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On generalized space of quaternions and its application to a class of Mellin transforms

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Abstract

The Mellin integral transform is an important tool in mathematics and is closely related to Fourier and bi-lateral Laplace transforms. In this article we aim to investigate the Mellin transform in a class of quaternions which are coordinates for rotations and orientations. We consider a set of quaternions as a set of generalized functions. Then we provide a new definition of the cited Mellin integral on the provided set of quaternions. The attributive Mellin integral is one-to-one, onto and continuous in the quaternion spaces. Further properties of the discussed integral are given on a quaternion context.

Keywords: Mellin transform, rotations, quaternions, Laplace transform, Boehmians.
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1. Preliminaries, definitions and notations

Quaternions were devised by William Hamilton in his extensions to vector algebras to satisfy the properties of division rings. Quaternions are four-element vectors that can be used to encode rotations in a 3-dimension coordinate system. Compared to vector calculus, quaternions have further advantages in physical laws as relativistic, classical and quantum mechanics that can nicely be written using quaternions. The ultimate reason for such attentiveness is attributed to the fact that that quaternionic multiplication turns the three-sphere of unit quaternions into a group, acting by rotations of the three-space of purely imaginary

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quaternions, by the relation $v \to qvq^{-1}$. A one more reason for the renewed interest is related to the fact that the resulting substitution of matrices by quaternions speeds up frequently the numerical calculations of the composition of rotations, their square roots, and some standard operations that must be performed when controlling everything from aircrafts to robots. The more interesting application of the quaternionic formalism is the motion of two spheres rolling on each other without slipping with infinite friction.

Numbers of the form $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$, with $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ are called *quaternions*. They are added, subtracted, and multiplied according to the usual laws of arithmetic, except for the commutative law of multiplication.

Quaternions are also written in the condensed notation $q = \alpha_0 + \hat{w}$, where $\hat{w} = \alpha_1 i + \alpha_2 j + \alpha_3 k$ is a vector in $\mathbb{R}^3$ called the pure part of the quaternion and α_0 being its real part.

The addition rule for quaternions is component-wise addition: If $p = \alpha_0 + \hat{w}$, $\hat{w} = \alpha_1 i + \alpha_2 j + \alpha_3 k$ and $q = \beta_0 + \hat{\vartheta}$; $\hat{\vartheta} = \beta_1 i + \beta_2 j + \beta_3 k$, then $p + q = (\alpha_0 + \beta_0) + \left(\hat{w} + \hat{\vartheta}\right)$, where $\left(\hat{w} + \hat{\vartheta}\right) = (\alpha_1 + \beta_1)\, i + (\alpha_2 + \beta_2)\, j + (\alpha_3 + \beta_3)\, k$.

The multiplication rule for quaternions is the same as for polynomials, extended by the multiplicative properties of $i; j; k$ given as

$$\begin{aligned} pq = &(\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) + (\alpha_2\beta_3 - \beta_2\alpha_3 + \alpha_0\beta_2 + \beta_0\alpha_1)\, i \\ &+ (\alpha_3\beta_2 - \beta_3\alpha_1 + \alpha_0\beta_2 + \beta_0\alpha_2)\, j + (\alpha_1\beta_2 - \beta_2\alpha_2 + \alpha_0\beta_3 + \beta_0\alpha_3)\, k. \end{aligned} \tag{1.1}$$

The set $(Q, +, .)$ of quaternions with the base $\{1, i, j, k\}$ and the identities $i^2 = j^2 = k^2 = -1$, $ij = k$; $ji = -k; j\ k = i$; $kj = -i$; $ki = j$, $ik = -j$; supplied with the usual operations $+$ and . defines a non-commutative division ring.

The *conjugate* element of a quaternion q is given as $q^* = \alpha_0 - \hat{w}$. The quantity $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$, also denoted by $N(q) = \|q\|^2$, is called the *reduced norm* of q. Clearly, q is nonnull if $N(q) \neq 0$, in which case $q^{-1} = \frac{q^*}{N(q)}$ is the *multiplicative inverse* of q. The norm is a real-valued function and the norm of a product of two quaternions satisfies the property

$$\|pq\|^2 = \|p\|^2 \|q\|^2 .$$

Division of a quaternion by a real-valued scalar is just componentwise division. The conjugate and magnitude of a product of two quaternions p and q satisfy the following properties

$$\|pq\|^2 = \|p\|^2 \|q\|^2 ; \;\; (p^*)^* = p; \;\; (p + q)^* = p^* + q^*. \tag{1.2}$$

To take possession of the complete account of quaternions we refer to [12, 14, 17].

