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Fixed point theorems by combining Jleli and Samet's, and Branciari's inequalities

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Abstract

The aim of this paper is to introduce a new class of generalized metric spaces (called *RS-spaces*) that unify and extend, at the same time, Branciari's generalized metric spaces and Jleli and Samet's generalized metric spaces. Both families of spaces seen to be different in nature: on the one hand, Branciari's spaces are endowed with a rectangular inequality and their metrics are finite valued, but they can contain convergent sequences with two different limits, or convergent sequences that are not Cauchy; on the other hand, in Jleli and Samet's spaces, although the limit of a convergent sequence is unique, they are not endowed with a triangular inequality and we can found two points at infinite distance. However, we overcome such drawbacks and we illustrate that many abstract metric spaces (like dislocated metric spaces, b-metric spaces, rectangular metric spaces, modular metric spaces, among others) can be seen as particular cases of RS-spaces. In order to show its great applicability, we present some fixed point theorems in the setting of RS-spaces that extend well-known results in this line of research.

Keywords: Generalized metric space, Branciari metric space, fixed point, contractive mapping.
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1. Introduction

Fixed point theory is currently one of the most active branches of nonlinear analysis. Although some results in this line of research had previously appeared, it is widely accepted that this discipline was motivated

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by the Banach contractive mapping principle [4]. After the publication of such theorem, many extensions have been presented in different ways: some authors have weaken the contractivity condition (for instance, by involving auxiliary functions), other researchers have took into account distance spaces more general than metric spaces, and a third way consists in considering additional algebraic structures on the underlying space (see [2, 10, 16, 18–20, 24–28, 30]). In this paper we focus on the second methodology.

In the last years, there have been introduced many fixed point results in the setting of natural extensions of metric spaces. Among others, it is worth noting the following classes of spaces: quasimetric-spaces [3], Mustafa and Sims' generalized metric spaces [22], Czerwik's b-metric spaces [7], Hitzler and Seda's dislocated metric spaces [11], Nakano's modular spaces [23], Musielak and Orlicz's spaces [21], Bakhtin b-metric spaces [3], etc.

Very recently, two very general families of generalized metric spaces have attracted the attention of researchers. On the one hand, Branciari's generalized metric spaces were introduced in [5] in order to show some fixed point theorems. However, the presented proofs became incorrect because these spaces have metrically non-intuitive properties: for instance, there exist convergent sequences that are not Cauchy, or there exist convergent sequences with two different limits (see [29]). Nevertheless, these drawbacks have not been a limitation for developing fixed point theory in this environment (see [14, 15, 17, 31, 32]). On the other hand, Jleli and Samet [13] introduced a kind of generalized metric spaces which are not endowed with a proper triangle inequality: it was replaced by a weaker condition involving convergent sequences. Their spaces are also singular because the metric can take the value ∞, which is forbidden in most of previous generalized metric spaces.

At a first sight, Branciari's spaces and Jleli and Samet's spaces seem to be incompatible: for instance, in the second kind of spaces, the limit of a convergent sequence is unique, and two points can be placed having infinite distance between them. However, in this manuscript, we introduce a new class of spaces, that we call *RS-spaces*, that are natural extensions of both Branciari's spaces and Jleli and Samet's spaces. We also show some fixed point results that extend and clarify the relationships between the presented statements in this line of research by using such kind of spaces.

2. Preliminaries

Henceforth, $\mathbb{N} = \{0, 1, 2, \ldots\}$ stands for the set of all non-negative integer numbers, and let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. From now on, X will denote a nonempty set and $T : X \to X$ will be a self-mapping. We say that a sequence $\{x_n\}$ in X is *infinite* if $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ such that $n \neq m$.

Given a point $x_0 \in X$, the *Picard sequence of* T *based on* x_0 is the sequence $\{x_n\}_{n \geq 0}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. In particular, $x_n = T^n x_0$ for all $n \in \mathbb{N}$, where T^n denotes the n^{th}-iterates of T. A Picard sequence satisfies $x_{n+m} = T^m x_n = T^n x_m$ for all $n, m \in \mathbb{N}$. The *orbit of* x_0 *by* T is the set $\mathcal{O}_T(x_0) = \{\, T^n x_0 : n \in \mathbb{N} \,\}$.

A *binary relation on* X is a nonempty subset $\mathcal{S}$ of the Cartesian product $X \times X$. For simplicity, we denote $x\mathcal{S}y$ if $(x, y) \in \mathcal{S}$. We say that x and y are $\mathcal{S}$*-comparable* if $x\mathcal{S}y$ or $y\mathcal{S}x$. A binary relation $\mathcal{S}$ on X is *reflexive* if $x\mathcal{S}x$ for all $x \in X$; it is *transitive* if $x\mathcal{S}z$ for all $x, y, z \in X$ such that $x\mathcal{S}y$ and $y\mathcal{S}z$; and it is *antisymmetric* if $x\mathcal{S}y$ and $y\mathcal{S}x$ imply $x = y$. Given a non-empty subset A of X, we will say that $\mathcal{S}$ is *transitive on* A if

$$x, y, z \in A, \quad x\mathcal{S}y, \quad y\mathcal{S}z \quad \Rightarrow \quad x\mathcal{S}z.$$

A sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$*-nondecreasing* if $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$.

A *preorder* (or a *quasiorder*) is a reflexive, transitive binary relation and a *partial order* is an antisymmetric preorder. The *trivial preorder on* X is denoted by $\mathcal{S}_X$, and is given by $x\mathcal{S}_X y$ for all $x, y \in X$.

If $\phi : [0, \infty] \to [0, \infty]$ is a nondecreasing function, then the n^{th}-iterates ϕ^n of ϕ is also a nondecreasing function for all $n \in \mathbb{N}$. Furthermore, for all $s_1, s_2, \ldots, s_m \in [0, \infty]$, it follows that

$$\phi\left(\max\{\, s_1, s_2, \ldots, s_m \,\}\right) = \max\{\, \phi(s_1), \phi(s_2), \ldots, \phi(s_m) \,\}.$$

An *extended comparison function* (or, simply, a *comparison function*) is a function $\phi : [0,\infty] \to [0,\infty]$ such that

($\mathcal{P}_1$) ϕ is nondecreasing;

($\mathcal{P}_2$) for all $t \in (0,\infty)$, $\lim_{n\to\infty} \phi^n(t) = 0$.

Let $\mathcal{F}_{\text{com}}$ be the family of all (extended) comparison functions. For any $\phi \in \mathcal{F}_{\text{com}}$, it follows that: (1) $\phi(t) < t$ for all $t \in (0,\infty)$; (2) $\phi(0) = 0$; (3) ϕ is continuous at $t = 0$; (4) $\phi(t) \le t$ for all $t \in [0,\infty]$; (5) If $\phi(t) \ge t$, then $t \in \{0,\infty\}$; (6) $\phi^m(t) \le \phi^n(t) \le t$ for all $t \in [0,\infty]$ and all $n, m \in \mathbb{N}$ such that $n \le m$; and (7) ϕ^n is nondecreasing for all $n \in \mathbb{N}$.

Proposition 2.1 ([15]). *Let $\{a_n\} \subset [0,\infty)$ be a sequence of non-negative real numbers such that $\{a_n\} \to 0$ and let $\phi \in \mathcal{F}_{\text{com}}$. If*

$$b_n = \max\left\{ \phi(a_n), \phi^2(a_{n-1}), \phi^3(a_{n-2}), \dots, \phi^n(a_1), \phi^{n+1}(a_0) \right\} \quad \text{for all } n \in \mathbb{N},$$

then $b_n < \infty$ for all $n \in \mathbb{N}$. Furthermore, $\{b_n\} \to 0$.

2.1. Convergent and Cauchy sequences in spaces without metric structures

The following notions are usually given in a metric space or, at least, in a space endowed with a function that serves as a metric. However, as they can be considered in general, we recall them here. Notice that we do not assume any condition on the function $d : X \times X \to [0,\infty]$, which can also take the value ∞.

Definition 2.2 ([27, 31]). Given a function $d : X \times X \to [0,\infty]$, a sequence $\{x_n\} \subseteq X$ is:

- *d-Cauchy* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;
- *d-convergent to $x \in X$* if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$ (in such a case, we will write $\{x_n\} \xrightarrow{d} x$ and we will say that x is a *d-limit of* $\{x_n\}$);
- *strongly d-convergent to $x \in X$* if $\{x_n\}$ is d-Cauchy and d-convergent to x.

We say that (X, d) is *complete* if every d-Cauchy sequence in X is d-convergent to a point of X.

We say that a sequence $\{x_n\}$ is *almost periodic* if there exists $n_0 \in \mathbb{N}$ and $N \in \mathbb{N}^*$ such that

$$x_{n_0+r+Nk} = x_{n_0+r} \quad \text{for all } k \in \mathbb{N} \text{ and all } r \in \{0, 1, 2, \dots, N-1\}.$$

This means that

$$\begin{aligned}
&x_{n_0} = x_{n_0+N} = x_{n_0+2N} = x_{n_0+3N} = \dots, \\
&x_{n_0+1} = x_{n_0+1+N} = x_{n_0+1+2N} = x_{n_0+1+3N} = \dots, \\
&x_{n_0+2} = x_{n_0+2+N} = x_{n_0+2+2N} = x_{n_0+2+3N} = \dots, \\
&\qquad \vdots \\
&x_{n_0+N-1} = x_{n_0+2N-1} = x_{n_0+3N-1} = x_{n_0+4N-1} = \dots .
\end{aligned}$$

Therefore

$$\{ x_n : n \ge n_0 \} = \{ x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots, x_{n_0+N-1} \} = \{ x_n : n \ge k_0 \}$$

for all $k_0 \ge n_0$. The following results are useful in order to describe Picard sequences and d-Cauchy Picard sequences.

Proposition 2.3. *Every Picard sequence is either infinite or almost periodic.*

Proof. Let $T : X \to X$ be a self-mapping and let $\{x_n\}$ be a Picard sequence of T. Suppose that $\{x_n\}$ is not infinite and we have to prove that it is almost periodic. If $\{x_n\}$ is not infinite, there exist $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. Let $N = m_0 - n_0 \in \mathbb{N}^*$. Firstly, we prove that

$$x_{n_0+Nk} = x_{n_0} \quad \text{for all } k \in \mathbb{N}. \tag{2.1}$$

We proceed by induction methodology. If $k = 0$, it is obvious, and if $k = 1$, then $x_{n_0+N} = x_{n_0+(m_0-n_0)} = x_{m_0} = x_{n_0}$. Assume that (2.1) holds for some $k \in \mathbb{N}$. Then

$$x_{n_0+N(k+1)} = x_{n_0+Nk+N} = T^N(x_{n_0+Nk}) = T^N x_{n_0} = x_{n_0+N} = x_{n_0}.$$

This completes the induction, so (2.1) holds. In particular, by (2.1), we have that

$$x_{n_0+Nk+r} = T^r(x_{n_0+Nk}) = T^r(x_{n_0}) = x_{n_0+r} \quad \text{for all } k \in \mathbb{N} \text{ and all } r \in \{0, 1, 2, \ldots, N-1\}. \tag{2.2}$$

Therefore, $\{x_n\}$ is almost periodic. □

Proposition 2.4. *If $d : X \times X \to [0, \infty]$ is a function and $\{x_n\}$ is a d-Cauchy Picard sequence on X, then at least one of the following conditions holds:*

(a) *$x_n \neq x_m$ for all $n, m \in \mathbb{N}$ such that $n \neq m$ (that is, $\{x_n\}$ is an infinite sequence);*

(b) *there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) = 0$ for all $n, m \geq n_0$.*

Furthermore, in case (b), if d satisfies the condition "$d(x, y) = 0 \Rightarrow x = y$", then there exists $z \in X$ such that $x_n = z$ for all $n \geq n_0$, $d(z, z) = 0$ and $\{x_n\}$ strongly d-converges to z.

Proof. Let $T : X \to X$ be a self-mapping such that $\{x_n\}$ is a Picard sequence associated to T. Assume that (a) is false and we have to prove that (b) holds. Following the proof of Proposition 2.3, there exist $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. If $N = m_0 - n_0 \in \mathbb{N}^*$, then

$$x_{n_0+r+Nk} = x_{n_0+r} \quad \text{for all } k \in \mathbb{N} \text{ and all } r \in \{0, 1, 2, \ldots, N-1\}.$$

This means that $\{x_n\}_{n \geq n_0}$ is a periodic sequence whose terms are $\{ x_n : n \geq n_0 \} = \{ x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots, x_{m_0-1} \} = Y$. The term after x_{m_0-1} is x_{n_0} and so on. Therefore

$$\{ x_n : n \geq k_0 \} = Y \quad \text{for all } k_0 \geq n_0$$

because the sequence is periodic. As Y is a finite set, we can consider

$$\begin{aligned} \delta &= \max \{ d(y, z) : y, z \in Y \} \\ &= \max \{ d(x_{n_0+i}, x_{n_0+j}) : i, j \in \{0, 1, 2, \ldots, N-1\}\} \in [0, \infty]. \end{aligned}$$

We claim that $\delta < \infty$. Taking into account that $\{x_n\}$ is a d-Cauchy sequence on X, given $\varepsilon = 1$, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $d(x_n, x_m) < 1$ for all $n, m \geq n_1$. Since $\{ x_n : n \geq n_1 \} = \{ x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots, x_{m_0-1} \} = Y$, then

$$\delta = \max \{ d(y, z) : y, z \in Y \} = \sup (d(x_n, x_m) : n, m \geq n_1) \leq 1.$$

Therefore, δ is finite. Let $i_0, j_0 \in \{0, 1, 2, \ldots, N-1\}$ be such that

$$\delta = d(x_{n_0+i_0}, x_{n_0+j_0}).$$

We announce that $\delta = 0$. We reason by contradiction. Assume that $\delta > 0$. Since $\{x_n\}$ is a d-Cauchy sequence on X, there exists $n_2 \in \mathbb{N}$ such that $n_2 \geq n_0$ and $d(x_n, x_m) < \delta/2$ for all $n, m \geq n_2$. As $x_{n_0+i_0}, x_{n_0+j_0} \in Y = \{ x_n : n \geq n_2 \}$, we deduce that

$$\delta = d(x_{n_0+i_0}, x_{n_0+j_0}) \leq \frac{\delta}{2},$$

which is a contradiction. As a result

$$\sup\left(d(x_n, x_m) : n, m \geq n_0\right) = \max\{ d(y,z) : y, z \in Y \} = \delta = 0.$$

Hence, $d(x_n, x_m) = 0$ for all $n, m \geq n_0$, so item (b) holds.

