1 w_2 \ldots w_{n-1}}, w_n x\right)$$

$$\times \prod_{s=1}^{n-1} \sum_{k_s=0}^{w_s-1} (-1)^{\left(\sum_{i=1}^{n-1} k_i\right)} q^{lw_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1} w_i\right)k_j} \lambda^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1} w_i\right)k_j} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i \right) k_j \right]^{m-l}_{q^{w_n}}$$

$$= \frac{2}{[2]_q} \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_j\right]^l_q [w_n]^{m-l}_q E_{l,q^{w_1 w_2 \cdots w_{n-1}}} \left(\lambda^{w_1 w_2 \ldots w_{n-1}}, w_n x\right) U_{m,q^{w_n},\lambda^{w_n}}(w_1, w_2, ..., w_{n-1} \mid l),$$

where

$$U_{m,q,\lambda}(w_1, w_2, ..., w_{n-1} \mid l) = \prod_{s=1}^{n-1} \sum_{k_s=0}^{w_s-1} (-1)^{\left(\sum_{l=1}^{n-1} k_l\right)} \lambda^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1} w_i\right)k_j}$$

$$\times q^{l\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i=j}}^{n-1} w_i\right)k_j} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i \right) k_j \right]^{m-l}_q .$$

Last of all, by (2.5), we get the following theorem.

Theorem 2.3. *Let w_i be odd natural numbers where $i = \{1, 2, \cdots, n\}$ and let $m \geq 0$. Then the following expression*

$$\sum_{l=0}^{m} \binom{n}{m} \left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]^l_q \left[w_{\sigma(n)}\right]^{m-l}_q$$

$$\times E_{l,q^{w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}}} \left(\lambda^{w_{\sigma(1)} w_{\sigma(2)} \ldots w_{\sigma(n-1)}}, w_{\sigma(n)} x\right) U_{m,q^{w_{\sigma(n)}},\lambda^{w_{\sigma(n)}}}(w_{\sigma(1)}, w_{\sigma(2)}, ..., w_{\sigma(n-1)} \mid l)$$

holds true for some $\sigma \in S_n$.

References

[1] E. Ağyüz, M. Acikgoz, S. Araci, *A symmetric identity on the q-Genocchi polynomials of higher-order under third dihedral group D_3*, Proc. Jangjeon Math. Soc., **18** (2015), 177–187. 1

[2] U. Duran, M. Acikgoz, S. Araci, *Symmetric identities involving weighted q-Genocchi polynomials under S_4*, Proc. Jangjeon Math. Soc., **18** (2015), 455–465. 1

[3] Y. He, S. Araci, *Sums of products of Apostol-Bernoulli and Apostol-Euler polynomials*, Adv. Difference Equ., **2014** (2014), 13 pages. 1

[4] Y. He, S. Araci, H. M. Srivastava, M. Acikgoz, *Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials*, Appl. Math. Comput., **262** (2015), 31–41. 1

[5] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), 288–299. 1

[6] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), 484–491.1, 1

[7] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), 93–96.

[8] D. S. Kim, T. Kim, *Some identities of symmetry for q-Bernoulli polynomials under symmetric group of degree n*, Ars Combin., **126** (2016), 435–441.

[9] T. Kim, Y. H. Kim, K. W. Hwang, *On the q-extensions of the Bernoulli and Euler numbers, related identities and Lerch zeta function*, Proc. Jangjeon Math. Soc., **12** (2009), 77–92. 1, 1

[10] T. Kim, D. S. Kim, H. I. Kwon, J. J. Seo, D. V. Dolgy, *Some identities of q-Euler polynomials under the symmetric group of degree n*, J. Nonlinear Sci. Appl., **9** (2016), 1077–1082. 1, 1

[11] D. Q. Lu, H. M. Srivastava, *Some series identities involving the generalized Apostol type and related polynomials*, Comput. Math. Appl., **62** (2011), 3591–3602. 1