The Mellin transform is a standard tool for analyzing behaviors of many functions in mathematics and mathematical physics, such as zeta functions and in connection with various spectral problems. It is closely related to the Fourier and bi-lateral Laplace transforms, and is used in many diverse areas of mathematics, including analytic number theory, the study of difference equations, asymptotic expansions, and the study of special functions.

The one-dimensional direct and inverse Mellin transforms are defined by [23]

$$\left.\begin{aligned} &\hat{\varphi}(y) = \int_0^\infty \varphi(x)\, x^{y-1} dx, \\ &\hat{\varphi}^{-1}(x) = (1/(2\pi i)) \int_{c-i\infty}^{c+i\infty} \hat{\varphi}(y)\, x^{-y} dy, \; c = \mathrm{Re}(y) \end{aligned}\right\} \tag{1.3}$$

provided the integrals exist.

Mellin transforms, among other integrals (see [6, 13, 20, 22]), were further employed in solving integral equations of fractional order. By using this theory, various explicit solutions of linear non-homogeneous ordinary differential equations with three left-hand sided Liouville derivatives of fractional order were established in literature. Indeed, as an example to this idea, if we consider the differential equation

$$\delta t^{\alpha+2} \left(D_-^{\alpha+2} y\right)(t) + \beta t^{\alpha+1} \left(D_-^{\alpha+2} y\right)(t) + \gamma t^{\alpha} \left(D_-^{\alpha} y\right)(t) = g(t),$$

where $\delta, \beta, \gamma \in \mathbb{C}$, $\alpha > 0$, D_{-}^{α} being the right-hand side Liouville fractional derivative of order α, then using the identity

$$\left(t^{\alpha+k}\widehat{\left(D_{-}^{\alpha+k}y\right)}\right)(z) = \Gamma(z+\alpha+k)\,\hat{y}(z)/\Gamma(z),$$

we write

$$\left(\delta\Gamma(z+\alpha+2)/\Gamma(z) + \gamma\Gamma(z+\alpha)/\Gamma(z) + \beta\Gamma(z+\alpha+1)/\Gamma(z)\right)\hat{y}(z) = \hat{g}(z),$$

that can be explicitly written in the form

$$P_{\alpha}(z)\,\hat{y}(z) = \hat{g}(z),$$

where

$$P_{\alpha}(z) = \left(\Gamma(z+\alpha)/\Gamma(z)\right)\left(\delta z^2 + \delta\alpha^2 + \gamma + (\delta+\beta+2\delta\alpha)z + (\delta+\beta)\alpha\right).$$

Therefore, when considering the inverse Mellin transform for the above equation, the solution is given in the form

$$y(t) = \left(\hat{g}\widehat{(z)/P_{\alpha}}(z)\right)^{-1}.$$

Over and above, an explicit solution to the Euler-type homogeneous differential equation having finite number of fractional derivatives has been given by a Mellin transform technique; see [16, 23] for further investigation.

The Mellin-type convolution product of two integrable functions φ and ψ is given by [21]

$$\left(\widehat{\varphi \ddagger \psi}\right)(y) = \int_0^{\infty} \varphi(\zeta)\,\psi\left(y\zeta^{-1}\right)\zeta^{-1}\mathtt{d}\zeta\ . \tag{1.4}$$

The Mellin relationship with the convolution product is given by [18, 21]

$$\widehat{\varphi \ddagger \psi} = \hat{\varphi}\hat{\psi}, \tag{1.5}$$

where $\hat{\varphi}$ and $\hat{\psi}$ are the Mellin transforms of φ and ψ, respectively.

However, the Mellin transforms were extended to distributions in [21]and to Boehmians in [4]. The modified Mellin transform was discussed in [19] and represented in the space of generalized functions in [10]. By combining Fourier and Mellin transforms, the Fourier-Mellin transforms have many applications in digital signals, image processing, and ship target recognition by sonar system and radar signals as well. In addition, in combining Fourier and modified Mellin transforms the Fourier-Modified Mellin transform has diverse applications in engineering and engineering mathematics as well; see [4, 19].

We organize this article as follows. In Section 2 we present a complex valued space Ω of quaternions and establish a convolution theorem with an assigned convolution products. In Sections 3 and 4 we generate the spaces $\mathfrak{ß}$ and $\mathfrak{ß}_{\mathfrak{r}}$ of quaternions in a generalized sense. In Section 5 we give the representative of the Mellin transform on the discussed spaces of quaternions. Further results are also established in this article.