Next, assume that the condition "$d(x,y) = 0 \Rightarrow x = y$" holds. Therefore, as $d(x_n, x_m) = 0$ for all $n, m \geq n_0$, it follows that $x_n = x_m$ for all $n, m \geq n_0$. If we call $z = x_{n_0}$, then $x_n = z$ for all $n \geq n_0$. Moreover, $d(z,z) = d(x_{n_0}, x_{n_0}) = 0$ and $d(z, x_n) = d(x_n, z) = 0$ for all $n \geq n_0$. In particular, $\{x_n\}$ d-converges to z. □

One of the most important properties of an abstract metric space is the uniqueness of the limit. Usually, a convergent sequence with two different limits produces non-intuitive results. For instance, this is the case of B_N-spaces. Nevertheless, in the following definitions, such property is not required.

Let $\mathcal{S}$ be a binary relation on X.

Definition 2.5. The pair (X, d) is *$\mathcal{S}$-nondecreasing-regular* if

$$\left.\begin{array}{l} \{x_n\} \overset{d}{\to} z \\ \{x_n\}\ \mathcal{S}\text{-nondecreasing} \end{array}\right\} \quad \Rightarrow \quad x_n \mathcal{S} z \ \text{ for all } n \in \mathbb{N}.$$

Definition 2.6. The pair (X, d) is *$\mathcal{S}$-nondecreasing-complete* if

$$\left.\begin{array}{l} \{x_n\} \subseteq X \text{ is } d\text{-Cauchy} \\ \{x_n\}\ \mathcal{S}\text{-nondecreasing} \end{array}\right\} \quad \Rightarrow \quad \{x_n\} \text{ is } d\text{-convergent in } X.$$

Definition 2.7. A self-mapping $T : X \to X$ is *$\mathcal{S}$-nondecreasing-continuous at $z \in X$* if $\{Tx_n\} \overset{d}{\to} Tz$ for all $\mathcal{S}$-nondecreasing sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \overset{d}{\to} z$. The mapping T is *$\mathcal{S}$-nondecreasing-continuous* if it is $\mathcal{S}$-nondecreasing-continuous at each point of X.

2.2. Branciari N-generalized metric spaces

The following notion was introduced by Branciari in [5].

Definition 2.8 ([5])**.** Given $N \in \mathbb{N}^*$, a *Branciari N-generalized metric space* (for short, a *B_N-space*) is a pair (X, d), where X is a non-empty set and $d : X \times X \to [0, \infty)$ is a function such that the following properties hold:

(B_1) $d(x,y) = 0$ if, and only if, $x = y$.

(B_2) $d(y,x) = d(x,y)$.

(B_3) $d(x,y) \leq d(x,u_1) + d(u_1,u_2) + d(u_2,u_3) + \ldots + d(u_{N-1},u_N) + d(u_N,y)$ for any $x, u_1, u_2, \ldots, u_N, y \in X$ such that $x, u_1, u_2, \ldots, u_N, y$ are all different.

If $N = 2$, then (X, d) is a *Branciari generalized metric space* (for short, a *B-space*).

A B_1-space is a metric space. However, if $N \geq 2$, it was proved that B_N-spaces can satisfy some properties that are not metrically desirable (see [14, 29]). For instance, in a B_N-space,

- there may exist convergent sequences that are not Cauchy sequences;
- there may exist convergent sequences with two different limits;
- the metric $d : X \times X \to [0, \infty)$ may not be a continuous function;
- there may exist open balls centered in different points that are never disjoint although their radius are arbitrarily small.

Example 2.9 ([29])**.** Let $A = \{0, 2\}$, $B = \{\, 1/n : n \in \mathbb{N}^* \,\}$ and $X = A \cup B$. Define $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y \text{ and } (\{x, y\} = A \text{ or } \{x, y\} \subset B); \\ y, & \text{if } x \in A \text{ and } y \in B; \\ x, & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Then (X, d) is a B-space. Although the sequence $\{1/n\}_{n \in \mathbb{N}^*}$ is not d-Cauchy, it d-converges, at the same time, to $x = 0$ and to $y = 2$. Hence, the d-limit of a d-convergent sequence in a B-space need not be unique.

Surprisingly, in [31], Suzuki *et al.* proved that, for $N \geq 2$, only B_3-spaces have a compatible symmetric topology.

2.3. Jleli and Samet's generalized metric spaces

Henceforth, let $\mathcal{D} : X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, define the set

$$C(\mathcal{D}, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n \to \infty} \mathcal{D}(x_n, x) = 0 \right\}. \tag{2.3}$$

Generalized metric and generalized metric space are defined as follows.

Definition 2.10 ([13], Definition 2.1)**.** Let X be a nonempty set and let $\mathcal{D} : X \times X \to [0, \infty]$ be a function which satisfies:

($\mathcal{D}_1$) $\mathcal{D}(x, y) = 0$ implies $x = y$;

($\mathcal{D}_2$) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$ for all $x, y \in X$;

($\mathcal{D}_3$) there exists $C > 0$ such that

$$\text{if } x, y \in X \text{ and } \{x_n\} \in C(\mathcal{D}, X, x), \quad \text{then} \quad \mathcal{D}(x, y) \leq C \limsup_{n \to \infty} \mathcal{D}(x_n, y). \tag{2.4}$$

Then $\mathcal{D}$ is called a *generalized metric* and the pair $(X, \mathcal{D})$ is called a *generalized metric space* (in the sense of Jleli and Samet; for short, a *JS-space*).

The following is an example of JS-space, where the metric takes the value ∞.

Example 2.11 ([15])**.** Let $X = \{0, 1\}$ be endowed with the function $\mathcal{D} : X \times X \to [0, \infty]$ given by

$$\mathcal{D}(0, 0) = 0 \quad \text{and} \quad \mathcal{D}(1, 0) = \mathcal{D}(0, 1) = \mathcal{D}(1, 1) = \infty.$$

Then $(X, \mathcal{D})$ is a JS-space and (2.4) holds with $C = 1$.

Given a JS-space $(X, \mathcal{D})$ and a point $x \in X$, a sequence $\{x_n\} \subseteq X$ is said to be:

- *$\mathcal{D}$-convergent to x* if $\{x_n\} \in C(\mathcal{D}, X, x)$ (in such a case, we will write $\{x_n\} \xrightarrow{\mathcal{D}} x$);
- *$\mathcal{D}$-Cauchy* if $\lim_{n,m \to \infty} \mathcal{D}(x_n, x_m) = 0$.

A JS-space $(X, \mathcal{D})$ is *complete* if every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent. Jleli and Samet proved that the limit of a $\mathcal{D}$-convergent sequence is unique.

Proposition 2.12 ([13], Proposition 2.4)**.** *Let $(X, \mathcal{D})$ be a JS-space. Let $\{x_n\}$ be a sequence in X and $(x, y) \in X \times X$. If $\{x_n\}$ $\mathcal{D}$-converges to x and $\{x_n\}$ $\mathcal{D}$-converges to y, then $x = y$.*

2.4. b-dislocated metric spaces

As in the case of partial metric spaces, the following classes were firstly considered by avoiding the condition $d(x,x)=0$ on a metric.

Definition 2.13 ([11])**.** Let X be a nonempty set. A mapping $d_\ell : X \times X \to [0,\infty)$ is called a *dislocated metric* (or simply *d_ℓ-metric*) if the following conditions hold for any $x,y,z \in X$:

(d_1) If $d_\ell(x,y)=0$, then $x=y$;

(d_2) $d_\ell(x,y)=d_\ell(y,x)$;

(d_3) $d_\ell(x,y) \leq d_\ell(x,z)+d_\ell(z,y)$.

The pair (X,d_ℓ) is called a *dislocated metric space* or a *d_ℓ-metric space.*

Definition 2.14 ([12])**.** Let X be a set and let $d_b : X \times X \to [0,\infty)$ be a mapping satisfying, for each $x,y,z \in X$ and $s \geq 1$:

(bd_1) $d_b(x,y)=0$ implies $x=y$;

(bd_2) $d_b(x,y)=d_b(y,x)$;

(bd_3) $d_b(x,y) \leq s(\,d_b(x,z)+d_b(z,y)\,)$.

Then the pair (X,d) is called a *b-dislocated metric space.*

2.5. Rectangular, b-metric and rectangular b-metric spaces

The following spaces were successively defined.

Definition 2.15 ([3])**.** A *b-metric on* X is a mapping $d : X \times X \to [0,\infty)$ satisfying the following properties:

(bM_1) $d(x,y)=0$ if, and only if, $x=y$;

(bM_2) $d(x,y)=d(y,x)$ for all $x,y \in X$;

(bM_3) there exists a real number $s \geq 1$ such that $d(x,y) \leq s\,[\,d(x,z)+d(z,y)\,]$ for all $x,y,z \in X$.

In such a case, (X,d) is called a *b-metric space* (in short *bM-space*) *with coefficient s.*

Definition 2.16 ([5])**.** A *rectangular metric on* X is a mapping $d : X \times X \to [0,\infty)$ satisfying the following properties:

(RM_1) $d(x,y)=0$ if, and only if, $x=y$;

(RM_2) $d(x,y)=d(y,x)$ for all $x,y \in X$;

(RM_3) $d(x,y) \leq d(x,u)+d\,(u,v)+d(v,y)$ for all $x,y \in X$ and all distinct points $u,v \in X \diagdown \{x,y\}$.

In such a case, (X,d) is called a *rectangular metric space* (in short *RM-space*).

Notice that rectangular metrics are B_2-metrics in the sense of Definition 2.8.

Definition 2.17 ([9])**.** A *rectangular b-metric on* X is a mapping $d : X \times X \to [0,\infty)$ satisfying the following properties:

(RbM_1) $d(x,y)=0$ if, and only if, $x=y$;

(RbM_2) $d(x,y)=d(y,x)$ for all $x,y \in X$;

(RbM_3) there exists a real number $s \geq 1$ such that $d(x,y) \leq s\,[\,d(x,u)+d\,(u,v)+d(v,y)\,]$ for all $x,y \in X$ and all distinct points $u,v \in X \diagdown \{x,y\}$.

In such a case, (X,d) is called a *rectangular b-metric space* (in short *RbM-space*) *with coefficient s.*

See [8, 9] for some results in these kind of spaces.

3. RS-generalized metric spaces

In this section we present a new class of generalized metric spaces and show that various well-known abstract metric spaces belong to this class.

Definition 3.1. An *RS-generalized metric space* (for short, an *RS-space*) is a pair $(X, \mathcal{D})$ where X is a non-empty set and $\mathcal{D} : X \times X \to [0, \infty]$ is a function such that the following properties are fulfilled:

($\mathcal{D}_1$) If $\mathcal{D}(x, y) = 0$ then $x = y$;

($\mathcal{D}_2$) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$ for all $x, y \in X$;

($\mathcal{D}_3'$) there exists $C > 0$ such that if $x, y \in X$ are two points and $\{x_n\}$ is a $\mathcal{D}$-Cauchy infinite sequence in X such that $\{x_n\} \xrightarrow{\mathcal{D}} x$ then

$$\mathcal{D}(x, y) \le C \limsup_{n\to\infty} \mathcal{D}(x_n, y). \tag{3.1}$$

If X is endowed with a binary relation $\mathcal{S}$, then an *RS-space* is a triple $(X, \mathcal{D}, \mathcal{S})$ satisfying ($\mathcal{D}_1$), ($\mathcal{D}_2$) and ($\mathcal{D}_3'$) assuming that the sequence $\{x_n\}$ in ($\mathcal{D}_3'$) is $\mathcal{S}$-nondecreasing.

Remark 3.2. If $(X, \mathcal{D})$ is an RS-space, then it is easy to see that the set of all constants for which ($\mathcal{D}_3'$) holds is a nonempty, non-upper-bounded interval of non-negative real numbers. Its infimum is the lowest (optimal) constant for which ($\mathcal{D}_3'$) holds. We will denote it by $C_{X,\mathcal{D}}$. Since the case $C_{X,\mathcal{D}} = 0$ leads to a trivial space, throughout we shall assume that $C_{X,\mathcal{D}} > 0$.

Let us show that the class of RS-spaces contains some important subclasses.

Lemma 3.3. *Every JS-space is an RS-space.*

Proof. It is obvious because ($\mathcal{D}_3$) implies ($\mathcal{D}_3'$). □

Corollary 3.4. *Every b-dislocated metric is a JS-space and so it is an RS-space.*

Jleli and Samet showed in [13] a great variety of JS-spaces: Czerwik *b-metric spaces* [7], Hitzler and Seda's *dislocated metric spaces* [11], Nakano's *modular spaces* [23], Musielak and Orlicz's spaces [21], *convex modular spaces* having the *Fatou property* [1, 13], etc.. Lemma 3.3 guarantees that such classes of spaces are also RS-spaces, which gives validity to our study.

