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A graphical version of Reich's fixed point theorem

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Abstract

In this paper, we discuss the definition of the Reich multivalued monotone contraction mappings defined in a metric space endowed with a graph. In our investigation, we prove the existence of fixed point results for these mappings. We also introduce a vector valued Bernstein operator on the space $C([0,1],X)$, where X is a Banach space endowed with a partial order. Then we give an analogue to the Kelisky-Rivlin theorem.

Keywords: Multivalued monotone contraction, graph theory, fixed point theory, partial order.
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1. Introduction

The fundamental fixed point theorem of Banach [2] has laid the foundation of metric fixed point theory for contraction mappings on a complete metric space. Fixed point theory of certain important single-valued mappings is very interesting in its own right due to their results having constructive proofs and applications in industrial fields such as image processing engineering, biology, physics, computer science, economics, and telecommunication.

Multivalued mappings are a useful tool in convex optimization, differential inclusions, control theory, financial mathematics, and mathematical biology. Following the Banach contraction principle, Nadler [19] gave the definition of multivalued contractions and established multivalued contraction version of the classical Banach's fixed point theorem. Subsequently many mathematicians generalized Nadler's fixed point theorem in different ways. In this regard, we mention the work of Reich [22] where he left open a problem on

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the existence of a fixed point of certain class of multivalued mappings (see Problem 3.1). Mizoguchi and Takahashi [18] gave a partial answer to Reich' s problem. Following the publication of Ran and Reuring's fixed point theorem [20] seen as the Banach contraction principle in metric spaces endowed with a partial order, Sultana and Vetrivel [26] tried to extend the main results of [18] to metric spaces endowed with a graph. In particular, they used their ideas to discuss the iterate of the Bernstein operator. Moreover, they gave an example of a nonlinear version of the Bernstein operator and establish the Kelisky and Rivlin's theorem [11] for such operator.

In this paper, we will revisit the definition of the Reich multivalued mappings given by the authors in [26]. Then we will prove a fixed point theorem of these mappings. Moreover, we will define a vector valued Bernstein operator and give a more general version of the Kelisky and Rivlin's theorem. In particular, we will improve the conclusion of [26] regarding the Bernstein operator.

For more on fixed point theory, we suggest the books [7, 12].

2. Preliminaries and Basic results

The multivalued version of the classical Banach's fixed point theorem was obtained by Nadler [19]. Extensions and generalizations of Nadler's fixed point theorem were obtained by many mathematicians [6, 16]. The case of the Banach contraction principle for monotone mappings couched within the case of partially ordered metric spaces was obtained by Ran and Reurings [20]. Following the publication of [20], it was natural to find a similar version of Theorem [19] to metric spaces endowed with partial order or more generally a graph. Beg and Butt [3] gave the first attempt. But their definition of multivalued monotone mappings was not correct which had the effect that the proof of their version of the fixed point theorem of Nadler was wrong (see for example [1]). The difficulty one meets when dealing with multivalued mappings defined in a partially ordered set $(X, \preceq)$ is the problem of comparing two subsets with respect to the order. In fact, there are mainly three well-known pre-orders (reflexive, transitive but not necessarily antisymmetric), namely the *Smyth ordering*, the *Hoare ordering*, and the *Egli-Milner ordering* [9, 13, 25], which have been proposed in the context of nondeterministic programming languages for example:

1. $A \preceq_S B$ if and only if for any $b \in B$, there exists $a \in A$ such that $a \preceq b$ (Smyth ordering);
2. $A \preceq_H B$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \preceq b$ (Hoare ordering);
3. $A \preceq_{EM} B$ if and only if $A \preceq_S B$ and $A \preceq_H B$ (Egli-Milner ordering)

for any nonempty subsets A and B of X. Clearly, the Hoare order is equivalent to the Smyth order in the dual underlying lattice. Similarly, we follow Jachymski's extension [10] of the fixed point theorem of Ran and Reurings [20] to a metric space endowed with a graph instead of a partial order. Recall that a directed graph G consists of two sets: $V(G)$ a nonempty set of elements called vertices, and $E(G)$ a possibly empty set of elements in $V(G) \times V(G)$ called edges. If $E(G)$ contains all the loops (u, u), then G is reflexive. Let X be a set endowed with a reflexive digraph G such that $V(G) = X$. Then, we will write