2. The space Ω

Let $q = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ be a quaternion in Q. Then q and its conjugate q^* can be written as

$$q = u + \upsilon j \text{ and } q^* = u^* - \upsilon j, \tag{2.1}$$

where $u = \alpha_0 + \alpha_1 i$, $\upsilon = \alpha_2 + i\alpha_3 \in \mathbb{C}$ and $\mathbb{C}$ is the field of complex numbers.

Remark 2.1. Let $\upsilon = \alpha_2 + i\alpha_3 \in \mathbb{C}$. Then we have

$$j\upsilon = \upsilon^* j \text{ and } j\upsilon^* = \upsilon j, \tag{2.2}$$

where υ^* is the conjugate element of υ.

Definition 2.2. Let q_1 and q_2 be in Q. Then $q_1 = u_1 + \upsilon_1 j$ and $q_2 = u_2 + \upsilon_2 j$. Hence, on Q we define the operator $\sharp : Q \times Q \to Q$ by

$$q_1 \sharp q_2 = q_1 q_2^*. \tag{2.3}$$

In view of the products assigned to Q and by (2.2) we get

$$q_1 \sharp q_2 = (u_1 u_2^* + \upsilon_1 \upsilon_2) - (u_1 \upsilon_2 - \upsilon_1 u_2^*)\, j.$$

Following (2.1), every function $\varphi : \mathbb{R} \to Q$ can be written in terms of its components as

$$\varphi = \varphi_1 + \varphi_2 j, \tag{2.4}$$

where φ_1 and φ_2 are complex-valued functions.

We give some auxiliary results we recall are needful to our investigation.

Lemma 2.3. *Let q_1, q_2, q_3, $q_4 = \alpha$ and $q_5 = \beta$ be in Q and $k_1, k_2 \in \mathbb{R}$. Then we have*

(i) $q_1 \sharp q_2 = (q_2 \sharp q_1)^*$;

(ii) $q_1 \sharp (k_1 q_2 + k_2 q_3) = k_1 q_1 \sharp q_2 + k_2 q_1 \sharp q_3$;

(iii) $q_1 \sharp (q_4 q_2 + q_5 q_3) = (q_1 \sharp q_2)\, q_4^* + (q_1 \sharp q_3)\, q_5^*$;

(iv) $(q_4 q_1 + q_5 q_2) \sharp q_3 = q_4 (q_1 \sharp q_3) + q_5 (q_2 \sharp q_3)$.

Then, the mapping $\sharp$ defines an inner product on Q by Lemma 2.3.

Definition 2.4. Let Ω be the space of all functions $\varphi = \varphi_1 + \varphi_2 j$, where φ_1 and φ_2 are complex valued such that $\|\varphi\|_\Omega < \infty$,

$$\|\varphi\|_\Omega = \left(\int_0^\infty \left(|\varphi_{1,2}(x)|^2 \right) dx \right)^{\frac{1}{2}}, \tag{2.5}$$

where $|\varphi_{1,2}|^2 = |\varphi_1|^2 + |\varphi_2|^2$.

Indeed, if φ_1 and φ_2 are complex valued functions, $\varphi_1 = \theta_1 + \theta_2 i$ and $\varphi_2 = \theta_3 + \theta_4 i$, then, of course every function φ can be written as $\varphi = \theta_1 + \theta_2 i + \theta_3 j + \theta_4 k \in \Omega$.

Definition 2.5. Every sequence $\{\varphi_n\} \in \Omega$ is said to converge to $\varphi \in \Omega$, that is, $\varphi_n \to \varphi$ in Ω as $n \to \infty$, if $\varphi_n = \varphi_{1n} + \varphi_{2n} j$, $\varphi = \varphi_1 + \varphi_2 j$, and $\varphi_{1n} \to \varphi_1$ and $\varphi_{2n} \to \varphi_2$ as $n \to \infty$.

Define $\diamond$ as

$$(\varphi \diamond \psi)(x) = \int_0^\infty (\varphi \sharp \psi)(x)\, dx. \tag{2.6}$$

Then $\varphi \diamond \varphi = \|\varphi\|_\Omega$.

Also, we can easily inspect that

$$(\varphi \diamond \psi)(x) = \int_0^\infty \left((\varphi_1^* \psi_1 + \varphi_2 \psi_2^*) - (\varphi_1 \psi_2 - \varphi_2 \psi_1^*)\, j \right)(x)\, dx.$$

Indeed, the space $(\Omega, \diamond)$ is a Hilbert space.

On behalf of the preceding analysis, we state the following definition.