Next, we show that, although we have slightly modified Jleli and Samet's axiom ($\mathcal{D}_3$), this subtle change is sufficient to cover Branciari's generalized metric spaces.

Lemma 3.5. *Every B_N-space is an RS-space (where $\mathcal{D} = d$ and $C = 1$).*

Proof. Let (X, d) be a B_N-space. If $N = 1$, then (X, d) is a metric space, so (X, d) is an RS-space (notice that condition ($\mathcal{D}_3'$) follows from the continuity of the metric). Assume that $N \ge 2$. If we take $\mathcal{D} = d$, then ($\mathcal{D}_1$) and ($\mathcal{D}_2$) follows from (B_1) and (B_2), respectively. To prove ($\mathcal{D}_3'$), let $x, y \in X$ and let $\{x_n\} \subseteq X$ be an infinite, strongly d-convergent sequence such that $\{x_n\} \xrightarrow{d} x$. If $x = y$, then $\mathcal{D}(x, y) = 0$ and ($\mathcal{D}_3'$) trivially holds. Assume that $\mathcal{D}(x, y) > 0$, that is, $x \ne y$. As $\{x_n\}$ is an infinite sequence, there exists $n_0 \in N$ such that $x_n \ne x$ and $x_n \ne y$ for all $n \ge n_0$. Therefore,

$$d(x, y) \le d(x, x_{n+N-1}) + d(x_{n+N-1}, x_{n+N-2}) + \ldots + d(x_{n+1}, x_n) + d(x_n, y)$$

for all $n \ge n_0$. Taking into account that $\{x_n\}$ is d-Cauchy and d-converges to x, letting $n \to \infty$ in the previous inequality, we deduce that

$$\begin{aligned} d(x, y) &\le \limsup_{n\to\infty} \Big(d(x, x_{n+N-1}) + d(x_{n+N-1}, x_{n+N-2}) + \ldots + d(x_{n+1}, x_n) + d(x_n, y) \Big) \\ &= 0 + 0 + \ldots + 0 + \limsup_{n\to\infty} d(x_n, y) = \limsup_{n\to\infty} d(x_n, y), \end{aligned}$$

which means that ($\mathcal{D}_3'$) holds with $C = 1$. □

Notice that the class of RS-spaces is larger than the classes of JS-spaces and B_N-spaces. On the one hand, in [29], the authors showed a B_2-space having a sequence with two different limits (see also [14]). By Proposition 2.12, this space cannot be a JS-space. On the other hand, the JS-space showed in Example 2.11 is not a B_N-space because the metric takes the value ∞. In Lemma 3.7 we will show a subclass of RS-spaces that are neither JS-spaces nor B_N-spaces.

Lemmas 3.3 and 3.5 advise us that RS-spaces can have all "metric drawbacks" of both classes of spaces: sequences converging to two different limits, distance ∞ between some points, absence of triangle inequality, etc. The main aim of this manuscript is to overcome such drawbacks in the field of *fixed point theory*.

By using the same arguments that we employ in the proof of Lemma 3.5, it is easy to check the following result.

Lemma 3.6. *Rectangular, b-metric and rectangular b-metric spaces are RS-spaces.*

Next we show an example of an RS-space which is neither a Branciari space nor a JS-space.

Lemma 3.7. *Let (Y,d) be a B_N-space and let $Z=\{z_0,z_1,z_2,\ldots,z_N,z_{N+1}\}$ be a set of $N+2$ different points such that $Y\cap Z=\varnothing$. Let $X=Y\cup Z$ and define $\mathcal{D}:X\times X\to[0,\infty]$ as follows:*

$$\mathcal{D}(x,y)=\begin{cases} d(x,y), & \text{if } x,y\in Y,\\ \infty, & \text{if } x\in Y \text{ and } y\in Z, \text{ or viceversa},\\ 0, & \text{if } x=y\in Z,\\ 3, & \text{if } x\neq y \text{ and } z_1\in\{x,y\},\\ M, & \text{if } \{x,y\}=\{z_0,z_{N+1}\},\\ 2, & \text{otherwise (with } x,y\in Z \text{ and } x\neq y), \end{cases}$$

where $2\leq M\leq 3N+2$. Then $(X,\mathcal{D})$ is an RS-space but it is not a B_N-space.

Furthermore, if (Y,d) contains a sequence with two different d-limits in Y, then $(X,\mathcal{D})$ is not a JS-space.

Notice that if $2N+4<M\leq 3N+2$, then $(X,\mathcal{D})$ does not satisfy the Branciari inequality

$$\mathcal{D}(x,y)\leq\mathcal{D}(x,u_1)+\mathcal{D}(u_1,u_2)+\mathcal{D}(u_2,u_3)+\ldots+\mathcal{D}(u_{N-1},u_N)+\mathcal{D}(u_N,y),$$

because

$$\begin{aligned}\mathcal{D}(z_0,z_{N+1})&=M>2N+4=3+3+2(N-1)=3+3+(2+2+\overset{N-1\text{ times}}{\cdots}+2+2)\\ &=\mathcal{D}(z_0,z_1)+\mathcal{D}(z_1,z_2)+\mathcal{D}(z_2,z_3)+\ldots+\mathcal{D}(z_{N-1},z_N)+\mathcal{D}(z_N,z_{N+1}).\end{aligned}\tag{3.2}$$

Proof. Let us prove the following three properties.

($\mathcal{D}_1$) Let $x,y\in X$ be such that $\mathcal{D}(x,y)=0$. Then $d(x,y)=0$ or $x=y\in Z$. Both cases leads to $x=y$, so ($\mathcal{D}_1$) holds.

($\mathcal{D}_2$) It is obvious that $\mathcal{D}$ is symmetric.

($\mathcal{D}_3'$) Let $x,y\in X$ and let $\{x_n\}\subseteq X$ be an infinite, strongly $\mathcal{D}$-convergent sequence such that $\{x_n\}\xrightarrow{\mathcal{D}}x$. As the set $Z\cup\{x,y\}$ is finite, there exists $n_0\in\mathbb{N}$ such that $x_n\in Y$, $x_n\neq x$ and $x_n\neq y$ for all $n\geq n_0$. If $x\notin Y$, then $\mathcal{D}(x_n,x)=\infty$ for all $n\geq n_0$, which is impossible because $\{\mathcal{D}(x_n,x)\}\to 0$. Hence $x\in Y$. Moreover, if $y\notin Y$, then $\mathcal{D}(x_n,y)=\infty$ for all $n\geq n_0$, so ($\mathcal{D}_3'$) trivially holds. Assume that $y\in Y$. In this case, $\mathcal{D}(x_n,y)=d(x_n,y)$ for all $n\geq n_0$. As (X,d) is a B_N-space, we have that

$$\mathcal{D}(x,y)=d(x,y)\leq d(x,x_{n+N})+d(x_{n+N},x_{n+N-1})+d(x_{n+N-1},x_{n+N-2})+\ldots+d(x_{n+1},x_n)+d(x_n,y)$$

for all $n\geq n_0$. Taking into account that $\{x_n\}$ is d-Cauchy (=$\mathcal{D}$-Cauchy on Y) and it d-converges (=$\mathcal{D}$-converges on Y) to x, letting $n\to\infty$ in the previous inequality, we deduce that

$$\mathcal{D}(x,y)\leq\limsup_{n\to\infty}\Big(d(x,x_{n+N})+d(x_{n+N},x_{n+N-1})+d(x_{n+N-1},x_{n+N-2})+\ldots+d(x_{n+1},x_n)+d(x_n,y)\Big)$$

$$= 0 + 0 + 0 + \ldots + 0 + \limsup_{n\to\infty} d\left(x_n, y\right) = \limsup_{n\to\infty} \mathcal{D}\left(x_n, y\right).$$

Hence, condition $(\mathcal{D}_3')$ also holds and $(X, \mathcal{D})$ is an RS-space. Notice that it is not a B_N-space: if $x \in Y$, then $\mathcal{D}(x, z_0) = \infty$, which is impossible in a B_N-space (its metric is finite valued). Furthermore, if (Y, d) contains a sequence with two different d-limits in Y, then $(X, \mathcal{D})$ contains a sequence with two different $\mathcal{D}$-limits in X (because $d = \mathcal{D}|_{Y\times Y}$). In such a case, $(X, \mathcal{D})$ is not a JS-space because of Proposition 2.12. □

Although the limit of convergent sequences in a B-space is not necessarily unique, we have the following property.

Proposition 3.8. *Let $\{x_n\}$ be a $\mathcal{D}$-Cauchy sequence in an RS-space $(X, \mathcal{D})$ such that $\{x_n\}$ is infinite or Picard. Then $\{x_n\}$ has, at most, a unique $\mathcal{D}$-limit (that is, if $\{x_n\} \xrightarrow{\mathcal{D}} x$ and $\{x_n\} \xrightarrow{\mathcal{D}} y$, where $x, y \in X$, then $x = y$).*

Proof. Suppose that $\{x_n\} \xrightarrow{\mathcal{D}} x$ and $\{x_n\} \xrightarrow{\mathcal{D}} y$, where $x, y \in X$.

Case 1. Assume that $\{x_n\}$ is infinite. By condition $(\mathcal{D}_3')$,

$$\mathcal{D}(x, y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y) = 0,$$

so $(\mathcal{D}_1)$ guarantees that $x = y$.

Case 2. Assume that $\{x_n\}$ is Picard. In this case, it follows from Proposition 2.4 that, if $\{x_n\}$ is not infinite (in other case, we can apply case 1), there exists $z \in X$ such that $x_n = z$ for all $n \geq n_0$, $\mathcal{D}(z, z) = 0$ and $\{x_n\}$ strongly $\mathcal{D}$-converges to z. Since $\mathcal{D}(x_n, x) = \mathcal{D}(z, x)$ is constant for all $n \geq n_0$ and $\lim_{n\to\infty} \mathcal{D}(x_n, x) = 0$, then $\mathcal{D}(z, x) = 0$, so $z = x$. Similarly $z = y$, so we conclude that $x = y$. □

Let $(X, \mathcal{D})$ be an RS-space and let $\{z_n\} \subseteq X$ be a sequence. Given $n_0 \in \mathbb{N}$, we denote

$$\delta_{n_0}(\mathcal{D}, \{z_n\}) = \sup\left(\{\, \mathcal{D}(z_n, z_m) : n, m \in \mathbb{N},\ n, m \geq n_0 \,\}\right).$$

Given a self-mapping $T : X \to X$ and a point $x_0 \in X$, we will use the notation:

$$\delta_{n_0}(\mathcal{D}, T, x_0) = \sup\left(\{\, \mathcal{D}(T^n x_0, T^m x_0) : n, m \in \mathbb{N},\ n, m \geq n_0 \,\}\right),$$

and

$$\delta(\mathcal{D}, T, x_0) = \delta_0(\mathcal{D}, T, x_0) = \sup\left(\{\, \mathcal{D}(T^n x_0, T^m x_0) : n, m \in \mathbb{N} \,\}\right).$$

Notice that $\delta_{n_0}(\mathcal{D}, T, x_0)$ is $\delta_{n_0}(\mathcal{D}, \{x_n\})$ where $\{x_n\}$ is the Picard sequence of T based on x_0. By the symmetry of $\mathcal{D}$, we can alternatively express

$$\delta_{n_0}(\mathcal{D}, T, x_0) = \sup\left(\{\, \mathcal{D}(T^n x_0, T^m x_0) : n, m \in \mathbb{N},\ m \geq n \geq n_0 \,\}\right).$$

Notice that if $n, m \in \mathbb{N}$ with $n \leq m$, then

$$\delta_m(\mathcal{D}, T, x_0) \leq \delta_n(\mathcal{D}, T, x_0) \leq \delta(\mathcal{D}, T, x_0). \tag{3.3}$$

Proposition 3.9. *A sequence $\{z_n\}$ in an RS-space $(X, \mathcal{D})$ is $\mathcal{D}$-Cauchy if, and only if, $\lim_{m\to\infty} \delta_m(\mathcal{D}, \{z_n\}) = 0$. In particular, there exists $n_0 \in \mathbb{N}$ such that $\delta_{n_0}(\mathcal{D}, \{z_n\}) < \infty$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\{z_n\}$ is $\mathcal{D}$-Cauchy, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{D}(z_n, z_m) < \varepsilon$ for all $n, m \in \mathbb{N}$ satisfying $n, m \geq n_0$. Hence $\delta_{n_0}(\mathcal{D}, \{z_n\}) \leq \varepsilon$. This means that $\lim_{m\to\infty} \delta_m(\mathcal{D}, \{z_n\}) = 0$. The converse is similar. In particular, for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $\delta_{n_0}(\mathcal{D}, \{z_n\}) \leq 1 < \infty$. □

Remark 3.10. The condition "$\delta_{n_0}(\mathcal{D}, \{z_n\}) < \infty$ for some $n_0 \in \mathbb{N}$" does not imply that the sequence $\{z_n\}$ is $\mathcal{D}$-Cauchy even in metric spaces: if $X = \mathbb{R}$ is endowed with the Euclidean metric d_E, the sequence $\{z_n\}$ given by $z_n = (-1)^n$ for all $n \in \mathbb{N}$ satisfies $\delta_{n_0}(d_E, \{z_n\}) < \infty$ for all $n_0 \in \mathbb{N}$. However, $\{z_n\}$ is not a d_E-Cauchy sequence. Therefore, the condition "$\delta_{n_0}(\mathcal{D}, \{z_n\}) < \infty$ for some $n_0 \in \mathbb{N}$" is more general than Cauchy's property.

