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Regularization iterative algorithms for monotone and strictly pseudocontractive mappings

Sun Young Cho[a], Abdul Latif[b], Xiaolong Qin[c,*]

[a]*Department of Mathematics, Gyeongsang National University, Jinju, South Korea.*
[b]*Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia.*
[c]*Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Sichuan, P. R. China.*

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Abstract

In this article, the sum of a monotone mapping, an inverse strongly monotone mapping, and a strictly pseudocontractive mapping are investigated based on two regularization iterative algorithms. Strong convergence analysis of the two iterative algorithms is obtained in the framework of real Hilbert spaces.

Keywords: Monotone mapping, iterative algorithm, zero point, variational inclusion, resolvent technique.
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1. Introduction and Preliminaries

In this paper, we always assume that H is a real Hilbert space with inner product $\langle x, y\rangle$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$ for $x, y \in H$ and C is a nonempty convex and closed subset of H.

Let $S : C \to C$ be a mapping. In this paper, we use $Fix(S)$ to stand for the set of fixed points of mapping S. Recall that S is said to be contractive iff

$$\|Sx - Sy\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

We also say that S is an α-contractive mapping. S is said to be an Meir-Keeler contraction iff for every $\epsilon > 0$, there exists $\eta > 0$ such that $\|x - y\| \le \epsilon + \eta$ implies $\|Sx - Sy\| \le \epsilon$, $\forall x, y \in C$. In 1969, Meir and

*Corresponding author
Email addresses: ooly61@hotmail.com (Sun Young Cho), alatif@kau.edu.sa (Abdul Latif), qxlxajh@163.com (Xiaolong Qin)

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Keeler [16] proved that every Meir-Keeler contraction has a unique fixed point in complete metric spaces; see [16] and the references therein. S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is bounded convex and closed, then the set of fixed points of S is not empty; see [5, 13] and the references therein. Fixed point theory of the class of nonexpansive mappings, which is powerful and efficient, has been applied to variational inclusion problems of maximal monotone operators in the framework of infinite dimensional Hilbert spaces. One of the most popular techniques for solving inclusion problems of nonlinear mapping B goes back to the work of Browder [6]. The basic ideas is to reduce the inclusion problems to a fixed point problem of mapping $(I + hB)^{-1}$, which is called the classical resolvent of B. If B has some monotonicity conditions, the classical resolvent of B is with full domain and nonexpansive, see [21, 22] and the references therein.

S is said to be strictly pseudocontractive iff there is a constant $\lambda \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \lambda\|x - y - Sx + Sy\|^2, \quad \forall x, y \in C.$$

We also say S is λ-strictly pseudocontractive. The class of λ-strictly pseudocontractive mappings was introduced by Browder and Petryshyn [7] in 1967. It is clear that the class of λ-strictly pseudocontractive mappings strictly include the class of nonexpansive mappings as a special cases. It is also known that every λ-strict pseudocontraction is Lipschitz continuous; see [7] and the references therein.

S is said to be pseudocontractive iff

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|x - y - Sx + Sy\|^2, \quad \forall x, y \in C.$$

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a positive constant $\kappa > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \kappa\|x - y\|^2, \quad \forall x, y \in C.$$

We also say that A is κ-strongly monotone. A is said to be inverse strongly monotone iff there exists a positive constant $\kappa > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \kappa\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

We also say that A is inverse κ-strongly monotone. From the above, we also see that A is inverse strongly monotone iff A^{-1} is strongly monotone. Every inverse strongly monotone mapping is monotone and Lipschitz continuous.

Recall that the classical variational inequality is to find a point $\bar{x}$ in C such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of variational inequality (1.1) is denoted by $VI(C, A)$ in this paper. Projection methods have been recently investigated for solving variational inequality (1.1). It is known that $\bar{x}$ is a solution to (1.1) iff $\bar{x}$ is a fixed point of mapping $Proj_C(I - hA)$, where $Proj_C$ is the metric projection from H onto C, h is some positive real number and I denotes the identity on H. If A is strongly monotone, then the existence of solutions of variational inequality (1.1) is guaranteed by the contractivity of the mapping $Proj_C(I - hA)$. If A is inverse strongly monotone, then $Proj_C(I - hA)$ is nonexpansive. Mann iterative process is an efficient and powerful process to study fixed points of nonexpansive mappings. However, Mann iterative process is only weak convergence in infinite dimensional spaces; see [2] and the references therein. Halpern iterative process (HIP) generates a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0,$$

where x is a fixed element, $\{\alpha_n\}$ is a control sequence in $(0, 1)$. HIP was initially introduced in [11]. Halpern showed that the following conditions

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$,

are necessary in the sense if Halpern iterative process is strongly convergent for all nonexpansive mappings, then $\{x_n\}$ must satisfy conditions (C1), and (C2). Recently, Halpbern iterative process has been extensively investigated by many author for solving solutions of variational inequality (1.1) and fixed points of nonexpansive mappings; see [8, 12, 15, 20, 25–27] and the references therein.