Definition 2.6. Let $\varphi \in \Omega$, $\varphi = \varphi_1 + \varphi_2 j$. Then we define the Mellin transform of φ as

$$\hat{\varphi} \triangleq \hat{\varphi}_1 + \hat{\varphi}_2 j,$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are the Mellin transforms of φ_1 and φ_2, respectively.

Theorem 2.7. *Let $\varphi, \psi \in \Omega$, $\varphi = \varphi_1 + \varphi_2 j$ and $\psi = \psi_1 + \psi_2 j$. Then we have*

$$\left(\widehat{\varphi\sharp\psi}\right)(y) = U\left(\hat{\varphi};\hat{\psi}\right)(y),$$

where

$$U\left(\hat{\varphi};\hat{\psi}\right)(y) = \left(\hat{\varphi}_1\hat{\psi}_1 - \hat{\varphi}_2\hat{\psi}_2^*\right)(y) + \left(\hat{\varphi}_1\hat{\psi}_1 + \hat{\varphi}_2\hat{\psi}_1^*\right)(y)\, j, \tag{2.7}$$

$\hat{\theta}$ and $\hat{\theta}_i^*$ are the Mellin transforms of θ and θ_i^*, respectively.

Proof. Proof of Theorem 2.7 follows from (2.3). Details are thus avoided. □

3. The generalized space ß

Quaternions have become a common part of mathematics and physics culture, but quaternions nowhere discussed in a generalized sense. In what follows we generate spaces of generalized functions named as Boehmians in a quaternion concept. The complete account of Boehmian spaces can be obtained from the cited papers [1–11, 15, 17].

Let D be the space of test functions of compact support over $(0,\infty)$ and $\{\delta_n\}$ be a sequence of D such that (3.1)–(3.3) are satisfied

$$\int_0^\infty \delta_n(\zeta)\, d\zeta = 1,\ n \in \mathbb{N}; \tag{3.1}$$

$$\int_0^\infty |\delta_n(\zeta)|\, d\zeta \le k,\ n \in \mathbb{N},\ k \in \mathbb{R}; \tag{3.2}$$

$$\operatorname{supp}\delta_n(\zeta) \subset (\alpha_n, \beta_n),\ \text{where } \alpha_n,\ \beta_n \to 1 \text{ as } n \to \infty. \tag{3.3}$$

Then $\{\delta_n\}$ is said to be delta sequence. The collection of all delta sequences is denoted by Δ.

Theorem 3.1. *Let φ, $\psi \in \Omega$, $\varphi = \varphi_1 + \varphi_2 j$, $\psi = \psi_1 + \psi_2 j$ and $\delta \in D$. Then we have*

$$(\varphi + \psi)\sharp\delta = \varphi\sharp\delta + \psi\sharp\delta.$$

Theorem 3.2. $k(\varphi\sharp\delta) = (k\varphi)\sharp\delta = \varphi\sharp(k\delta) = (\varphi\sharp\delta)k,\ k \in \mathbb{R}.$

Proofs of Theorems 3.1 and 3.2 are straightforward. Proofs are therefore deleted.

Theorem 3.3. *Let $\varphi_n\{\varphi_n = \varphi_{1n} + \varphi_{2n} j\}$ and $\varphi\{\varphi = \varphi_1 + \varphi_2 j\}$ be in Ω for every $n \in \mathbb{N}$ and $\varphi_n \to \varphi$ as $n \to \infty$. For an arbitrary $\delta \in D$, we have*

$$\varphi_n\sharp\delta \to \varphi\sharp\delta \text{ in } \Omega \text{ as } n \to \infty.$$

Proof. As the proof of this theorem is straightforward, we omit the details. □

Theorem 3.4. *Let $\varphi_n \to \varphi$ in Ω as $n \to \infty$, where $\varphi = \varphi_1 + \varphi_2 j$, $\varphi_n = \varphi_{1n} + \varphi_{2n} j$ are in Ω. Let $\{\delta_n\} \in \Delta$. Then we have*

$$\varphi_n\sharp\delta_n \to \varphi \text{ as } n \to \infty.$$

Proof. On aid of the hypothesis of the theorem we write

$$\begin{aligned}
\|\varphi_n\sharp\delta_n(x) - \varphi(x)\|_\Omega^2 &= \int_0^\infty \left(|\varphi_{1n}\sharp\delta_n(x) - \varphi_1(x)|^2 + |\varphi_{2n}\sharp\delta_n(x) - \varphi_2(x)|^2\right) dx \\
\text{i.e.} \qquad &= \int_0^\infty \left|\varphi_{1n}\left(x\zeta^{-1}\right)\zeta^{-1}\delta_n(\zeta)\, d\zeta - \varphi_1(x)\int_0^\infty \delta_n(\zeta)\, d\zeta\right|^2 dx \\
&\quad + \int_0^\infty \left|\varphi_{2n}\left(x\zeta^{-1}\right)\zeta^{-1}\delta_n(\zeta)\, d\zeta - \varphi_2(x)\int_0^\infty \delta_n(\zeta)\, d\zeta\right|^2 dx.
\end{aligned}$$