4. Ćirić type fixed point theorems in the context of RS-generalized metric spaces

This section is dedicated to introduce, in the setting of RS-spaces, the main results of this manuscript inspired by the Ćirić type contractivity condition presented in [6]. We employ a similar scheme to that the authors used in [15] but taking into account that in RS-spaces we have to deal with several drawbacks.

Remark 4.1.

1. The results, we present in this section, can be similarly stated by considering a binary relation on X that needs only to be reflexive and transitive on the orbit $\mathcal{O}_T(x_0)$ or in the set of comparable pairs. This would lead to weaker statements. However, for simplicity, we involve a preorder $\mathcal{S}$ on X.
2. Notice that the following notions are given involving nondecreasing sequences. Similar concepts can be introduced for non-increasing sequences (we leave this task to the reader).
3. Throughout this section, given an initial point $x_0 \in X$ and a mapping $T : X \to X$, we will always denote by $\{x_n\}_{n\in\mathbb{N}}$ the Picard sequence of T based on x_0.

4.1. Some common properties

In the next subsections, we will present some fixed point theorems under Ćirić contractivity conditions by involving different initial hypotheses (continuity, regularity, etc.) Nevertheless, there is a common part in the proofs of our main results. In this subsection we describe such common properties in the following theorem.

Theorem 4.2. *Let $(X, \mathcal{D}, \mathcal{S})$ be an RS-space with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} Tx_0$ and $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \le \phi\left(\max\left\{\mathcal{D}(x, y), \mathcal{D}(x, Tx), \mathcal{D}(y, Ty), \mathcal{D}(x, Ty), \mathcal{D}(y, Tx)\right\}\right) \quad \text{for all } x, y \in \mathcal{O}_T(x_0). \tag{4.1}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 is an $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy sequence.

Furthermore, if $(X, \mathcal{D})$ is $\mathcal{S}$-nondecreasing-complete, then $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{D}$-converges to a point $\omega \in X$ that satisfies

$$\mathcal{D}(\omega, \omega) = 0 \quad \text{and} \tag{4.2}$$

$$\mathcal{D}(x_n, \omega) \le C\, \phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D}, T, x_0)\right) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \ge n_0, \tag{4.3}$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X, \mathcal{D})$ satisfies property $(\mathcal{D}'_3)$.

Proof. Let us consider the Picard sequence $\{x_n = T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0. We divide the proof into four steps.

Step 1. We claim that $\{x_n\}$ is an $\mathcal{S}$-nondecreasing sequence. Since $x_0 \mathcal{S} Tx_0 = x_1$ (the case $Tx_0 \mathcal{S} x_0$ is similar), as T is $\mathcal{S}$-nondecreasing, then $x_1 = Tx_0 \mathcal{S} Tx_1 = x_2$. Repeating this argument, $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is an $\mathcal{S}$-nondecreasing sequence.

Step 2. We claim that

$$\delta_{k+1}(\mathcal{D}, T, x_0) \le \phi\left(\delta_k(\mathcal{D}, T, x_0)\right) \quad \text{for all } k \in \mathbb{N},\ k \ge n_0. \tag{4.4}$$

To prove it, let $k \in \mathbb{N}$ be an arbitrary integer number such that $k \ge n_0$. By (3.3),

$$\delta_{k+1}(\mathcal{D}, T, x_0) \le \delta_k(\mathcal{D}, T, x_0) \le \delta_{n_0}(\mathcal{D}, T, x_0) < \infty.$$

Let $n, m \in \mathbb{N}$ be such that $m \ge n \ge k+1$, and let us define $m' = m-1$ and $n' = n-1$. Then $m' \ge n' \ge k$. In view of (4.1), we have

$$\mathcal{D}(x_n, x_m) = \mathcal{D}(x_{n'+1}, x_{m'+1}) = \mathcal{D}(Tx_{n'}, Tx_{m'})$$

$$\leq \phi\left(\max\left\{\, \mathcal{D}(x_{n'},x_{m'}),\, \mathcal{D}(x_{n'},x_{n'+1}),\, \mathcal{D}(x_{m'},x_{m'+1}), \mathcal{D}(x_{n'},x_{m'+1}),\, \mathcal{D}(x_{m'},x_{n'+1}) \,\right\}\right).$$

If we set

$$\Delta_k = \left\{\, \mathcal{D}\left(T^p x_0, T^q x_0\right) : p,q \in \mathbb{N},\ p,q \geq k \,\right\},$$

then

$$\mathcal{D}(x_{n'},x_{m'}),\, \mathcal{D}(x_{n'},x_{n'+1}),\, \mathcal{D}(x_{m'},x_{m'+1}),\, \mathcal{D}(x_{n'},x_{m'+1}),\, \mathcal{D}(x_{m'},x_{n'+1}) \in \Delta_k.$$

So,

$$\max\left\{\, \mathcal{D}(x_{n'},x_{m'}),\, \mathcal{D}(x_{n'},x_{n'+1}),\, \mathcal{D}(x_{m'},x_{m'+1}), \mathcal{D}(x_{n'},x_{m'+1}),\, \mathcal{D}(x_{m'},x_{n'+1}) \,\right\} \leq \sup \Delta_k = \delta_k\left(\mathcal{D},T,x_0\right).$$

Consequently, as ϕ is nondecreasing, for all $m \geq n \geq k+1$,

$$\begin{aligned} \mathcal{D}\left(x_n,x_m\right) &\leq \phi\left(\max\left\{\, \mathcal{D}(x_{n'},x_{m'}),\, \mathcal{D}(x_{n'},x_{n'+1}),\, \mathcal{D}(x_{m'},x_{m'+1}), \mathcal{D}(x_{n'},x_{m'+1}),\, \mathcal{D}(x_{m'},x_{n'+1}) \,\right\}\right) \\ &\leq \phi\left(\delta_k\left(\mathcal{D},T,x_0\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned} \delta_{k+1}\left(\mathcal{D},T,x_0\right) &= \sup\left(\left\{\, \mathcal{D}\left(x_n,x_m\right) : n,m \in \mathbb{N},\ m \geq n \geq k+1 \,\right\}\right) \\ &\leq \phi\left(\delta_k\left(\mathcal{D},T,x_0\right)\right). \end{aligned}$$

This prove that (4.4) holds. Repeating this argument and taking into account that ϕ is nondecreasing, it follows that for all $k \in \mathbb{N}$,

$$\delta_{n_0+k}\left(\mathcal{D},T,x_0\right) \leq \phi\left(\delta_{n_0+k-1}\left(\mathcal{D},T,x_0\right)\right) \leq \phi^2\left(\delta_{n_0+k-2}\left(\mathcal{D},T,x_0\right)\right) \leq \ldots \leq \phi^k\left(\delta_{n_0}\left(\mathcal{D},T,x_0\right)\right).$$

As a result, as $\phi(t) \leq t$ (in particular, $\phi^k(t) \leq t < \infty$) for all $t \in (0,\infty)$,

$$\delta_{n_0+k}\left(\mathcal{D},T,x_0\right) \leq \phi^k\left(\delta_{n_0}\left(\mathcal{D},T,x_0\right)\right) < \infty \quad \text{for all } k \in \mathbb{N}. \tag{4.5}$$

Step 3. We claim that $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence. Let $t_0 = \delta_{n_0}\left(\mathcal{D},T,x_0\right)$. If $t_0 = 0$, then $\mathcal{D}\left(x_n,x_m\right) = 0$ for all $n,m \geq n_0$. In particular, $\lim_{n,m\to\infty} \mathcal{D}\left(x_n,x_m\right) = 0$, thus $\{x_n\}$ is $\mathcal{D}$-Cauchy and we are done. So, assume that $t_0 = \delta_{n_0}\left(\mathcal{D},T,x_0\right) \in (0,\infty)$. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \phi^n\left(t_0\right) = 0$, there exists $k_0 \in \mathbb{N}$ such that $\phi^k\left(t_0\right) < \varepsilon$ for all $k \geq k_0$. By the symmetry of $\mathcal{D}$,

$$\begin{aligned} &\sup\left(\left\{\, \mathcal{D}\left(x_n,x_m\right) : n,m \in \mathbb{N},\ n,m \geq n_0+k_0 \,\right\}\right) \\ &\quad = \delta_{n_0+k_0}\left(\mathcal{D},T,x_0\right) \leq \phi^{k_0}\left(\delta_{n_0}\left(\mathcal{D},T,x_0\right)\right) = \phi^{k_0}\left(t_0\right) < \varepsilon. \end{aligned}$$

This implies that $\lim_{n,m\to\infty} \mathcal{D}\left(x_n,x_m\right) = 0$, so $\{x_n\}$ is a $\mathcal{D}$-Cauchy sequence. Hence, step 3 holds.

Next, suppose that $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-complete. Hence $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{D}$-converges to a point $\omega \in X$.

Step 4. We check that (4.2) *and* (4.3)*hold.* We consider the cases given in Proposition 2.4.

Case (4.a). Suppose that $\{x_n\}$ is infinite. In this case, by using $(\mathcal{D}_3')$,

$$\mathcal{D}\left(\omega,\omega\right) \leq C \limsup_{m\to\infty} \mathcal{D}(x_m,\omega) = 0,$$

so $\mathcal{D}\left(\omega,\omega\right) = 0$. Moreover, it follows from $(\mathcal{D}_3')$ and (4.5) that, for all $n \in \mathbb{N}$ such that $n \geq n_0$,

$$\mathcal{D}\left(\omega,x_n\right) \leq C \limsup_{m\to\infty} \mathcal{D}(x_{m+n_0},x_n) \leq C\, \delta_n\left(\mathcal{D},T,x_0\right) \leq C\, \phi^{n-n_0}\left(\delta_{n_0}\left(\mathcal{D},T,x_0\right)\right).$$

Case (4.b). Suppose that there exists $n_0 \in \mathbb{N}$ such that $\mathcal{D}\left(x_n,x_m\right) = 0$ for all $n,m \geq n_0$. In this case, Proposition 2.4 also guarantees that there exists $z \in X$ such that $x_n = z$ for all $n \geq n_0$, $\mathcal{D}\left(z,z\right) = 0$ and $\{x_n\}$ strongly $\mathcal{D}$-converges to z. Taking into account that $\{x_n\}$ is Picard and $\mathcal{D}$-Cauchy, Proposition 3.8 ensures us that it has a unique $\mathcal{D}$-limit. Therefore, $\omega = z$, so $\mathcal{D}\left(\omega,\omega\right) = \mathcal{D}\left(\omega,x_n\right) = \mathcal{D}\left(z,z\right) = 0$ for all $n \geq n_0$. Then (4.2) and (4.3) are obvious. □

Corollary 4.3. *Let $(X,\mathcal{D},\mathcal{S})$ be an RS-space with respect to a preorder $\mathcal{S}$ and let $T: X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty) \leq \phi(\max\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\}) \quad \text{for all } x,y \in X \text{ such that } x\mathcal{S}y. \tag{4.6}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 is an $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy sequence.

Furthermore, if $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-complete, then $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{D}$-converges to a point $\omega \in X$ that satisfies $\mathcal{D}(\omega,\omega) = 0$ and

$$\mathcal{D}(x_n,\omega) \leq C\,\phi^{n-n_0}(\delta_{n_0}(\mathcal{D},T,x_0)) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \geq n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3')$.

Proof. We only have to prove that (4.6) implies (4.1), that is, if the contractivity condition holds for all $x,y \in X$ such that $x\mathcal{S}y$, then it also holds for all $x,y \in \mathcal{O}_T(x_0)$. Indeed, let us consider the Picard sequence $\{x_n = T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0. Since $x_0 \mathcal{S} T x_0 = x_1$and T is $\mathcal{S}$-nondecreasing, then $x_1 = Tx_0 \mathcal{S} T x_1 = x_2$. Repeating this argument, $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$. Furthermore, as $\mathcal{S}$ is a preorder, then $x_n \mathcal{S} x_m$ for all $n,m \in \mathbb{N}$ such that $n \leq m$. In addition to this, as the condition (4.6) is symmetric on x and y (because $\mathcal{D}$ is symmetric), then (4.6) holds for all x_n and x_m (being $n,m \in \mathbb{N}$ arbitrary), so it holds for all $x,y \in \mathcal{O}_T(x_0)$. As a consequence, Theorem 4.2 is applicable. □

4.2. Fixed point theorems under $\mathcal{S}$-nondecreasing-continuity

In this subsection, we show that the limit ω of the Picard sequence is a fixed point of T. Here we shall assume that T is $\mathcal{S}$-nondecreasing-continuous.

Theorem 4.4. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T: X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty) \leq \phi(\max\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\}) \quad \text{for all } x,y \in \mathcal{O}_T(x_0). \tag{4.7}$$

Additionally, assume that

(a) T is $\mathcal{S}$-nondecreasing-continuous.

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega) = 0$ and

$$\mathcal{D}(x_n,\omega) \leq C\,\phi^{n-n_0}(\delta_{n_0}(\mathcal{D},T,x_0)) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \geq n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3')$.

In addition to this, if condition (4.7) holds for all $x,y \in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega \mathcal{S} \omega'$, $\mathcal{D}(\omega,\omega') < \infty$ and $\mathcal{D}(\omega',\omega') < \infty$, then $\omega = \omega'$.

This theorem improves earlier results in several senses: (1) we do not assume any kind of triangle inequality on the space $(X,\mathcal{D})$; (2) we replace $\delta(\mathcal{D},T,x_0) < \infty$ by the weaker condition "$\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$"; (3) we assume the mapping T is $\mathcal{S}$-nondecreasing-continuous (in this case T may be discontinuous); (4) we assume the space $(X,\mathcal{D})$ is only $\mathcal{S}$-nondecreasing-complete; (5) we do not have to check that the contractivity condition (4.7) holds for all $x,y \in X$ as it holds only for pairs in the orbit of a point; (6) $\mathcal{S}$ is not necessarily a partial order: we only assume it is a preorder; and (7) we involve a general kind of auxiliary functions: $\mathcal{F}_{\text{com}}$.