Recall that an operator $B : H \rightrightarrows H$ is said to be monotone iff, for all $x, y \in H$, $x' \in Bx$ and $y' \in By$ imply $\langle x - y, x' - y'\rangle \geq 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of operator B. A monotone mapping $B : H \rightrightarrows H$ is maximal iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, x') \in H \times H$, $\langle x - y, x' - y'\rangle \geq 0$, for all $(y, y') \in Graph(B)$ implies $y' \in Bx$. If B is maximal monotone, then $(I + hB)^{-1} : H \to Domain(B)$, where $Domain(B)$ denote the domain of B, is single-valued and firmly nonexpansive. Moreover, $B^{-1}(0) = Fix((I + hB)^{-1})$. One known example of maximal monotone mapping is $N + M$, where N is the normal cone mapping

$$Nx := \{x^* \in H : \langle x^*, y - x\rangle \leq 0, \forall y \in C\},$$

for $x \in C$ and is empty otherwise, and M is a single valued maximal monotone mapping that is continuous on C. Then, $0 \in Nx + Mx$ iff $x \in C$ satisfies variational inequalities of $\langle Mx, y - x\rangle \geq 0$ for all $y \in C$. Another example of maximal monotone mapping is ∂B, the subdifferential of a proper closed convex function $B : H \to (-\infty, \infty]$ which is defined by

$$\partial Bx := \{x^* \in H : Bx + \langle y - x, x^*\rangle \leq By, \forall y \in H\}, \quad \forall x \in H.$$

Rockafellar [22] proved that ∂B is a maximal monotone operator. It is easy to verify that $0 \in \partial Bv$ if and only if $Bv = \min_{x\in H} Bx$. Recently, zero points of the sum of two monotone operators have been extensively investigated based on iterative techniques since the problem is applicable in image recovery, signal processing, and machine learning, which are mathematically modeled as a monotone mapping equation and this mapping is decomposed as the sum of two monotone mappings; see [3, 4, 9, 10, 17–19] and the references therein.

In this article, we study the sum of a monotone mapping, an inverse strongly monotone mapping, and a strictly pseudocontractive mapping based on two viscosity regularization iterative algorithms. Strong convergence analysis of the iterative algorithms is obtained in the framework of real Hilbert spaces.

Before giving the main results, we provide the following lemmas which play an important role in this article.

Lemma 1.1 ([14])**.** *Let $\{\mu_n\}$ be a sequence of nonnegative numbers satisfying the condition $\mu_{n+1} \leq (1 - s_n)\mu_n + s_n a_n + b_n$, $\forall n \geq 0$, where $\{s_n\}$ is a number sequence in $(0, 1)$ such that $\sum_{n=0}^{\infty} s_n = \infty$, $\lim_{n\to\infty} s_n = 0$, $\{a_n\}$ is a sequence such that $\limsup_{n\to\infty} a_n \leq 0$, and $\{b_n\}$ is a positive sequence such that $\sum_{n=0}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty}\mu_n = 0$.*

Lemma 1.2 ([3])**.** *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $A : C \to H$ be a monotone mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $Fix((I + hB)^{-1}(I - hA)) = (B + A)^{-1}(0)$, where h is some positive real number.*

Lemma 1.3 ([1])**.** *Let H be a Hilbert space, and B an maximal monotone mapping on H. For $x \in E$, $h > 0$ and $h' > 0$, we have $(I + hB)^{-1}x = (I + h'B)^{-1}\left(\frac{h'}{h}x + \left(1 - \frac{h'}{h}\right)(I + hB)^{-1}x\right)$.*

Lemma 1.4 ([7])**.** *Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudo-contraction. Then $I - T$ is demiclosed at zero, that is, $\{x_n\}$ converges weakly to some point $\bar{x}$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $\bar{x} \in Fix(T)$.*

Lemma 1.5 ([23]). *Let H be a real Hilbert space and let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ with $1 > \limsup_{n\to\infty} \lambda_n \geq \liminf_{n\to\infty} \lambda_n > 0$. Suppose $x_{n+1} = \lambda_n y_n + (1-\lambda_n)x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.6 ([24]). *Let S be an Meir-Keeler contraction on a convex subset C of a Banach space E. Then for each $\epsilon > 0$, there exists $Q \in (0,1)$ such that*

$$\|x - y\| \geq \epsilon \text{ implies } \|Sx - Sy\| \leq Q\|x - y\|, \forall x, y \in C. \tag{1.2}$$

2. Main results

First, we give a strong convergence theorem with the aid of contractions.