By the parity of (3.1), it follows that

$$\begin{aligned}\|\varphi_n\sharp\delta_n(x)-\varphi(x)\|_\Omega^2 &= \int_0^\infty\left|\int_0^\infty\left(\varphi_{1n}\left(x\zeta^{-1}\right)\zeta^{-1}-\varphi_1(x)\right)\delta_n(\zeta)\,d\zeta\right|^2dx\\ &+\int_0^\infty\left|\int_0^\infty\left(\varphi_{2n}\left(x\zeta^{-1}\right)\zeta^{-1}-\varphi_2(x)\right)\delta_n(\zeta)\,d\zeta\right|^2dx.\end{aligned}$$

By aid of Jensens inequality we regulate the above equation to have

$$\begin{aligned}\|\varphi_n\sharp\delta_n(x)-\varphi(x)\|_\Omega^2 &\leq \int_0^\infty\int_{\alpha_n}^{\beta_n}\left(\left|\varphi_{1n}\left(x\zeta^{-1}\right)\zeta^{-1}\right|^2+|\varphi_1(x)|^2\right)|\delta_n(\zeta)|\,d\zeta dx\\ &+\int_0^\infty\int_{\alpha_n}^{\beta_n}\left(\left|\varphi_{2n}\left(x\zeta^{-1}\right)\zeta^{-1}\right|^2+|\varphi_2(x)|^2\right)|\delta_n(\zeta)|\,d\zeta dx.\end{aligned}$$

By taking into account of (3.3), by simple computation we have $\|\varphi_n\sharp\delta_n(x)-\varphi(x)\|_\Omega^2\to 0$ as $n\to\infty$. Hence,

$$\varphi_n\sharp\delta_n(x)\to\varphi(x)\ \text{ as } n\to\infty \text{ in } \Omega.$$

The previous formula finishes the proof of this theorem. □

The Boehmian space **ß** is therefore described where every typical element in **ß** will be written as

$$\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right].$$

Differentiation in **ß** is given as

$$D^k\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\{D^k\varphi_n\}}{\{\delta_n\}}\right],\ k\in\mathbb{N},\ D^k=\frac{d^k}{dx^k}.$$

Addition in **ß** is given as

$$\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]+\left[\frac{\{g_n\}}{\{\alpha_n\}}\right]=\left[\frac{\{\varphi_n\}\sharp\{\alpha_n\}+\{g_n\}\sharp\{\delta_n\}}{\{\delta_n\}\sharp\{\alpha_n\}}\right].$$

Scalar multiplication in **ß** is given as

$$k_1\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\{k_1\varphi_n\}}{\{\delta_n\}}\right],\ k_1\in\mathbb{C}.$$

δ convergence in **ß** is given as : $\beta_n\xrightarrow{\delta}\beta$ in **ß**, if there exists a delta sequence $\{\delta_n\}$ such that

$$(\beta_n\sharp\delta_n),\ \ (\beta\sharp\delta_n)\in \text{ß for every } k,n\in\mathbb{N},$$

and

$$(\beta_n\sharp\delta_k)\to(\beta\sharp\delta_k)\ \text{ as } n\to\infty \text{ in ß, for every } k\in\mathbb{N}.$$

The equivalent statement for δ convergence is given as :

$\beta_n\xrightarrow{\Delta}\beta$ as $(n\to\infty)$ in **ß** if and only if there is $\varphi_{n,k},\varphi_k\in\mathbf{\Omega}$ and $\{\delta_k\}\in\Delta$ such that $\left[\frac{\{\varphi_{n,k}\}}{\{\delta_k\}}\right]$, $\beta=\left[\frac{\{\varphi_k\}}{\{\delta_k\}}\right]$ and, $\forall k\in\mathbb{N}$, $\varphi_{n,k}\to\varphi_k$ in $\mathbf{\Omega}$ as $n\to\infty$.

The concept of Δ convergence in **ß** is given as: $\beta_n\xrightarrow{\Delta}\beta$ in **ß** if there exists $\{\delta_n\}\in\Delta$ such that

$$(\beta_n-\beta)\sharp\delta_n\in\text{ß}$$

for all $n\in\mathbb{N}$, and

$$(\beta_n-\beta)\sharp\delta_n\to 0 \text{ in ß}$$

as $n\to\infty$.