Proof. By Theorem 4.2, the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 is $\mathcal{S}$-nondecreasing and it converges to a point $\omega \in X$ verifying (4.2) and (4.3). Furthermore, as we additionally assume that T is $\mathcal{S}$-nondecreasing-continuous, $\{x_{n+1} = Tx_n\} \xrightarrow{\mathcal{D}} T\omega$. Proposition 3.8 further implies that $T\omega = \omega$, so ω is a fixed point of T.

Next suppose that condition (4.7) holds for all $x, y \in X$ such that $x\mathcal{S}y$, and assume that ω' is another fixed point of T satisfying $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega') < \infty$ and $\mathcal{D}(\omega',\omega') < \infty$. Since $\mathcal{S}$ is reflexive, we have $\omega'\mathcal{S}\omega'$, and condition (4.7) gives

$$\begin{aligned}\mathcal{D}(\omega',\omega') = \mathcal{D}(T\omega',T\omega') &\le \phi\left(\max\left\{\mathcal{D}(\omega',\omega'), \mathcal{D}(\omega',T\omega'), \mathcal{D}(\omega',T\omega'), \mathcal{D}(\omega',T\omega'), \mathcal{D}(\omega',T\omega')\right\}\right)\\ &= \phi\left(\mathcal{D}(\omega',\omega')\right).\end{aligned}$$

Since $\mathcal{D}(\omega',\omega') < \infty$, it follows that $\mathcal{D}(\omega',\omega') = 0$. As a result,

$$\begin{aligned}\mathcal{D}(\omega,\omega') = \mathcal{D}(T\omega,T\omega') &\le \phi\left(\max\left\{\mathcal{D}(\omega,\omega'), \mathcal{D}(\omega,T\omega), \mathcal{D}(\omega',T\omega'), \mathcal{D}(\omega,T\omega'), \mathcal{D}(\omega',T\omega)\right\}\right)\\ &= \phi\left(\max\left\{\mathcal{D}(\omega,\omega'), \mathcal{D}(\omega,\omega), \mathcal{D}(\omega',\omega')\right\}\right) = \phi\left(\mathcal{D}(\omega,\omega')\right).\end{aligned}$$

Again, since $\mathcal{D}(\omega,\omega') < \infty$, we have $\mathcal{D}(\omega,\omega') = 0$, so $\omega = \omega'$. □

The following statements follow from Theorem 4.4 by swapping an hypothesis by a stronger one.

Corollary 4.5. *Let $(X,\mathcal{D})$ be a complete RS-space and let $T : X \to X$ be a self-mapping. Let $x_0 \in X$ be a point such that $\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty) \le \phi\left(\max\left\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\right\}\right) \quad \text{for all } x,y \in \mathcal{O}_T(x_0). \tag{4.8}$$

Additionally, assume that

(a) *T is continuous.*

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega) = 0$ and

$$\mathcal{D}(x_n,\omega) \le C\,\phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D},T,x_0)\right) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \ge n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3')$.

In addition to this, if condition (4.8) holds for all $x, y \in X$, and ω' is another fixed point of T such that $\mathcal{D}(\omega,\omega') < \infty$ and $\mathcal{D}(\omega',\omega') < \infty$, then $\omega = \omega'$.

Corollary 4.6. *Let $(X,\mathcal{D})$ be an $\preceq$-nondecreasing-complete RS-space with respect to a partial order $\preceq$ on X and let $T : X \to X$ be an $\preceq$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \preceq Tx_0$ and $\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty) \le \phi\left(\max\left\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\right\}\right) \quad \text{for all } x,y \in \mathcal{O}_T(x_0). \tag{4.9}$$

Additionally, assume that

(a) *T is $\preceq$-nondecreasing-continuous.*

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega) = 0$ and

$$\mathcal{D}(x_n,\omega) \le C\,\phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D},T,x_0)\right) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \ge n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3')$.

In addition to this, if condition (4.9) holds for all $x, y \in X$ such that $x \preceq y$, and ω' is another fixed point of T such that $\omega \preceq \omega'$, $\mathcal{D}(\omega,\omega') < \infty$ and $\mathcal{D}(\omega',\omega') < \infty$, then $\omega = \omega'$.

After Theorem 5.9 in [15], the authors have shown a list of possible changes in hypotheses so that their main results remained true. The same commentaries can be done here with respect to Theorem 4.4.

4.3. Fixed point theorems under $\mathcal{S}$-nondecreasing-regularity

In this subsection we analyze the case in which the operator T is not necessarily continuous. In this line, we highlight that, in order to guarantee the existence of fixed points, it is not sufficient to assume the regularity (or $\mathcal{S}$-regularity) of the space (see, for instance, [15]). Let us introduce the following notation. Given a self-mapping $T : X \to X$ of an RS-space $(X, \mathcal{D})$ and a point $x_0 \in X$, let

$$\mathcal{O}'_T(x_0) = \mathcal{O}_T(x_0) \cup \left\{ \omega \in X : \lim_{n\to\infty} \mathcal{D}(T^n x_0, \omega) = 0 \right\}.$$

In a JS-space, Proposition 2.12 guarantees that the second part of $\mathcal{O}'_T(x_0)$ contains, at most, a single point. However, in general RS-spaces (like B_N-spaces), the uniqueness of the limit is not guaranteed. For simplicity, we assume $\delta(\mathcal{D}, T, x_0) < \infty$ in the following statement.

Theorem 4.7. *Let $(X, \mathcal{D}, \mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta(\mathcal{D}, T, x_0) < \infty$. Suppose that there exists $\phi \in \mathcal{F}_{\mathrm{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \phi\left(\max\{\mathcal{D}(x, y), \mathcal{D}(x, Tx), \mathcal{D}(y, Ty), \mathcal{D}(x, Ty), \mathcal{D}(y, Tx)\}\right) \quad \text{for all } x, y \in \mathcal{O}'_T(x_0). \tag{4.10}$$

Then the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega \in X$ that satisfies $\mathcal{D}(\omega, \omega) = 0$ and

$$\mathcal{D}(T^n x_0, \omega) \leq C_{X,\mathcal{D}}\, \phi^n(\delta(\mathcal{D}, T, x_0)) \quad \text{for all } n \in \mathbb{N}.$$

Additionally, assume that

(b) *$\mathcal{D}(\omega, T\omega) < \infty$, $\mathcal{D}(x_0, T\omega) < \infty$ and, if $\phi(\mathcal{D}(\omega, T\omega)) > 0$, then $C_{X,\mathcal{D}}\, \phi(\mathcal{D}(\omega, T\omega)) < \mathcal{D}(\omega, T\omega)$.*

Then ω is a fixed point of T.

Furthermore, if condition (4.10) holds for all $x, y \in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega \mathcal{S} \omega'$, $\mathcal{D}(\omega, \omega') < \infty$ and $\mathcal{D}(\omega', \omega') < \infty$, then $\omega = \omega'$.

Proof. Since $\mathcal{O}_T(x_0) \subseteq \mathcal{O}'_T(x_0)$, Theorem 4.2 guarantees that the Picard sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ of T based on x_0 is $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy, and it strongly $\mathcal{D}$-converges to a point $\omega \in X$ satisfying (4.2) and (4.3). Suppose that $\mathcal{D}(\omega, T\omega) < \infty$ and $\mathcal{D}(x_0, T\omega) < \infty$. As $\{x_n\} \xrightarrow{\mathcal{D}} \omega$, then $\omega \in \mathcal{O}'_T(x_0)$. Define

$$a_n = \max\{\mathcal{D}(x_n, \omega), \mathcal{D}(x_n, x_{n+1}), \mathcal{D}(\omega, x_{n+1})\} \quad \text{for all } n \in \mathbb{N}.$$

Since $\{x_n\} \xrightarrow{\mathcal{D}} \omega$ and $\{x_n\}$ is $\mathcal{D}$-Cauchy, we have $\{a_n\} \to 0$. Notice that

$$\begin{aligned} &\mathcal{D}(x_n, x_{n+1}) \leq \delta(\mathcal{D}, T, x_0) < \infty \quad \text{and} \\ &\mathcal{D}(x_n, \omega) = \mathcal{D}(T^n x_0, \omega) \leq C_{X,\mathcal{D}}\, \phi^n(\delta(\mathcal{D}, T, x_0)) < \infty \end{aligned}$$

for all $n \in \mathbb{N}$, then $a_n < \infty$ for all $n \in \mathbb{N}$. By Proposition 2.1,

$$\begin{aligned} &\left\{ b_n = \max\left\{ \phi(a_n), \phi^2(a_{n-1}), \phi^3(a_{n-2}), \ldots, \phi^n(a_1), \phi^{n+1}(a_0) \right\} \right\} \to 0 \\ &\text{and} \quad b_n < \infty \text{ for all } n \in \mathbb{N}. \end{aligned}$$

We claim that $\mathcal{D}(x_n, T\omega) < \infty$ for all $n \in \mathbb{N}$. Indeed, by hypothesis, $\mathcal{D}(x_0, T\omega) < \infty$. Assume that $\mathcal{D}(x_n, T\omega) < \infty$ for some $n \in \mathbb{N}$. So, as $\omega \in \mathcal{O}'_T(x_0)$,

$$\begin{aligned} &\mathcal{D}(x_{n+1}, T\omega) \\ &= \mathcal{D}(Tx_n, T\omega) \\ &\leq \phi\left(\max\{\mathcal{D}(x_n, \omega), \mathcal{D}(x_n, x_{n+1}), \mathcal{D}(\omega, T\omega), \mathcal{D}(x_n, T\omega), \mathcal{D}(\omega, x_{n+1})\}\right) \\ &= \max\{\phi(\max\{\mathcal{D}(x_n, \omega), \mathcal{D}(x_n, x_{n+1}), \mathcal{D}(\omega, x_{n+1})\}), \phi(\mathcal{D}(\omega, T\omega)), \phi(\mathcal{D}(x_n, T\omega))\} \\ &= \max\{\phi(a_n), \phi(\mathcal{D}(\omega, T\omega)), \phi(\mathcal{D}(x_n, T\omega))\}. \end{aligned} \tag{4.11}$$

Since all terms in the maximum are finite, $\mathcal{D}(x_{n+1}, T\omega) < \infty$. This completes the induction.

Notice that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathcal{D}(x_n, T\omega) < \infty \quad \text{and} \\
&\mathcal{D}(x_{n+1}, T\omega) \leq \max\{\phi(a_n), \phi(\mathcal{D}(\omega, T\omega)), \phi(\mathcal{D}(x_n, T\omega))\}.
\end{aligned} \tag{4.12}$$

Since ϕ is nondecreasing, applying (4.12), we get that

$$\begin{aligned}
\phi(\mathcal{D}(x_n, T\omega)) &\leq \phi\left(\max\{\phi(a_{n-1}), \phi(\mathcal{D}(\omega, T\omega)), \phi(\mathcal{D}(x_{n-1}, T\omega))\}\right) \\
&= \max\left\{\phi^2(a_{n-1}), \phi^2(\mathcal{D}(\omega, T\omega)), \phi^2(\mathcal{D}(x_{n-1}, T\omega))\right\}.
\end{aligned} \tag{4.13}$$

In view of (4.12) and (4.13), since $\phi^2(\mathcal{D}(\omega, T\omega)) \leq \phi(\mathcal{D}(\omega, T\omega))$, we have that

$$\begin{aligned}
\mathcal{D}(x_{n+1}, T\omega) &\leq \max\{\phi(a_n), \phi(\mathcal{D}(\omega, T\omega)), \phi(\mathcal{D}(x_n, T\omega))\} \\
&\leq \max\left\{\phi(a_n), \phi(\mathcal{D}(\omega, T\omega)), \phi^2(a_{n-1}), \phi^2(\mathcal{D}(\omega, T\omega)), \phi^2(\mathcal{D}(x_{n-1}, T\omega))\right\} \\
&= \max\left\{\phi(a_n), \phi^2(a_{n-1}), \phi(\mathcal{D}(\omega, T\omega)), \phi^2(\mathcal{D}(x_{n-1}, T\omega))\right\}.
\end{aligned}$$

Repeating this process n times, we derive that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{D}(x_{n+1}, T\omega) &\leq \max\left\{\phi(a_n), \phi^2(a_{n-1}), \ldots, \phi^n(a_1), \phi^{n+1}(a_0), \phi(\mathcal{D}(\omega, T\omega)), \phi^{n+1}(\mathcal{D}(x_0, T\omega))\right\} \\
&= \max\left\{b_n, \phi(\mathcal{D}(\omega, T\omega)), \phi^{n+1}(\mathcal{D}(x_0, T\omega))\right\}.
\end{aligned} \tag{4.14}$$

Next, we consider two cases.

Case 1. If $\phi(\mathcal{D}(\omega, T\omega)) = 0$, it follows from (4.14) that

$$0 \leq \mathcal{D}(x_{n+1}, T\omega) \leq \max\left\{b_n, \phi^{n+1}(\mathcal{D}(x_0, T\omega))\right\} \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we obtain that $\lim_{n\to\infty} \mathcal{D}(x_n, T\omega) = 0$. Thus, $(\mathcal{D}_3')$ leads to

$$\mathcal{D}(\omega, T\omega) \leq C \limsup_{n\to\infty} \mathcal{D}(x_{n+1}, T\omega) = C \lim_{n\to\infty} \mathcal{D}(x_n, T\omega) = 0.$$

As a result, $\mathcal{D}(\omega, T\omega) = 0$, so $T\omega = \omega$ by $(\mathcal{D}_1)$.