Theorem 2.1. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a λ-strictly pseudocontractive mapping on C and S a fixed α-contractive mapping on C. Let B be a maximal monotone operator on H and $A : C \to H$ an inverse κ-strongly monotone mapping. Assume $(A+B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $[0,1]$ and let $\{h_n\}$ be a positive real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and*

$$\begin{cases} z_n = \beta_n(1-\alpha_n)x_n + \alpha_n Sx_n + (1-\alpha_n)(1-\beta_n)Tx_n, \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n, \quad \forall n \geq 0, \end{cases}$$

where $\{y_n\}$ is a sequence in C such that $\|y_n - (I + h_n B)^{-1}(z_n - h_n A z_n)\| \leq \mu_n$, where $\sum_{n=1}^{\infty} \mu_n < \infty$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{h_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} |h_n - h_{n-1}| = \lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \gamma \leq \gamma_n$, $0 < h \leq h_n \leq h' < 2\kappa$, $0 \leq \lambda \leq \beta_n \leq \beta < 1$, where β, γ, h and h' are four real numbers. Then $\{x_n\}$ converges in norm to a point $\bar{x} \in (A+B)^{-1}(0) \cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)} S\bar{x}$.

Proof. First, we show that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded sequences. Since A is inverse κ-strongly monotone, we have

$$\begin{aligned} \|(I - h_n A)x - (I - h_n A)y\|^2 &= \|x - y\|^2 - 2h_n\langle x - y, Ax - Ay\rangle + {h_n}^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - h_n(2\kappa - h_n)\|Ax - Ay\|^2. \end{aligned}$$

By using the restriction imposed on $\{h_n\}$, we have $\|(I - h_n A)x - (I - h_n A)y\| \leq \|x - y\|$. That is, $I - h_n A$ is a nonexpansive mapping for every n. Fixing $p \in (A+B)^{-1}(0) \cap Fix(T)$, we find that

$$\begin{aligned} \|\beta_n x_n + (1-\beta_n)Tx_n - p\|^2 &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p\|^2 - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - (1-\beta_n)(\beta_n - \lambda)\|Tx_n - x_n\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|z_n - p\| &\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|\beta_n x_n + (1-\beta_n)Tx_n - p\| \\ &\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \big(1 - \alpha_n(1-\alpha)\big)\|x_n - p\| + \alpha_n\|Sp - p\|. \end{aligned} \tag{2.1}$$

It follows from Lemma 1.2 that

$$\|x_{n+1} - p\| \leq \gamma_n\|y_n - p\| + (1-\gamma_n)\|x_n - p\|$$

$$
\begin{aligned}
&\leq \gamma_n\|y_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|+\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\| \\
&\quad +(1-\gamma_n)\|x_n-p\| \\
&\leq \gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+(1-\gamma_n)\|x_n-p\|+\gamma_n\mu_n \\
&\leq \gamma_n\|z_n-p\|+(1-\gamma_n)\|x_n-p\|+\gamma_n\mu_n \\
&\leq \big(1-\alpha_n\gamma_n(1-\alpha)\big)\|x_n-p\|+\alpha_n\gamma_n\|Sp-p\|+\gamma_n\mu_n \\
&\leq \max \|x_n-p\|, \frac{\|Sp-p\|}{1-\alpha}+\mu_n \\
&\ \vdots \\
&\leq \max\{\|x_0-p\|, \frac{\|Sp-p\|}{1-\alpha}\}+\sum_{i=0}^{\infty}\mu_i<\infty.
\end{aligned}
$$

This proves that sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Since $Proj_{(A+B)^{-1}(0)\cap Fix(T)}S$ is α-contractive, we find that it has a unique fixed point. Next, we denote the unique fixed point by $\bar{x}$. We are in a position to show that

$$\limsup_{n\to\infty}\langle \bar{x}-S\bar{x}, \bar{x}-z_n\rangle \leq 0.$$

To show this inequality, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle \bar{x}-S\bar{x}, \bar{x}-z_n\rangle = \lim_{i\to\infty}\langle \bar{x}-S\bar{x}, \bar{x}-z_{n_i}\rangle \leq 0.$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we assume that $z_{n_i} \rightharpoonup \hat{x}$.

Notice that

$$
\begin{aligned}
\|z_n-z_{n-1}\| &\leq \big(1-\alpha_n(1-\alpha)\big)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|Sx_{n-1}\|+\|\beta_{n-1}x_{n-1}+(1-\beta_{n-1})Tx_{n-1}\|) \\
&\quad +(1-\alpha_n)|\beta_n-\beta_{n-1}|\|x_{n-1}-Tx_{n-1}\| \\
&\leq \big(1-\alpha_n(1-\alpha)\big)\|x_n-x_{n-1}\|+(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M,
\end{aligned}
$$

where M is an appropriate constant such that $M \geq \sup_{n\geq 1}\{\|Sx_{n-1}\|+\|x_{n-1}\|+\|Tx_{n-1}\|\}$. Setting $w_n = z_n-h_nAz_n$, one further has

$$
\begin{aligned}
\|w_n-w_{n-1}\| &\leq \|h_n-h_{n-1}\|\|Az_{n-1}\|+\|z_n-z_{n-1}\| \\
&\leq (1-\alpha_n(1-\alpha))\,\|x_n-x_{n-1}\|+|h_n-h_{n-1}|\|Az_{n-1}\| \\
&\quad +M(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|).
\end{aligned}
\tag{2.2}
$$