4. The Boehmian space ßᴙ

Denote by $A_{\varsigma} = \{w : w = \hat{\varphi} \text{ for some } \varphi \in \Omega\}$, $E_{\varsigma} = \left\{\{\epsilon_n\} : \{\epsilon_n\} = \left\{\hat{\delta}_n\right\} \text{ for some } \{\delta_n\} \in \Delta\right\}$, and $R_{\varsigma} = \left\{\epsilon : \epsilon = \hat{\delta} \text{ for some } \delta \in D\right\}$.

We state the following definition.

Definition 4.1. Let $w \in A_{\varsigma}$ and $\epsilon \in R_{\varsigma}$. Then we define $(w \star \epsilon)(\xi) = U(w, \epsilon)(\xi)$.

Theorem 4.2. *Let $w \in A_{\varsigma}$ and $\epsilon \in R_{\varsigma}$. Then we have $w \star \epsilon \in A_{\varsigma}$.*

Proof. For each $w \in A_{\varsigma}$ and $\epsilon \in R_{\varsigma}$ there are $\varphi \in \Omega$ and $\delta \in D$ such that $w = \hat{\varphi}$ and $\epsilon = \hat{\delta}$. Hence by Definition 4.1 and (2.7) we get

$$(w \star \epsilon)(\xi) = U(w, \epsilon)(\xi) = U\left(\hat{\varphi}, \hat{\delta}\right)(\xi) = \left(\widehat{\varphi \sharp \delta}\right)(\xi).$$

But since $f \sharp \delta \in \Omega$, it follows $w \star \epsilon \in A_{\varsigma}$. This completes the proof of the theorem. □

Theorem 4.3. *The mapping $\star : A_{\varsigma} \times R_{\varsigma} \to A_{\varsigma}$ obeys the following identities:*

(i) *Let ϵ_1 and $\epsilon_2 \in R_{\varsigma}$. Then we have $\epsilon_1 \star \epsilon_2 \in R_{\varsigma}$.*

(ii) *Let ϵ_1 and $\epsilon_2 \in R_{\varsigma}$. Then we have $\epsilon_1 \star \epsilon_2 = \epsilon_2 \star \epsilon_1$.*

(iii) *Let $w_1, w_2 \in A_{\varsigma}$ and $\epsilon \in R_{\varsigma}$. Then we have $(w_1 + w_2) \star \epsilon = w_1 \star \epsilon + w_2 \star \epsilon$.*

(vi) *Let $w \in A_{\varsigma}$ and $\epsilon_1, \epsilon_2 \in R_{\varsigma}$. Then we have $w \star (\epsilon_1 \star \epsilon_2) = (w \star \epsilon_1) \star \epsilon_2$.*

Proof. Proof of this theorem can be obtained by similar computation to that of [8, 10]. We prefer to delete the details. □

Theorem 4.4. *Let $w_1, w_2 \in A_{\varsigma}$ and $\{\epsilon_n\} \in E_{\varsigma}$ be such that $w_1 \star \epsilon_n = w_2 \star \epsilon_n, n \in \mathbb{N}$. Then we have $w_1 = w_2$.*

Proof. Assume for every $n \in \mathbb{N}$, $w_1 \star \epsilon_n = w_2 \star \epsilon_n$. Then we have

$$(w_1 - w_2) \star \epsilon_n = 0 \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

Let $\varphi_1, \varphi_2 \in \Omega$ and $\{\delta_n\} \in \Delta$ such that $\hat{\varphi}_1 = w_1$, $\hat{\varphi}_2 = w_2$ and $\{\epsilon_n\} = \left\{\hat{\delta}_n\right\}$. From (4.1), Definition 4.1 and Theorem 3.4 we get

$$0 = \left(\widehat{\varphi_1 - \varphi_2}\right) \star \epsilon_n = \left(\widehat{(\varphi_1 - \varphi_2) \sharp \delta_n}\right) \to \left(\widehat{(\varphi_1 - \varphi_2)}\right) \text{ as } n \to \infty.$$

Hence

$$\hat{\varphi}_1 - \hat{\varphi}_2 \to 0 \text{ as } n \to \infty.$$

Therefore $w_1 \to w_2$ as $n \to \infty$. This completes the proof of the theorem. □