Case 2. Assume that $\phi(\mathcal{D}(\omega, T\omega)) \in (0, \infty)$. By hypothesis,

$$\phi(\mathcal{D}(\omega, T\omega)) < \frac{\mathcal{D}(\omega, T\omega)}{C_{X,\mathcal{D}}}. \tag{4.15}$$

We claim that $\mathcal{D}(\omega, T\omega) = 0$. Suppose not. Then taking into account that $\mathcal{D}(\omega, T\omega) < \infty$, we have that $\mathcal{D}(\omega, T\omega) \in (0, \infty)$. Taking $\varepsilon = \phi(\mathcal{D}(\omega, T\omega)) > 0$, we can find $n_0 \in \mathbb{N}$ such that $b_n \leq \phi(\mathcal{D}(\omega, T\omega))$ and $\phi^{n+1}(\mathcal{D}(x_0, T\omega)) \leq \phi(\mathcal{D}(\omega, T\omega))$ for all $n \geq n_0$. In view of (4.14), we get

$$\mathcal{D}(x_{n+1}, T\omega) \leq \max\left\{b_n, \phi(\mathcal{D}(\omega, T\omega)), \phi^{n+1}(\mathcal{D}(x_0, T\omega))\right\} = \phi(\mathcal{D}(\omega, T\omega))$$

for all $n \geq n_0$. Using $(\mathcal{D}_3')$ together with (4.15), we conclude that

$$\mathcal{D}(\omega, T\omega) \leq C \limsup_{n\to\infty} \mathcal{D}(x_{n+1}, T\omega) \leq C\,\phi(\mathcal{D}(\omega, T\omega)) < C\,\frac{\mathcal{D}(\omega, T\omega)}{C} = \mathcal{D}(\omega, T\omega),$$

which is a contradiction. Thus, $\mathcal{D}(\omega, T\omega) = 0$, so $T\omega = \omega$.

The rest of the proof follows, point by point, as in the proof of Theorem 4.4. □

If $\phi(t) < t/C_{X,\mathcal{D}}$ for all $t \in (0, \infty)$ then we can avoid the assumption "if $\phi(\mathcal{D}(\omega, T\omega)) > 0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega, T\omega)) < \mathcal{D}(\omega, T\omega)$", so that the following consequence is immediate.

Corollary 4.8. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\phi\in\mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty)\leq\phi\left(\max\{\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\mathcal{D}(y,Tx)\}\right)$$
$$\text{for all } x,y\in\mathcal{O}'_T(x_0).$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that satisfies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^n(\delta(\mathcal{D},T,x_0))\quad\text{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b') $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and $\phi(t)<t/C_{X,\mathcal{D}}$ for all $t\in(0,\infty)$.

Then ω is a fixed point of T.

Furthermore, if condition (4.10) holds for all $x,y\in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

The following statement can be demonstrated by analogous arguments we employed in the proof of Theorem 4.7.

Theorem 4.9. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\phi\in\mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty)\leq\phi\left(\max\{\mathcal{D}(x,y),\mathcal{D}(x,Tx),\mathcal{D}(y,Ty),\mathcal{D}(x,Ty),\mathcal{D}(y,Tx)\}\right)\tag{4.16}$$
$$\text{for all } x,y\in X \text{ such that } x\mathcal{S}y.$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that satisfies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^n(\delta(\mathcal{D},T,x_0))\quad\text{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b'') $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-regular, $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and, if $\phi(\mathcal{D}(\omega,T\omega))>0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega,T\omega))<\mathcal{D}(\omega,T\omega)$ (this last condition can be replaced by the fact that $\phi(t)<t/C_{X,\mathcal{D}}$ for all $t\in(0,\infty)$).

Then ω is a fixed point of T.

Furthermore, if ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

Proof. Since the contractivity condition (4.16) holds for all $x,y\in X$ such that $x\mathcal{S}y$, repeating the arguments of the proof of Corollary 4.3, we deduce that it also holds for all $x,y\in\mathcal{O}_T(x_0)$. Now Theorem 4.2 guarantees that the Picard sequence $\{T^nx_0\}_{n\in\mathbb{N}}$ of T based on x_0 is $\mathcal{S}$-nondecreasing, $\mathcal{D}$-Cauchy, and it $\mathcal{D}$-converges to a point $\omega\in X$ satisfying (4.2) and (4.3). Since $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-regular, we obtain that $x_n\mathcal{S}\omega$ for all $n\in\mathbb{N}$. Thus, as the contractivity condition (4.16) is applicable to x_n and ω, we can repeat, point by point, the arguments of the proof of Theorem 4.7 in order to get that ω is a fixed point of T. □

4.4. Fixed point theorems under a stronger contractive condition

By considering the stronger contractivity condition $\mathcal{D}(Tx,Ty)\leq\phi(\mathcal{D}(x,y))$ in an appropriate subset of X, we can avoid some hypotheses in the previous results. For instance, the following affirmation is an immediate consequence of Corollary 4.5.

Corollary 4.10. *Let $(X,\mathcal{D})$ be a complete RS-space and let $T:X\to X$ be a continuous self-mapping. Let $x_0\in X$ be a point such that $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ for some $n_0\in\mathbb{N}$. Suppose that there exists $\phi\in\mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty)\leq\phi(\mathcal{D}(x,y))\quad\text{for all } x,y\in\mathcal{O}_T(x_0). \tag{4.17}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C\,\phi^{n-n_0}(\delta_{n_0}(\mathcal{D},T,x_0))\quad\text{for all } n\in\mathbb{N}\text{ such that } n\geq n_0,$$

where $C=C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3')$.

In addition to this, if condition (4.17) holds for all $x,y\in X$, and ω' and ω'' are two fixed points of T such that $\mathcal{D}(\omega',\omega'')<\infty$, then $\omega'=\omega''$.

The continuity of T can be deduced when the contractivity condition is assumed for all $x,y\in\mathcal{O}_T'(x_0)$ as in the following result.

Theorem 4.11. *Let $(X,\mathcal{D})$ be a complete RS-space and let $T:X\to X$ be a self-mapping. Let $x_0\in X$ be a point such that $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ for some $n_0\in\mathbb{N}$. Suppose that there exists $\phi\in\mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty)\leq\phi(\mathcal{D}(x,y))\quad\text{for all } x,y\in\mathcal{O}_T'(x_0). \tag{4.18}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega)=0$.

In addition to this, if (4.18) holds for all $x,y\in X$, then T is continuous. Moreover, if ω' and ω'' are two fixed points of T such that $\mathcal{D}(\omega',\omega'')<\infty$, then $\omega'=\omega''$.

Proof. Consider on X the trivial preorder $\mathcal{S}_X$ given by $x\mathcal{S}_Xy$ for all $x,y\in X$. Then T is $\mathcal{S}_X$-nondecreasing and $(X,\mathcal{D})$ is $\mathcal{S}_X$-nondecreasing-complete. Now Theorem 4.2 guarantees that $\{x_n=T^nx_0\}_{n\in\mathbb{N}}$ is an $\mathcal{S}_X$-nondecreasing, $\mathcal{D}$-Cauchy sequence. As $(X,\mathcal{D})$ is complete, there is a $\omega\in X$ such that $\{T^nx_0\}\xrightarrow{\mathcal{D}}\omega$. Since $\omega\in\mathcal{O}_T'(x_0)$, we note that

$$\mathcal{D}\left(T^{n+1}x_0,T\omega\right)=\mathcal{D}(TT^nx_0,T\omega)\leq\phi(\mathcal{D}(T^nx_0,\omega))\quad\text{for all } n\in\mathbb{N}.$$

Using $\{\mathcal{D}(T^nx_0,\omega)\}\to0$ and ϕ is continuous at $t=0$, with $\phi(0)=0$, we deduce that $\{T^{n+1}x_0\}\xrightarrow{\mathcal{D}}\omega$. Hence $T\omega=\omega$ by Proposition 3.8, and ω is a fixed point of T. By using $(\mathcal{D}_3')$,

$$\mathcal{D}(\omega,\omega)\leq C\limsup_{m\to\infty}\mathcal{D}(T^mx_0,\omega)=0.$$

As a result, $\mathcal{D}(\omega,\omega)=0$.

Next, assume that (4.18) holds for all $x,y\in X$, and let $z\in X$ be an arbitrary point. Then T is continuous at z. Indeed, let $\{y_n\}\subseteq X$ be a sequence such that $\{y_n\}\xrightarrow{\mathcal{D}}z$. Then $\{\mathcal{D}(y_n,z)\}\to0$. By (4.18), we derive that $\mathcal{D}(Ty_n,Tz)\leq\phi(\mathcal{D}(y_n,z))$ for all $n\in\mathbb{N}$. Since ϕ is continuous at $t=0$, with $\phi(0)=0$, we deduce that $\{Ty_n\}\xrightarrow{\mathcal{D}}Tz$. So, T is continuous at z.

Finally, if ω' and ω'' are two fixed points of T satisfying $\mathcal{D}(\omega',\omega'')<\infty$, then $\mathcal{D}(\omega',\omega'')=\mathcal{D}(T\omega',T\omega'')\leq\phi(\mathcal{D}(\omega',\omega''))$, which yields that $\omega'=\omega''$. □

5. Consequences

In this section we illustrate how some well known fixed point results can be seen as a particular cases of our main theorems by using several types of contractivity conditions and completeness. We point out that the following results are valid for $\mathcal{S}$-nondecreasing-complete RS-space, where $\mathcal{S}$ is a preorder on X. However, we will write some corollaries by using the trivial preorder $\mathcal{S}_X$ and complete RS-spaces.

5.1. Banach type fixed point theorems in RS-generalized metric spaces

A mapping $T : X \to X$ satisfies a Banach type contractivity condition if there is $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda\, d(x, y) \quad \text{for all } x, y \in X.$$

Since the function $\phi = \phi_\lambda$, given by $\phi_\lambda(t) = \lambda\, t$ for all $t \in [0, \infty]$, belongs to $\mathcal{F}_{\text{com}}$, we can particularize Theorems 4.4, 4.7 and 4.9 in the following way.

Corollary 5.1. *Let $(X, \mathcal{D}, \mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\mathcal{D}(Tx, Ty) \leq \lambda\, \mathcal{D}(x, y) \qquad \text{for all } x, y \in \mathcal{O}_T(x_0). \tag{5.1}$$

Additionally, assume that, at least, one of the following conditions holds:

(a) T is $\mathcal{S}$-nondecreasing-continuous.

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega, \omega) = 0$ and

$$\mathcal{D}(x_n, \omega) \leq C\, \phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D}, T, x_0)\right) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \geq n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X, \mathcal{D})$ satisfies property $(\mathcal{D}'_3)$.

In addition to this, if condition (5.1) holds for all $x, y \in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega \mathcal{S} \omega'$, $\mathcal{D}(\omega, \omega') < \infty$ and $\mathcal{D}(\omega', \omega') < \infty$, then $\omega = \omega'$.

The following one is a particularization of the previous corollary by using the trivial preorder $\mathcal{S}_X$.

Corollary 5.2. *Let $(X, \mathcal{D})$ be a complete RS-space and let $T : X \to X$ be a self-mapping. Let $x_0 \in X$ be a point such that $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\mathcal{D}(Tx, Ty) \leq \lambda\, \mathcal{D}(x, y) \qquad \text{for all } x, y \in \mathcal{O}_T(x_0). \tag{5.2}$$

Additionally, assume that:

(a) T is continuous.

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega, \omega) = 0$ and

$$\mathcal{D}(x_n, \omega) \leq C\, \phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D}, T, x_0)\right) \quad \text{for all } n \in \mathbb{N} \text{ such that } n \geq n_0,$$

where $C = C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X, \mathcal{D})$ satisfies property $(\mathcal{D}'_3)$.

In addition to this, if condition (5.2) holds for all $x, y \in \mathcal{O}_T(x_0) \cup \mathrm{Fix}(T)$, then

- *$\mathcal{D}(z, z) = 0$ for all $z \in \mathrm{Fix}(T)$ such that $\mathcal{D}(z, z) < \infty$;*
- *if $z, z' \in \mathrm{Fix}(T)$ are two fixed points of T such that $\mathcal{D}(z, z') < \infty$, then $z = z'$.*

Proof. The first part follows from Corollary 5.1, and the last part follows from the contractivity condition (5.2). □

Corollary 5.3. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\lambda\in[0,1)$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\,\mathcal{D}(x,y)\qquad\text{for all } x,y\in\mathcal{O}_T(x_0). \tag{5.3}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that satisfies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^n(\delta(\mathcal{D},T,x_0))\quad\text{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b) $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and, if $\phi(\mathcal{D}(\omega,T\omega))>0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega,T\omega))<\mathcal{D}(\omega,T\omega)$.

Then ω is a fixed point of T.

Furthermore, if condition (5.3) holds for all $x,y\in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

Corollary 5.4. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\lambda\in[0,1)$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\,\mathcal{D}(x,y)\qquad\text{for all } x,y\in X \text{ such that } x\mathcal{S}y.$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that satisfies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^n(\delta(\mathcal{D},T,x_0))\quad\text{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b′′) $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-regular, $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and, if $\phi(\mathcal{D}(\omega,T\omega))>0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega,T\omega))<\mathcal{D}(\omega,T\omega)$ (this last condition can be replaced by the fact that $\phi(t)<t/C_{X,\mathcal{D}}$ for all $t\in(0,\infty)$).

Then ω is a fixed point of T.