On the other hand, one has from Lemma 1.3

$$
\begin{aligned}
\|y_{n-1}-y_n\| &\leq \|(I+h_nB)^{-1}w_n-(I+h_{n-1}B)^{-1}w_{n-1}\|+\mu_{n-1}+\mu_n \\
&= \|(I+h_{n-1}B)^{-1}w_{n-1}-(I+h_{n-1}B)^{-1}\left(\frac{r_{n-1}}{h_n}w_n+(1-\frac{h_{n-1}}{h_n})(I+h_nB)^{-1}w_n\right)\| \\
&\leq \|(1-\frac{h_{n-1}}{h_n})\left((I+h_nB)^{-1}w_n-w_{n-1}\right)+\frac{h_{n-1}}{h_n}\left(w_{n-1}-w_n\right)\| \\
&\leq \frac{|h_{n-1}-h_n|}{h_n}\|(I+h_nB)^{-1}w_n-w_n\|+\|w_n-w_{n-1}\|.
\end{aligned}
\tag{2.3}
$$

Combining (2.2) with (2.3), one finds that

$$
\begin{aligned}
\|y_{n-1}-y_n\|-\|x_n-x_{n-1}\| &\leq |h_{n-1}-h_n|\|Az_{n-1}\|+(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|)M \\
&\quad +\frac{|h_{n-1}-h_n|\|(I+h_nB)^{-1}w_n-w_n\|}{h_n}.
\end{aligned}
$$

Using the restrictions imposed on the control sequences, one finds

$$\limsup_{n\to\infty}(\|y_{n-1}-y_n\|-\|x_n-x_{n-1}\|)\le 0.$$

Using Lemma (1.5), one has $\lim_{n\to\infty}\|y_n-x_n\|=0$. Since

$$\begin{aligned}\|z_n-p\|^2&\le\alpha_n\|Sx_n-p\|^2+(1-\alpha_n)\|\beta_nx_n+(1-\beta_n)Tx_n-p\|^2\\&\le\alpha_n\|Sx_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2,\end{aligned}$$

one has from (2.1)

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\gamma_n\|y_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2\\&\le\gamma_n\|y_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2+\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2\\&\quad+(1-\gamma_n)\|x_n-p\|^2+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|\\&\le\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2+(1-\gamma_n)\|x_n-p\|^2\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2\\&\le\alpha_n\gamma_n\|Sx_n-p\|^2+\|x_n-p\|^2-h_n(2\kappa-h_n)\gamma_n\|Az_n-Ap\|^2\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2.\end{aligned}$$

It follows that

$$\begin{aligned}h_n(2\kappa-h_n)\gamma_n\|Az_n-Ap\|^2&\le\alpha_n\gamma_n\|Sx_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2.\\&\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|Sx_n-p\|^2\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2.\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty}\|Ap-Az_n\|=0. \tag{2.4}$$

In view of the firm nonexpansivity of $(I+h_nB)^{-1}$, one has

$$\begin{aligned}&\|(I+h_nB)^{-1}(z_n-h_nAz_n)-(I+h_nB)^{-1}(p-h_nAp)\|^2\\&\le\langle(z_n-h_nAz_n)-(p-h_nAp),(I+h_nB)^{-1}(z_n-h_nAz_n)-p\rangle\\&\le\frac{1}{2}\big(\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2+\|z_n-p\|^2\\&\quad-\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)-h_n(Az_n-Ap)\|^2\big)\\&\le\frac{1}{2}\big(\|z_n-p\|^2+\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2\\&\quad-\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2-h_n^2\|Az_n-Ap\|^2\\&\quad+2h_n\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|\|Az_n-Ap\|\big)\\&\le\frac{1}{2}\big(\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2-\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2\\&\quad+2h_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-z_n\|\|Az_n-Ap\|+\|z_n-p\|^2\big).\end{aligned}$$

It follows that

$$\begin{aligned}\|(I+h_nB)^{-1}(z_n-h_nAz_n)_n-p\|^2&\le\|z_n-p\|^2-\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2\\&\quad+2h_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-z_n\|\|Az_n-Ap\|.\end{aligned}\tag{2.5}$$