Theorem 4.5. *Let $w_n \to w$ as $n \to \infty$ in A_{ς} and $\epsilon \in R_{\varsigma}$. Then we have*

$$w_n \star \epsilon \to w \star \epsilon \text{ as } n \to \infty.$$

Proof. Let $w_n, w \in A_{\varsigma}$ for all $n \in \mathbb{N}$ and $\epsilon \in R_{\varsigma}$. Then there are $\{\varphi_n\}$, $\varphi \in \Omega$, and $\delta \in D$ such that

$$\{\hat{\varphi}_n\} = \{w_n\}, \ \hat{\varphi} = w \text{ and } \hat{\delta} = \epsilon.$$

Hence we simply write

$$w_n \star \epsilon - w \star \epsilon = (\widehat{(\varphi_n - \varphi) \sharp \delta}) \to 0 \text{ (as } n \to \infty).$$

Therefore, our theorem has been completely proved. □

Theorem 4.6. *Let $w_n \to w$ as $n \to \infty$ in Ą and $\{\epsilon_n\} \in$ Ę. Then we have*

$$w_n \star \epsilon_n \to w \text{ as } n \to \infty.$$

Proof. As the proof of this theorem is similar to that of Theorem 4.5, we prefer not to add more details. □

The Boehmian space **ß𝔯** is defined where every typical element in **ß𝔯** will be written as

$$\left[\frac{\{w_n\}}{\{\epsilon_n\}}\right].$$

Differentiation in **ß𝔯** is given by

$$D^k \left[\frac{\{w_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{D^k w_n\}}{\{\epsilon_n\}}\right], \; D^k = \frac{d^k}{dx^k}.$$

Addition in **ß𝔯** is given by

$$\left[\frac{\{w_n\}}{\{\epsilon_n\}}\right] + \left[\frac{\{w_n^*\}}{\{\epsilon_n^*\}}\right] = \left[\frac{\{w_n\} \star \{\epsilon_n^*\} + \{w_n^*\} \star \{\epsilon_n\}}{\{\epsilon_n\} \star \{\epsilon_n^*\}}\right].$$

Scalar multiplication in **ß𝔯** is given by

$$k \left[\frac{\{w_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{k w_n\}}{\{\epsilon_n\}}\right], \; k \in \mathbb{C}.$$

Convergence of type δ : A sequence of Boehmians $\{\beta_n\}$ in **ß𝔯** is said to be δ convergent to a Boehmian β in **ß𝔯**, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence $\{\epsilon_n\}$ such that

$$(\beta_n \star \epsilon_n), \; (\beta \star \epsilon_n) \in \text{Ą for every } k, n \in \mathbb{N}$$

and

$$(\beta_n \star \epsilon_k) \to (\beta \star \epsilon_k) \text{ as } n \to \infty \text{ in Ą for every } k \in \mathbb{N}.$$

Equivalent for convergence of type δ :

$\beta_n \xrightarrow{\delta} \beta$ as $(n \to \infty)$ in **ß𝔯** if and only if there is $w_{n,k}, w_k \in$ Ą and $\{\epsilon_k\} \in \delta$ such that $\left[\frac{\{w_{n,k}\}}{\{\epsilon_k\}}\right]$, $\beta = \left[\frac{\{w_k\}}{\{\epsilon_k\}}\right]$, and for each $k \in \mathbb{N}$,

$$w_{n,k} \to w_k$$

as $n \to \infty$ in Ą.

Convergence of type Δ : A sequence of Boehmians $\{\beta_n\}$ in **ß𝔯** is said to be Δ convergent to a Boehmian β in **ß𝔯**, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $\{\epsilon_n\} \in$ Ŗ such that

$$(\beta_n - \beta) \star \epsilon_n \in \text{ß𝔯}$$

for all $n \in \mathbb{N}$, and $(\beta_n - \beta) \star \epsilon_n \to 0$ as $n \to \infty$ in **ß𝔯**.

Since the convolutions and the delta sequences used in **ß𝔯** and **ß** are different, it is not possible to say one space is contained into the other.

5. Mellin transform associated with quaternions

Definition 5.1. Let $x = \left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right] \in$ **ß**. Then we define its extended Mellin transform by

$$\mathcal{F} \left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right] = \left[\frac{\{w_n\}}{\{\epsilon_n\}}\right] \in \text{ß𝔯},$$

where $\{w_n\} = \{\hat{\varphi}_n\}$ and $\{\epsilon_n\} = \left\{\hat{\delta}_n\right\}$.