Furthermore, if ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

5.2. Kannan type fixed point theorems in RS-generalized metric spaces

A Kannan type contractivity condition can be expressed as follows:

$$d(Tx,Ty)\leq\lambda\Big(d(Tx,x)+d(Ty,y)\Big)\quad\text{for all } x,y\in X,$$

where $\lambda\in[0,1/2)$.

Corollary 5.5. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ for some $n_0\in\mathbb{N}$. Suppose that there exists $\lambda\in[0,1/2)$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\Big(\mathcal{D}(Tx,x)+\mathcal{D}(Ty,y)\Big)\quad\text{for all } x,y\in\mathcal{O}_T(x_0). \tag{5.4}$$

Additionally, assume that

(a) T is $\mathcal{S}$-nondecreasing-continuous.

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C\,\phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D},T,x_0)\right)\quad \textit{for all } n\in\mathbb{N} \textit{ such that } n\geq n_0,$$

where $C=C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}_3')$.

In addition to this, if condition (5.4) holds for all $x,y\in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

Proof. If we take $\phi=\phi_{2\lambda}$, where $\phi_{2\lambda}(t)=2\lambda\,t$ for all $t\in[0,\infty]$, then $\phi_{2\lambda}\in\mathcal{F}_{\mathrm{com}}$. Using the fact that $r+s\leq 2\max\{r,s\}$ for all $r,s\in[0,\infty]$, we deduce that

$$\begin{aligned} d(Tx,Ty)&\leq\lambda\Big(d(Tx,x)+d(Ty,y)\Big)\leq\lambda\,2\max\Big\{\,d(Tx,x),\,d(Ty,y)\Big\}\\ &=\phi_{2\lambda}\left(\max\{\,d(Tx,x),\,d(Ty,y)\,\}\right). \end{aligned}$$

Hence, Theorem 4.4 is applicable. □

The same reasoning is valid in the following statements using Theorems 4.7 and 4.9.

Corollary 5.6. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\lambda\in[0,1/2)$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\Big(\mathcal{D}(Tx,x)+\mathcal{D}(Ty,y)\Big)\qquad \textit{for all } x,y\in\mathcal{O}_T(x_0). \tag{5.5}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that verifies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^{n}\left(\delta(\mathcal{D},T,x_0)\right)\quad \textit{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b) $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and, if $\phi(\mathcal{D}(\omega,T\omega))>0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega,T\omega))<\mathcal{D}(\omega,T\omega)$.

Then ω is a fixed point of T.

Furthermore, if condition (5.5) holds for all $x,y\in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

Corollary 5.7. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\phi\in\mathcal{F}_{\mathrm{com}}$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\Big(\mathcal{D}(Tx,x)+\mathcal{D}(Ty,y)\Big)\qquad \textit{for all } x,y\in X \textit{ such that } x\mathcal{S}y.$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that verifies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^{n}\left(\delta(\mathcal{D},T,x_0)\right)\quad \textit{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b″) $(X,\mathcal{D})$ is $\mathcal{S}$-nondecreasing-regular, $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and, if $\phi(\mathcal{D}(\omega,T\omega))>0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega,T\omega))<\mathcal{D}(\omega,T\omega)$ (this last condition can be replaced by the fact that $\phi(t)<t/C_{X,\mathcal{D}}$ for all $t\in(0,\infty)$).

Then ω is a fixed point of T.

Furthermore, if ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

In the next results, we particularize the previous corollaries by employing the trivial preorder $x\mathcal{S}_X y$ for all $x,y\in X$.

Corollary 5.8. *Let $(X,\mathcal{D})$ be a complete RS-space and let $T:X\to X$ be a self-mapping. Let $x_0\in X$ be a point such that $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$ for some $n_0\in\mathbb{N}$. Suppose that there exists $\lambda\in[0,1/2)$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\Big(\mathcal{D}(Tx,x)+\mathcal{D}(Ty,y)\Big)\qquad\text{for all } x,y\in\mathcal{O}_T(x_0). \tag{5.6}$$

Additionally, assume that T is continuous. Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 $\mathcal{D}$-converges to a fixed point ω of T. Furthermore, $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C\,\phi^{n-n_0}(\delta_{n_0}(\mathcal{D},T,x_0))\quad\text{for all } n\in\mathbb{N}\text{ such that } n\geq n_0,$$

where $C=C_{X,\mathcal{D}}$ is the (lowest) constant for which $(X,\mathcal{D})$ satisfies property $(\mathcal{D}'_3)$.

In addition to this, if condition (5.6) holds for all $x,y\in X$, and ω' is another fixed point of T such that $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

Corollary 5.9. *Let $(X,\mathcal{D},\mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete RS-space with respect to a preorder $\mathcal{S}$ and let $T:X\to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0\in X$ be a point such that $x_0\mathcal{S}Tx_0$ and $\delta(\mathcal{D},T,x_0)<\infty$. Suppose that there exists $\lambda\in[0,1/2)$ such that*

$$\mathcal{D}(Tx,Ty)\leq\lambda\Big(\mathcal{D}(Tx,x)+\mathcal{D}(Ty,y)\Big)\qquad\text{for all } x,y\in\mathcal{O}_T(x_0). \tag{5.7}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of T based on x_0 converges to a point $\omega\in X$ that verifies $\mathcal{D}(\omega,\omega)=0$ and

$$\mathcal{D}(x_n,\omega)\leq C_{X,\mathcal{D}}\,\phi^n(\delta(\mathcal{D},T,x_0))\qquad\text{for all } n\in\mathbb{N}.$$

Additionally, assume that

(b) $\mathcal{D}(\omega,T\omega)<\infty$, $\mathcal{D}(x_0,T\omega)<\infty$ and, if $\phi(\mathcal{D}(\omega,T\omega))>0$, then $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega,T\omega))<\mathcal{D}(\omega,T\omega)$.

Then ω is a fixed point of T.

Furthermore, if condition (5.7) holds for all $x,y\in X$ such that $x\mathcal{S}y$, and ω' is another fixed point of T such that $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

5.3. Jleli and Samet type fixed point theorems in RS-generalized metric spaces

In this section we deduce some fixed point theorems of Jleli and Samet [13]. We point out that they are simple consequences of our main results. Assume that $(X,\mathcal{D})$ is a JS-space throughout this section.

Corollary 5.10 ([13], Theorem 3.3)**.** *Suppose that the following conditions hold:*

(i) *$(X,\mathcal{D})$ is complete;*

(ii) *f is a k-contraction for some $k\in(0,1)$, that is, $\mathcal{D}(f(x),f(y))\leq k\,\mathcal{D}(x,y)$ for all $(x,y)\in X\times X$;*

(iii) *there exists $x_0\in X$ such that $\delta(\mathcal{D},f,x_0)<\infty$.*

Then $\{f^n(x_0)\}$ converges to $\omega\in X$, a fixed point of f. Moreover, if $\omega'\in X$ is another fixed point of f such that $\mathcal{D}(\omega,\omega')<\infty$, then $\omega=\omega'$.

Corollary 5.11 ([13], Theorem 4.3)**.** *Suppose that the following conditions hold:*

(i) *$(X,\mathcal{D})$ is complete;*

(ii) *f is a k-quasicontraction for some $k\in(0,1)$, that is,*

$$\mathcal{D}(f(x),f(y))\leq k\,\max\{\,\mathcal{D}(x,y),\,\mathcal{D}(x,f(x)),\,\mathcal{D}(y,f(y)),\,\mathcal{D}(x,f(y)),\,\mathcal{D}(y,f(x))\,\}$$

for all $(x,y)\in X\times X$;

(iii) *there exists $x_0\in X$ such that $\delta(\mathcal{D},f,x_0)<\infty$.*

Then $\{f^n(x_0)\}$ converges to some $\omega\in X$. If $\mathcal{D}(x_0,f(\omega))<\infty$ and $\mathcal{D}(\omega,f(\omega))<\infty$, then ω is a fixed point of f. Moreover, if $\omega'\in X$ is another fixed point of f such that $\mathcal{D}(\omega,\omega')<\infty$ and $\mathcal{D}(\omega',\omega')<\infty$, then $\omega=\omega'$.

Definition 5.12 ([13], Definition 5.1)**.** A mapping $f:X\to X$ is *weak continuous* if the following condition holds: if $\{x_n\}\subset X$ is $\mathcal{D}$-convergent to $x\in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{f(x_{n_q})\}$ is $\mathcal{D}$-convergent to $f(x)$ (as $q\to\infty$).

Given a partial order $\preceq$ on X, let $E_{\preceq}=\{\,(x,y)\in X\times X : x\preceq y\,\}$.

Definition 5.13 ([13], Definition 5.3)**.** We say that the pair $(X,\mathcal{D})$ is *$\mathcal{D}$-regular* if the following condition holds: for every sequence $\{x_n\}\subset X$ satisfying $(x_n,x_{n+1})\in E_{\preceq}$, for every n large enough, if $\{x_n\}$ is $\mathcal{D}$-convergent to $x\in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $(x_{n_q},x)\in E_{\preceq}$, for every q large enough.

Definition 5.14 ([13], Definition 5.4)**.** We say that $f:X\to X$ is a *weak k-contraction* for some $k\in(0,1)$ if the following condition holds:

$$(x,y)\in E_{\preceq}\quad\Rightarrow\quad\mathcal{D}(f(x),f(y))\leq k\,\mathcal{D}(x,y).$$

Corollary 5.15 ([13], Theorem 5.5)**.** *Suppose that the following conditions hold:*

(i) *$(X,\mathcal{D})$ is complete;*

(ii) *f is weak continuous;*

(iii) *f is a weak k-contraction for some $k\in(0,1)$, that is,*

$$\mathcal{D}(f(x),f(y))\leq k\,\mathcal{D}(x,y)\quad\textit{for all }\ (x,y)\in E_{\preceq};$$

(iv) *there exists $x_0\in X$ such that $\delta(\mathcal{D},f,x_0)<\infty$ and $(x_0,f(x_0))\in E_{\preceq}$;*

(v) *f is $\preceq$-monotone.*

Then $\{f^n(x_0)\}$ converges to $\omega\in X$ such that ω is a fixed point of f. Moreover, if $\mathcal{D}(\omega,\omega)<\infty$, then $\mathcal{D}(\omega,\omega)=0$.

Corollary 5.16 ([13], Theorem 5.7)**.** *Suppose that the following conditions hold:*

(i) *$(X,\mathcal{D})$ is complete;*

(ii) *$(X,\mathcal{D})$ is $\mathcal{D}$-regular;*

(iii) *f is a weak k-contraction for some $k\in(0,1)$, that is,*

$$\mathcal{D}(f(x),f(y))\leq k\,\mathcal{D}(x,y)\quad\textit{for all }\ (x,y)\in E_{\preceq};$$

(iv) *there exists $x_0\in X$ such that $\delta(\mathcal{D},f,x_0)<\infty$ and $(x_0,f(x_0))\in E_{\preceq}$;*

(v) *f is $\preceq$-monotone.*

Then $\{f^n(x_0)\}$ converges to $\omega\in X$ such that ω is a fixed point of f. Moreover, if $\mathcal{D}(\omega,\omega)<\infty$, then $\mathcal{D}(\omega,\omega)=0$.

5.4. Branciari type fixed point theorems in RS-generalized metric spaces

One can believe that Corollary 5.2 is a generalization of Banach contractive mapping principle. However, this is false. In Corollary 5.2 we are assuming that there exists a point $x_0 \in X$ such that

$$\delta_{n_0}(\mathcal{D},T,x_0) < \infty \quad \text{for some } n_0 \in \mathbb{N}. \tag{5.8}$$

As we have shown in Proposition 3.9, this condition is necessary if we want to prove that the Picard sequence $\{x_n = T^n x_0\}_{n\in\mathbb{N}}$ is Cauchy. However, in Remark 3.10 we pointed out that it is not sufficient in order to guarantee the Cauchy's property. In a general RS-space, if $\delta_n(\mathcal{D},T,x_0) = \infty$ for all $x_0 \in X$ and all $n \in \mathbb{N}$, there can exist Ćirić type contractions without fixed points, as we show in the following example.

Example 5.17. Let $X = \mathbb{N}$ be endowed with the Euclidean metric $d_E(x,y) = |\, x-y \,|$ for all $x,y \in X$. Hence (X,d_E) is a complete metric space. Let $T : X \to X$ and $\phi : [0,\infty] \to [0,\infty]$ be defined as

$$T(x) = x+1, \quad \phi(t) = \begin{cases} 0, & \text{if } t \in [0,1], \\ t-1, & \text{if } t \in (1,\infty), \\ \infty, & \text{if } t = \infty. \end{cases}$$

Hence $\phi \in \mathcal{F}_{\text{com}}$ and T is fixed point free. Let us show that T satisfies

$$d_E(Tx,Ty) \le \phi(\max\{d_E(x,Ty), d_E(y,Tx)\}) \qquad \text{for all } x,y \in X. \tag{5.9}$$

Indeed, let $x,y \in X$ be arbitrary. If $d_E(Tx,Ty) = 0$, then (5.9) is trivial. Suppose that $d_E(Tx,Ty) > 0$. Therefore, $|\, x-y \,| = d_E(Tx,Ty) \ge 1$. Without loss of generality, assume that $x < y$. Then

$$d_E(x,Ty) = d_E(x,y+1) = |\, x-(y+1) \,| = (y-x)+1 > 1.$$

Hence

$$\phi(\max\{d_E(x,Ty), d_E(y,Tx)\}) \ge \phi(d_E(x,Ty)) = \phi((y-x)+1) = y-x = d_E(Tx,Ty),$$

so (5.9) holds. In particular, T is a Ćirić type contraction because

$$d_E(Tx,Ty) \le \phi(\max\{d_E(x,y), d_E(x,Tx), d_E(y,Ty), d_E(x,Ty), d_E(y,Tx)\})$$

for all $x,y \in X$. In this case, Corollary 5.2 is not applicable since there do not exist $x_0 \in X$ and $n_0 \in \mathbb{N}$ such that $\delta_{n_0}(d_E,T,x_0) < \infty$ because

$$\begin{aligned} \delta_{n_0}(d_E,T,x_0) &= \sup(\{d_E(T^n x_0, T^m x_0) : n,m \in \mathbb{N},\ m \ge n \ge n_0\}) \\ &= \sup(\{m-n \in \mathbb{N} : n,m \in \mathbb{N},\ m \ge n \ge n_0\}) = \infty. \end{aligned}$$

As a result, condition (5.8) cannot be avoided in the following statement in Branciari generalized metric spaces (for a better readability, we employ the trivial preorder $\mathcal{S}_X$).