Thanks to (2.5), one sees

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \gamma_n\|y_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2+\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2\\&\quad+(1-\gamma_n)\|x_n-p\|^2+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|\\&\le \gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|^2+(1-\gamma_n)\|x_n-p\|^2\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2\\&\le \gamma_n\|z_n-p\|^2-\gamma_n\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2\\&\quad+2h_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-z_n\|\|Az_n-Ap\|+(1-\gamma_n)\|x_n-p\|^2\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2\\&\le \alpha_n\gamma_n\|Sx_n-p\|^2+\|x_n-p\|^2-\gamma_n\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2\\&\quad+2h_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-z_n\|\|Az_n-Ap\|\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2.\end{aligned}$$

This yields that

$$\begin{aligned}\gamma_n\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|^2 &\le \alpha_n\gamma_n\|Sx_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2h_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-z_n\|\|Az_n-Ap\|\\&\quad+2\mu_n\gamma_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\gamma_n\mu_n^2\\&\le \alpha_n\|Sx_n-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\&\quad+2h_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-z_n\|\|Az_n-Ap\|\\&\quad+2\mu_n\|(I+h_nB)^{-1}(z_n-h_nAz_n)-p\|+\mu_n^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|y_n-x_n\|=0$, we find that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.6}$$

Using (2.4) and (2.6), we have

$$\lim_{n\to\infty}\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|=0. \tag{2.7}$$

On the other hand, one has

$$\|z_n-x_n\|\le\|z_n-(I+h_nB)^{-1}(z_n-h_nAz_n)\|+\|y_n-x_n\|+\mu_n.$$

From (2.7), one sees

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{2.8}$$

It follows that $x_{n_i}\rightharpoonup\hat{x}$. Note that

$$\|Tx_n-x_n\|\le\frac{1}{(1-\alpha_n)(1-\beta_n)}\|z_n-x_n\|+\frac{\alpha_n}{(1-\alpha_n)(1-\beta_n)}\|Sx_n-x_n\|.$$

This yields from (2.8) and the restrictions imposed on control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0.$$

From Lemma 1.4, we conclude $\hat{x}\in Fix(T)$. We are in a position to conclude $\hat{x}\in(A+B)^{-1}(0)$. Putting $y_n'=(I+h_nB)^{-1}(z_n-h_nAz_n)$, one has $z_n-h_nAz_n-y_n'\in h_nBy_n'$. Let w_1 be an element in Bw_2. Since B is maximal monotone, we find that

$$\left\langle w_1-\frac{z_n-y_n'}{h_n}+Az_n,w_2-y_n'\right\rangle\ge 0.$$

Hence, one has $0 \le \langle w_1 + A\hat{x}, w_2 - \hat{x}\rangle$. This implies that $B\hat{x} \ni -A\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}(0)$. This proves $\bar{x} \in (A+B)^{-1}(0)$. Hence, we have

$$\limsup_{n\to\infty}\langle \bar{x} - S\bar{x}, \bar{x} - z_n\rangle \le 0, \tag{2.9}$$

Notice that

$$\begin{aligned}\|z_n - \bar{x}\|^2 &\le \alpha_n\langle Sx_n - S\bar{x}, z_n - \bar{x}\rangle + \alpha_n\langle S\bar{x} - \bar{x}, z_n - \bar{x}\rangle + (1-\alpha_n)\|x_n - p\|\|z_n - \bar{x}\|\\ &\le (1-\alpha_n(1-\alpha))\,\|x_n - \bar{x}\|\|z_n - \bar{x}\| + \alpha_n\langle \bar{x} - S\bar{x}, \bar{x} - z_n\rangle,\end{aligned}$$

from which it follows that

$$\|z_n - \bar{x}\|^2 \le (1-\alpha_n(1-\alpha))\|x_n - \bar{x}\|^2 + 2\alpha_n\langle \bar{x} - S\bar{x}, \bar{x} - z_n\rangle.$$

This yields that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le \gamma_n\|y_n - \bar{x}\|^2 + (1-\gamma_n)\|x_n - \bar{x}\|^2\\ &\le \gamma_n\|y_n - y_n'\|^2 + \gamma_n\|y_n' - \bar{x}\|^2 + (1-\gamma_n)\|x_n - \bar{x}\|^2 + 2\gamma_n\mu_n\|y_n' - \bar{x}\|\\ &\le \gamma_n\|z_n' - \bar{x}\|^2 + (1-\gamma_n)\|x_n - \bar{x}\|^2 + 2\gamma_n\mu_n\|y_n' - \bar{x}\| + \gamma_n\mu_n^2\\ &\le (1-\alpha_n\gamma_n(1-\alpha))\|x_n - \bar{x}\|^2 + 2\alpha_n\gamma_n\langle \bar{x} - S\bar{x}, \bar{x} - z_n\rangle\\ &\quad + 2\mu_n\|y_n' - \bar{x}\| + \mu_n^2.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\mu_n < \infty$, we conclude from Lemma 1.1 that $\{x_n\}$ converges in norm to $\bar{x}$, where

$$\bar{x} = Proj_{Fix(T)\cap(A+B)^{-1}(0)}S\bar{x},$$

that is, $\bar{x}$ is the unique solution to the following variational inequality:

$$\langle S\bar{x} - \bar{x}, \bar{x} - x'\rangle \ge 0, \forall x' \in Fix(T)\cap(A+B)^{-1}(0).$$

□

Next, we give another strong convergence theorem with the aid of Meir-Keeler contractions.