To show this definition is well defined, we assume $\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right] = \left[\frac{\{\varphi_n^*\}}{\{\delta_n^*\}}\right]$ in the sense of $\mathfrak{ß}$. Then

$$\varphi_n \sharp \delta_m^* = \varphi_m^* \sharp \delta_n.$$

Applying Mellin transform to both sides of the preceding equation yields

$$w_n \star \epsilon_n^* = w_m^* \star \epsilon_m, \text{ where } w_n = \hat{\varphi}_n, \; w_m^* = \varphi_m^*, \; \epsilon_n^* = \delta_n^* \text{ and } \epsilon_n = \hat{\delta}_n.$$

Hence $\frac{\{w_n\}}{\{\epsilon_n\}}$ and $\frac{\{w_n^*\}}{\{\epsilon_n^*\}}$ are two equivalent classes in $\mathfrak{ßr}$. This proves our assertion.

Theorem 5.2. *The Mellin transform $F : \mathfrak{ß} \to \mathfrak{ßr}$ is one-to-one.*

Proof. Assume that $\left[\frac{\{w_n\}}{\{\epsilon_n\}}\right] = \left[\frac{\{w_n^*\}}{\{\epsilon_n^*\}}\right]$ in $\mathfrak{ßr}$ then by the concept of equivalence classes we have

$$w_n \star \epsilon_n^* = w_m^* \star \epsilon_n.$$

Let φ_n, φ_n^* and δ_n^*, δ_n be the preimages of w_n, w_n^* and ϵ_n^*, ϵ_n, respectively. Then we get $\widehat{\varphi_n \sharp \delta_n^*} = \widehat{\varphi_n^* \sharp \delta_n}$ for each $n \in \mathbb{N}$, and hence $\varphi_n \sharp \delta_n^* = \varphi_n^* \sharp \delta_n$ which is interpreted to mean

$$\frac{\{\varphi_n\}}{\{\delta_n\}} \sim \frac{\{\varphi_n^*\}}{\{\delta_n^*\}}.$$

Therefore, the theorem is proved. □

Theorem 5.3. *The Mellin transform $F : \mathfrak{ß} \to \mathfrak{ßr}$ is continuous with respect to convergence of type δ.*

Proof. If $x_n \in \mathfrak{ß}$ is such that $x_n \to 0$ as $n \to \infty$, then, by [17] we have

$$x_n = \left[\frac{\{\varphi_{n,i}\}}{\{\delta_i\}}\right], \; \varphi_{n,i} \to 0 \text{ as } n \to \infty, \text{ where } \varphi_{n,i} \in \Omega.$$

Employing the Mellin transform yields

$$w_{n,i} \to 0 \text{ as } n \to \infty, w_{n,i} = \hat{\varphi}_{n,i}.$$

The above formula completes the proof of our theorem. □

Theorem 5.4. *The mapping $F : \mathfrak{ß} \to \mathfrak{ßr}$ is onto and linear.*

Proof. Linearity is obvious. Let $\left[\frac{\{w_n\}}{\{\epsilon_n\}}\right]$ be in $\mathfrak{ßr}$. Then $w_n \star \epsilon_m = w_m \star \epsilon_n \; \forall n \in \mathbb{N}$.

Hence, it follows $\widehat{\varphi_n \sharp \delta_m} = \widehat{\varphi_m \sharp \delta_n}$ for some $\{\varphi_n\}, \{\varphi_m\} \in \Omega$ and $\{\delta_n\}, \{\delta_m\} \in \Delta$.
Therefore,

$$\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right] \in \mathfrak{ß}$$

is the preimage of $\left[\frac{\{w_n\}}{\{\epsilon_n\}}\right]$.

Thus, the theorem is completely proved. □

Theorem 5.5. *The Mellin transform $F : \mathfrak{ß} \to \mathfrak{ßr}$ is continuous with respect to some convergence of type Δ.*

Proof. Assume $x_n \overset{\Delta}{\to} x \in \mathfrak{ß}$. Then Δ-convergence implies that there is $\varphi_n \in \Omega$ and $\{\delta_n\} \in \Delta$ such that

$$(x_n - x) \sharp \delta_n = \left[\frac{\{\varphi_n\} \sharp \{\delta_n\}}{\{\delta_n\}} \right]$$

and $\varphi_n \to 0$ as $n \to \infty$. Applying Mellin transform implies

$$\widehat{(x_n - x)} \star \epsilon_n = \left[\frac{\{w_n\} \star \{\epsilon_n\}}{\{\epsilon_n\}} \right] \approx w_n \to 0 \text{ as } n \to \infty, \tag{5.1}$$

where $\{\epsilon_n\} = \left\{ \hat{\delta}_n \right\}$ and $\{w_n\} = \{\hat{\varphi}_n\}$.

Hence from (5.1), $\hat{x}_n \to \hat{x}$ as $n \to \infty$. This finishes the proof of the above theorem. □

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