Corollary 5.18. *Let $(X,\mathcal{D})$ be a complete B_N-space and let $T : X \to X$ be a self-mapping. Let $x_0 \in X$ be a point such that $\delta_{n_0}(\mathcal{D},T,x_0) < \infty$ for some $n_0 \in \mathbb{N}$. Suppose that there exists $\phi \in \mathcal{F}_{\text{com}}$ such that*

$$\mathcal{D}(Tx,Ty) \le \phi(\max\{\mathcal{D}(x,y), \mathcal{D}(x,Tx), \mathcal{D}(y,Ty), \mathcal{D}(x,Ty), \mathcal{D}(y,Tx)\}) \tag{5.10}$$
$$\textit{for all } x,y \in \mathcal{O}_T(x_0).$$

Additionally, assume that

(a) T is continuous.

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 $\mathcal{D}$*-converges to a fixed point* ω *of* T*. Furthermore,* $\mathcal{D}(\omega,\omega)=0$ *and*

$$\mathcal{D}(x_n,\omega)\leq C\,\phi^{n-n_0}\left(\delta_{n_0}(\mathcal{D},T,x_0)\right)\quad \text{for all } n\in\mathbb{N} \text{ such that } n\geq n_0,$$

where $C=C_{X,\mathcal{D}}$ *is the (lowest) constant for which* $(X,\mathcal{D})$ *satisfies property* $(\mathcal{D}_3')$*.*

In addition to this, if condition (5.10) *holds for all* $x,y\in X$*, and* ω' *is another fixed point of* T *such that* $\mathcal{D}(\omega,\omega')<\infty$ *and* $\mathcal{D}(\omega',\omega')<\infty$*, then* $\omega=\omega'$*.*

Proof. It follows from Theorem 4.4. □

Nevertheless, condition (5.8) can be deduced in some cases. In the next result, we show how it can be derived by involving a triangle inequality and assuming the finiteness of the metric between some points.

Lemma 5.19. *Let* $\mathcal{D}:X\times X\to[0,\infty]$ *be a function satisfying* $(\mathcal{D}_2)$ *and the Branciari condition* (B_3) *for some* $N\in\mathbb{N}^*$ *(see Definition* 2.8*). Let* $x_0\in X$*, let* $\lambda\in[0,1)$ *and let* $T:X\to X$ *be a mapping satisfying*

$$\mathcal{D}(Tx,Ty)\leq\lambda\,\mathcal{D}(x,y)\quad \text{for all } x,y\in\mathcal{O}_T(x_0).$$

If $\mathcal{D}(x_0,T^r x_0)<\infty$ *for all* $r\in\{0,1,2,\ldots,N-1\}$*, then there exists* $n_0\in\mathbb{N}$ *such that* $\delta_{n_0}(\mathcal{D},T,x_0)<\infty$*.*

Proof. Let $\{x_n\}$ denote the Picard sequence of T based on x_0. Then $\mathcal{O}_T(x_0)=\{\,x_n:n\in\mathbb{N}\,\}$. Let

$$\Lambda=\max_{0\leq r\leq N-1}\mathcal{D}(x_0,x_r).$$

By hypothesis, $\Lambda<\infty$. Using the contractivity condition, for all $n,m\in\mathbb{N}$,

$$\mathcal{D}(x_{n+1},x_{m+1})=\mathcal{D}(Tx_n,Tx_m)\leq\lambda\,\mathcal{D}(x_n,x_m).$$

In particular, if $n,m\in\mathbb{N}$ are such that $n\leq m$, then

$$\mathcal{D}(x_n,x_m)\leq\lambda\,\mathcal{D}(x_{n-1},x_{m-1})\leq\lambda^2\,\mathcal{D}(x_{n-2},x_{m-2})\leq\ldots\leq\lambda^n\,\mathcal{D}(x_0,x_{m-n}).\tag{5.11}$$

Furthermore,

$$\mathcal{D}(x_n,x_{n+1})\leq\lambda^n\,\mathcal{D}(x_0,x_1)\quad \text{for all } n\in\mathbb{N}.\tag{5.12}$$

Next, we consider the two cases given in Proposition 2.3.

Case (a). Assume that $\{x_n\}$ *is infinite.* Let $n,m\in\mathbb{N}$ be arbitrary such that $n\leq m$. Let c and r be the quotient and the rest of the Euclidean division of $m-n$ over N, that is

$$m-n=Nc+r,\quad r\in\{0,1,\ldots,N-1\}.$$

Since the terms $\{x_n,x_{n+1},\ldots,x_m\}$ are all different, we can use c times the inequality (B_3) of Definition 2.8, deducing that

$$\begin{aligned}\mathcal{D}(x_n,x_m)=\mathcal{D}(x_n,x_{n+Nc+r})&\leq\sum_{i=n}^{n+N-1}\mathcal{D}(x_i,x_{i+1})+\mathcal{D}(x_{n+N},x_{n+Nc+r})\\&\leq\sum_{i=n}^{n+2N-1}\mathcal{D}(x_i,x_{i+1})+\mathcal{D}(x_{n+2N},x_{n+Nc+r})\leq\ldots\\&\leq\sum_{i=n}^{n+Nc-1}\mathcal{D}(x_i,x_{i+1})+\mathcal{D}(x_{n+Nc},x_{n+Nc+r}).\end{aligned}$$

By (5.11) and (5.12),

$$\mathcal{D}(x_n,x_m)\leq\sum_{i=n}^{n+Nc-1}\mathcal{D}(x_i,x_{i+1})+\mathcal{D}(x_{n+Nc},x_{n+Nc+r})\leq\sum_{i=n}^{n+Nc-1}\lambda^i\,\mathcal{D}(x_0,x_1)+\lambda^{n+Nc}\,\mathcal{D}(x_0,x_r)$$

$$\begin{aligned}&= \left(\lambda^n + \lambda^{n+1} + \ldots + \lambda^{n+Nc-1}\right) \mathcal{D}(x_0, x_1) + \lambda^{n+Nc} \mathcal{D}(x_0, x_r)\\ &= \lambda^n \frac{1-\lambda^{Nc}}{1-\lambda} \mathcal{D}(x_0, x_1) + \lambda^{n+Nc} \mathcal{D}(x_0, x_r) \leq \frac{\lambda^n}{1-\lambda} \mathcal{D}(x_0, x_1) + \lambda^{n+Nc} \mathcal{D}(x_0, x_r).\end{aligned}$$

Since $\lambda \in [0,1)$, then $\lambda^{n+Nc} \leq \lambda^n < 1$. Thus, for all $n, m \in \mathbb{N}$ such that $n \leq m$,

$$\mathcal{D}(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda} \mathcal{D}(x_0, x_1) + \lambda^{n+Nc} \mathcal{D}(x_0, x_r) \leq \frac{1}{1-\lambda} \mathcal{D}(x_0, x_1) + \Lambda,$$

which means that

$$\delta(\mathcal{D}, T, x_0) \leq \frac{1}{1-\lambda} \mathcal{D}(x_0, x_1) + \Lambda < \infty.$$

Case (b). Assume that $\{x_n\}$ is almost periodic. In this case, there exist $n_0 \in \mathbb{N}$ and $N' \in \mathbb{N}^*$ such that

$$x_{n_0+r+N'k} = x_{n_0+r} \quad \text{for all } k \in \mathbb{N} \text{ and all } r \in \{0, 1, 2, \ldots, N'-1\}.$$

Hence, the set

$$\{\, \mathcal{D}(x_n, x_m) : m \geq n \geq n_0 \,\} = \left\{ \mathcal{D}(x_{n_0+i}, x_{n_0+j}) : i, j \in \{0, 1, \ldots, N'-1\},\, i \leq j \right\}$$

is finite. By (5.11), for all $i, j \in \{0, 1, \ldots, N'-1\}$ such that $i \leq j$,

$$\mathcal{D}(x_{n_0+i}, x_{n_0+j}) \leq \lambda^{n_0+i} \mathcal{D}(x_0, x_{j-i}) \leq \mathcal{D}(x_0, x_{j-i}) \leq \Lambda.$$

Hence

$$\delta_{n_0}(\mathcal{D}, T, x_0) = \sup \{\, \mathcal{D}(x_n, x_m) : m \geq n \geq n_0 \,\} \leq \Lambda < \infty,$$

which concludes the proof. □

In a B_N-space (X, d), the metric only takes finite values. Hence, for all $x_0 \in X$, the condition "$d(x_0, T^r x_0) < \infty$ for all $r \in \{0, 1, 2, \ldots, N-1\}$ " is guaranteed, so we deduce the following consequence.

Corollary 5.20. *Let (X, d) be a B_N-space, let $\lambda \in [0,1)$ and let $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) \leq \lambda\, d(x, y) \quad \textit{for all } x, y \in X. \tag{5.13}$$

Then for all $x_0 \in X$ there exists $n_0 \in \mathbb{N}$ such that $\delta_{n_0}(d, T, x_0) < \infty$. Furthermore, T is continuous and it has a unique fixed point.

Proof. Given an arbitrary point $x_0 \in X$, Lemma 5.19 guarantees that there exists $n_0 \in \mathbb{N}$ such that $\delta_{n_0}(d, T, x_0) < \infty$. If $\{x_n\} \subseteq X$ is a sequence such that $\{x_n\} \xrightarrow{d} z \in X$, then condition (5.13) implies that $\{Tx_n\} \xrightarrow{d} Tz$ (although the d-limit of a d-convergent sequence need not be unique). Hence, T is continuous. Corollary 5.18 guarantees that T has a fixed point. And the uniqueness of the fixed point follows from the fact that $d(\omega, \omega') < \infty$ for all $\omega, \omega' \in \operatorname{Fix}(T)$. □

The following results are particularizations of Theorems 4.7 and 4.11 to Branciari generalized metric spaces.

Corollary 5.21. *Let $(X, \mathcal{D}, \mathcal{S})$ be an $\mathcal{S}$-nondecreasing-complete B_N-space with respect to a preorder $\mathcal{S}$ and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping. Let $x_0 \in X$ be a point such that $x_0 \mathcal{S} T x_0$ and $\delta(\mathcal{D}, T, x_0) < \infty$. Suppose that there exists $\phi \in \mathcal{F}_{\mathrm{com}}$ such that*

$$\mathcal{D}(Tx, Ty) \leq \phi\left(\max\{\, \mathcal{D}(x, y),\, \mathcal{D}(x, Tx),\, \mathcal{D}(y, Ty),\, \mathcal{D}(x, Ty),\, \mathcal{D}(y, Tx) \,\}\right) \quad \textit{for all } x, y \in \mathcal{O}'_T(x_0). \tag{5.14}$$

Then the Picard sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ of T based on x_0 converges to a point $\omega \in X$ that verifies $\mathcal{D}(\omega, \omega) = 0$ and

$$\mathcal{D}(T^n x_0, \omega) \leq C_{X,\mathcal{D}}\, \phi^n\left(\delta(\mathcal{D}, T, x_0)\right) \quad \textit{for all } n \in \mathbb{N}.$$

Additionally, assume that

(b) $\mathcal{D}(\omega, T\omega) < \infty$, $\mathcal{D}(x_0, T\omega) < \infty$ *and, if* $\phi(\mathcal{D}(\omega, T\omega)) > 0$, *then* $C_{X,\mathcal{D}}\,\phi(\mathcal{D}(\omega, T\omega)) < \mathcal{D}(\omega, T\omega)$.

Then ω *is a fixed point of* T.

Furthermore, if condition (5.14) *holds for all* $x, y \in X$ *such that* $x\mathcal{S}y$, *and* ω' *is another fixed point of* T *such that* $\omega\mathcal{S}\omega'$, $\mathcal{D}(\omega, \omega') < \infty$ *and* $\mathcal{D}(\omega', \omega') < \infty$, *then* $\omega = \omega'$.

Corollary 5.22. *Let* $(X, \mathcal{D})$ *be a complete* B_N*-space and let* $T : X \to X$ *be a self-mapping. Let* $x_0 \in X$ *be a point such that* $\delta_{n_0}(\mathcal{D}, T, x_0) < \infty$ *for some* $n_0 \in \mathbb{N}$. *Suppose that there exists* $\phi \in \mathcal{F}_{\text{com}}$ *such that*

$$\mathcal{D}(Tx, Ty) \le \phi(\mathcal{D}(x, y)) \quad \text{for all } x, y \in \mathcal{O}'_T(x_0). \tag{5.15}$$

Then the Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ *of* T *based on* x_0 $\mathcal{D}$*-converges to a fixed point* ω *of* T. *Furthermore,* $\mathcal{D}(\omega, \omega) = 0$.

In addition to this, if (5.15) *holds for all* $x, y \in X$, *then* T *is continuous. Moreover, if* ω' *and* ω'' *are two fixed points of* T *such that* $\mathcal{D}(\omega', \omega'') < \infty$, *then* $\omega' = \omega''$.

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