Theorem 2.2. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a λ-strictly pseudocontractive mapping on C and $\bar{S}$ a Meir-Keeler contraction on C. Let B be a maximal monotone operator on H and $A : C \to H$ an inverse κ-strongly monotone mapping. Assume $(A+B)^{-1}(0)\cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $[0,1]$ and let $\{h_n\}$ be a positive real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and*

$$\begin{cases} z_n = \beta_n(1-\alpha_n)x_n + \alpha_n\bar{S}x_n + (1-\alpha_n)(1-\beta_n)Tx_n,\\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n, \quad \forall n \ge 0,\end{cases}$$

where $\{y_n\}$ is a sequence in C such that $\|y_n - (I + h_nB)^{-1}(z_n - h_nAz_n)\| \le \mu_n$, where $\sum_{n=1}^{\infty}\mu_n < \infty$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{h_n\}$ satisfy the following restrictions: $\lim_{n\to\infty}|h_n - h_{n-1}| = \lim_{n\to\infty}|\beta_n - \beta_{n-1}| = 0$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $0 < \gamma \le \gamma_n$, $0 < h \le h_n \le h' < 2\kappa$, $0 \le \lambda \le \beta_n \le \beta < 1$, where β, γ, h, and h' are four real numbers. Then $\{x_n\}$ converges in norm to a point $\bar{x} \in (A+B)^{-1}(0)\cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)}\bar{S}\bar{x}$.

Proof. Set

$$\begin{cases} \bar{z}_n = \beta_n(1-\alpha_n)\bar{x}_n + \alpha_n S\bar{x} + (1-\alpha_n)(1-\beta_n)T\bar{x}_n,\\ \bar{x}_{n+1} = \gamma_n\bar{y}_n + (1-\gamma_n)\bar{x}_n, \quad \forall n \ge 0,\end{cases}$$

where $\{\bar{y}_n\}$ is a sequence in C such that $\|\bar{y}_n - (I + h_nB)^{-1}(\bar{z}_n - h_nA\bar{z}_n)\| \le \mu_n$ and $\bar{x}$ is a fixed element in C. From Theorem 2.1, one sees that $\{\bar{x}_n\}$ converges in norm to $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)}S\bar{x}$.

Finally, we prove that $x_n \to \bar{x}$ as $n \to \infty$. To end this, we need to show that $x_n - \bar{x}_n \to 0$ as $n \to \infty$. Arguing by contradiction, we assume $\limsup_{n\to\infty} \|x_n - \bar{x}_n\| = d > 0$. Then we choose ϵ with $0 < \epsilon < d$. Using Lemma 1.6, for such ϵ, there exists $Q \in (0,1)$ satisfying (1.2). We also choose m such that $\frac{Q\|\bar{x}_n - \bar{x}\|}{1-Q} < \epsilon$ for all $n \geq m$. Note that $\|Sx - Sy\| \leq \max\{Q\|x-y\|, \epsilon\}$, $\forall x, y \in C$.

We now consider the following two possible cases.

Case 1. There exists some $m' \geq m$ such that $\|x_{m'} - \bar{x}_{m'}\| \leq \epsilon$. Then,

$$\begin{aligned}
\|x_{m'+1} - \bar{x}_{m'+1}\| &\leq \gamma_{m'}\|y_{m'} - \bar{y}_{m'}\| + (1-\gamma_{m'})\|x_{m'} - \bar{x}_{m'}\| \\
&\leq \gamma_{m'}\|y_{m'} - (I + h_{m'}B)^{-1}(z_{m'} - h_{m'}Az_{m'})\| \\
&\quad + \gamma_{m'}\|(I + h_{m'}B)^{-1}(z_{m'} - h_{m'}Az_{m'}) - (I + h_{m'}B)^{-1}(\bar{z}_{m'} - h_{m'}A\bar{z}_{m'})\| \\
&\quad + \gamma_{m'}\|(I + h_{m'}B)^{-1}(\bar{z}_{m'} - h_{m'}A\bar{z}_{m'}) - \bar{y}_{m'}\| + (1-\gamma_{m'})\|x_{m'} - \bar{x}_{m'}\| \\
&\leq 2\gamma_{m'}\mu_{m'} + \gamma_{m'}\|z_{m'} - \bar{z}_{m'}\| + (1-\gamma_{m'})\|x_{m'} - \bar{x}_{m'}\| \\
&\leq 2\gamma_{m'}\mu_{m'} + (1-\alpha_{m'}\gamma_{m'})\|x_{m'} - \bar{x}_{m'}\| + \gamma_{m'}\alpha_{m'}\|Sx_{m'} - S\bar{x}\| \\
&\leq 2\gamma_{m'}\mu_{m'} + (1-\alpha_{m'}\gamma_{m'})\|x_{m'} - \bar{x}_{m'}\| + \gamma_{m'}\alpha_{m'}\max\{Q\|x_{m'} - \bar{x}\|, \epsilon\} \\
&\leq 2\gamma_{m'}\mu_{m'} + \max\{(1-\alpha_{m'}\gamma_{m'}(1-Q))\|x_{m'} - \bar{x}_{m'}\| + \gamma_{m'}\alpha_{m'}(1-Q)\frac{Q\|\bar{x}_{m'} - \bar{x}\|}{1-Q}, \\
&\quad (1-\alpha_{m'}\gamma_{m'})\|x_{m'} - \bar{x}_{m'}\| + \gamma_{m'}\alpha_{m'}\epsilon\}.
\end{aligned}$$

By induction, one finds that $\|\bar{x}_n - x_n\| \leq \epsilon$ for all $n \geq m'$. This contradicts $\epsilon < d$.

Case 2. $\|x_{m'} - \bar{x}_{m'}\| \geq \epsilon$, $\forall n \geq m$.

For each $n \geq m$, one sees that

$$\begin{aligned}
\|x_{n+1} - \bar{x}_{n+1}\| &\leq \gamma_n\|y_n - \bar{y}_n\| + (1-\gamma_n)\|x_n - \bar{x}_n\| \\
&\leq \gamma_n\|y_n - (I + h_nB)^{-1}(z_n - h_nAz_n)\| \\
&\quad + \gamma_n\|(I + h_nB)^{-1}(z_n - h_nAz_n) - (I + h_nB)^{-1}(\bar{z}_n - h_nA\bar{z}_n)\| \\
&\quad + \gamma_n\|(I + h_nB)^{-1}(\bar{z}_n - h_nA\bar{z}_n) - \bar{y}_n\| + (1-\gamma_n)\|x_n - \bar{x}_n\| \\
&\leq 2\gamma_n\mu_n + \gamma_n\|z_n - \bar{z}_n\| + (1-\gamma_n)\|x_n - \bar{x}_n\| \\
&\leq 2\gamma_n\mu_n + (1-\alpha_n\gamma_n)\|x_n - \bar{x}_n\| + \gamma_n\alpha_n\|Sx_n - S\bar{x}\| \\
&\leq 2\gamma_n\mu_n + (1-\alpha_n\gamma_n(1-Q))\|x_n - \bar{x}_n\| + \gamma_n\alpha_n\|S\bar{x}_n - S\bar{x}\| \\
&\leq 2\gamma_n\mu_n + (1-\alpha_n\gamma_n(1-Q))\|x_n - \bar{x}_n\| + \gamma_n\alpha_n(1-Q)\frac{\|\bar{x}_n - \bar{x}\|}{1-Q}.
\end{aligned}$$

It follows that $\lim_{n\to\infty} \|x_n - \bar{x}_n\| = 0$. This yields a contradiction. Hence, $x_n - \bar{x}_n \to 0$ as $n \to \infty$. Using the fact that $\|x_n - \bar{x}\| \leq \|x_n - \bar{x}_n\| + \|\bar{x}_n - \bar{x}\|$, we find the desired conclusion immediately.

□

Since every nonexpansive mapping is a 0-strictly pseudocontractive mapping, we have the following result.

Corollary 2.3. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a nonexpansive mapping on C and S a fixed α-contractive mapping on C. Let B be a maximal monotone operator on H and $A : C \to H$ an inverse κ-strongly monotone mapping. Assume $(A + B)^{-1}(0) \cap Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $[0,1]$ and let $\{h_n\}$ be a positive real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $z_n = \alpha_nSx_n + (1-\alpha_n)Tx_n$, $x_{n+1} = \gamma_ny_n + (1-\gamma_n)x_n$, $\forall n \geq 0$, where $\{y_n\}$ is a sequence in C such that $\|y_n - (I + h_nB)^{-1}(z_n - h_nAz_n)\| \leq \mu_n$ where $\sum_{n=1}^{\infty} \mu_n < \infty$. Assume that the control sequences $\{\alpha_n\}$, $\{\gamma_n\}$, and $\{h_n\}$ satisfy the following restrictions:*

$\lim_{n\to\infty} |h_n - h_{n-1}| = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \gamma \le \gamma_n$, and $0 < h \le h_n \le h' < 2\kappa$, where γ, h and h' are three real numbers. Then $\{x_n\}$ converges in norm to a point $\bar{x} \in (A+B)^{-1}(0) \cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)} S\bar{x}$.

Corollary 2.4. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a nonexpansive mapping on C and S a Meir-Keeler contraction on C. Let B be a maximal monotone operator on H and $A : C \to H$ an inverse κ-strongly monotone mapping. Assume $(A+B)^{-1}(0) \cap Fix(T) \ne \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $[0,1]$ and let $\{h_n\}$ be a positive real number sequence in $(0, 2\kappa)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$, $z_n = \alpha_n Sx_n + (1-\alpha_n)Tx_n$, $x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n$, $\forall n \ge 0$, where $\{y_n\}$ is a sequence in C such that $\|y_n - (I+h_nB)^{-1}(z_n - h_nAz_n)\| \le \mu_n$, where $\sum_{n=1}^{\infty} \mu_n < \infty$. Assume that the control sequences $\{\alpha_n\}$, $\{\gamma_n\}$, and $\{h_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} |h_n - h_{n-1}| = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \gamma \le \gamma_n$, $0 < h \le h_n \le h' < 2\kappa$, where γ, h, and h' are three real numbers. Then $\{x_n\}$ converges in norm to a point $\bar{x} \in (A+B)^{-1}(0) \cap Fix(T)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)\cap Fix(T)} S\bar{x}$.*

Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} \infty & x \notin C, \\ 0 & x \in C. \end{cases}$$

Since i_C is a proper lower and semicontinuous convex function on H, the subdifferential ∂i_C of i_C is maximal monotone. So, we can define the resolvent of ∂i_C for $h > 0$, i.e., $(I + h\partial i_C)^{-1}$. Letting $x = (I + h\partial i_C)^{-1} y$, we find that

$$y \in x + r\partial i_C x \Longleftrightarrow y \in hN_C x + x \Longleftrightarrow Proj_C y = x,$$

where $Proj_C$ is the metric projection from H onto C and $N_C x := \{e \in H : \langle e, v - x\rangle, \forall v \in C\}$.

Corollary 2.5. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a λ-strictly pseudocontractive mapping on C and S a fixed α-contractive mapping on C. Let $A : C \to H$ be an inverse κ-strongly monotone mapping. Assume $VI(C,A) \cap Fix(T) \ne \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $[0,1]$ and let $\{h_n\}$ be a positive real number sequence in $(0,2\kappa)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and*

$$\begin{cases} z_n = \beta_n(1-\alpha_n)x_n + \alpha_n Sx_n + (1-\alpha_n)(1-\beta_n)Tx_n, \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n, \quad \forall n \ge 0, \end{cases}$$

where $\{y_n\}$ is a sequence in C such that $\|y_n - Proj_C(z_n - h_nAz_n)\| \le \mu_n$, where $\sum_{n=1}^{\infty} \mu_n < \infty$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{h_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} |h_n - h_{n-1}| = \lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \gamma \le \gamma_n$, $0 < h \le h_n \le h' < 2\kappa$, and $0 \le \lambda \le \beta_n \le \beta < 1$, where β, γ, h and h' are four real numbers. Then $\{x_n\}$ converges in norm to a point $\bar{x} \in VI(C,A) \cap Fix(T)$, where $\bar{x} = Proj_{VI(C,A)\cap Fix(T)} S\bar{x}$.

Corollary 2.6. *Let C be a nonempty convex closed subset of a real Hilbert space H. Let T be a λ-strictly pseudocontractive mapping on C and S a Meir-Keeler contraction on C. Let $A : C \to H$ be an inverse κ-strongly monotone mapping. Assume $VI(C,A) \cap Fix(T) \ne \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $[0,1]$ and let $\{h_n\}$ be a positive real number sequence in $(0,2\kappa)$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and*

$$\begin{cases} z_n = \beta_n(1-\alpha_n)x_n + \alpha_n Sx_n + (1-\alpha_n)(1-\beta_n)Tx_n, \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n, \quad \forall n \ge 0, \end{cases}$$

where $\{y_n\}$ is a sequence in C such that $\|y_n - Proj_C(z_n - h_nAz_n)\| \le \mu_n$ where $\sum_{n=1}^{\infty} \mu_n < \infty$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{h_n\}$ satisfy the following restrictions: $\lim_{n\to\infty} |h_n - h_{n-1}| = \lim_{n\to\infty} |\beta_n - \beta_{n-1}| = 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $0 < \gamma \le \gamma_n$, $0 < h \le h_n \le h' < 2\kappa$, and $0 \le \lambda \le \beta_n \le \beta < 1$, where β, γ, h and h' are four real numbers. Then $\{x_n\}$ converges in norm to a point $\bar{x} \in VI(C,A) \cap Fix(T)$, where $\bar{x} = Proj_{VI(C,A)\cap Fix(T)} S\bar{x}$.

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