1. $(A, B)_S \in E(G)$ if and only if for any $b \in B$, there exists $a \in A$ such that $(a, b) \in E(G)$;
2. $(A, B)_H \in E(G)$ if and only if for any $a \in A$, there exists $b \in B$ such that $(a, b) \in E(G)$;
3. $(A, B)_{EM} \in E(G)$ if and only if $(A, B)_S \in E(G)$ and $(A, B)_H \in E(G)$

for any nonempty subsets A and B of X. Throughout, we will use the Hoare relationship and will omit to write the subscript H. Before, we give the definition of the Reich multivalued mappings, we will need some basic definitions and notations. Let (X, d) be a metric space. The Hausdorff-Pompeiu distance H defined on $\mathcal{CB}(\mathcal{X})$, the set of nonempty bounded and closed subsets of X, is defined by

$$H(A, B) = \max\left\{\sup_{b\in B}\inf_{a\in A} d(b, a),\ \sup_{a\in A}\inf_{b\in B} d(a, b)\right\}$$

for any $A, B \in \mathcal{CB}(\mathcal{X})$. The following technical result is useful to explain our definition later on.

Lemma 2.1 ([19]). *Let (X,d) be a metric space. For any $A,B \in \mathcal{CB}(\mathcal{X})$ and $\varepsilon > 0$, and for any $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \leq H(A,B) + \varepsilon.$$

Using Lemma 2.1, we are able to give a simpler formulation of what is a multivalued contraction which avoids the use of Hausdorff-Pompeiu distance. Indeed, let (X,d) be a metric space. The mapping $T : X \to \mathcal{CB}(\mathcal{X})$ is a contraction mapping if there exists $\alpha \in [0,1)$ such that for any $x,y \in X$ and $a \in T(x)$, there exists $b \in T(y)$ such that

$$d(a,b) \leq \alpha\ d(x,y).$$

Clearly this definition does not use the boundedness assumption of the considered subsets of X. Instead, we will consider $\mathcal{C}(X)$ the set of all nonempty closed subsets of X.

In their attempt to extend Mizoguchi-Takahashi's fixed point theorem for Reich multivalued contraction mappings to metric spaces endowed with a graph, Sultana and Vetrivel [26] introduced the concept of Reich G-contractions.

Definition 2.2 ([26]). Let (X,d,G) be a metric space endowed with a reflexive directed graph G with no-parallel edges. The multivalued map $T : X \to \mathcal{CB}(\mathcal{X})$ is called a Reich G-contraction if for any different $x,y \in X$ with $(x,y) \in E(G)$, we have

(i) $H(T(x),T(y)) \leq k(d(x,y))\ d(x,y)$,

(ii) if $(u,v) \in T(x) \times T(y)$ is such that $d(u,v) \leq d(x,y)$, we have $(u,v) \in E(G)$, for some $k : (0,+\infty) \to [0,1)$ which satisfies $\limsup\limits_{s \to t+} k(s) < 1$ for any $t \in [0,+\infty)$.

This definition is not appropriate because of the condition (ii). The following example explains our reasoning.

Example 2.3. Consider the space $\mathbb{R}^2$ endowed with the Euclidean distance d. Then, consider the graph G obtained by the pointwise ordering of $\mathbb{R}^2$ defined by

$$(x,y) = ((x_1,x_2),(y_1,y_2)) \in E(G) \ \text{ iff } \ x_1 \leq y_1 \ \&\ x_2 \leq y_2.$$

Let A be the unit ball of $\mathbb{R}^2$, that is, $A = \{(x_1,x_2) \in \mathbb{R}^2;\ d^2(x,0) = x_1^2 + x_2^2 \leq 1\}$. Consider the multivalued map $T : \mathbb{R}^2 \to \mathcal{CB}(\mathbb{R}^\in)$ defined by $T(x) = A$. Then, we have $H(T(x),T(y) = 0$ for any $x,y \in \mathbb{R}^2$. Since T is a constant multivalued mapping, then it is a contraction according to Nadler's definition. Therefore, T must be a Reich G-contraction. The condition (i) is obviously satisfied but the condition (ii) fails. Indeed, set $x = (2,0)$ and $y = (2,2)$. Then $x \neq y$ and $(x,y) \in E(G)$. Since $d(x,y) = 2$, (ii) will hold if and only if for any $u,v \in A$ such that $d(u,v) \leq 2$, we must have $(u,v) \in E(G)$. This is not the case, if we take

$$u = (1,0) \ \text{ and } \ v = (0,1),$$

then $u,v \in A$, $d(u,v) = \sqrt{2}$, $(u,v) \notin E(G)$ and $(v,u) \notin E(G)$.

Before we give the correct definition of Reich multivalued G-contractions, we need the following remark.

Remark 2.4. Let (X,d) be a metric space. Let $T : X \to \mathcal{CB}(\mathcal{X})$. Assume there exists $\alpha : (0,+\infty) \to [0,1)$ with $\limsup\limits_{s \to t+} \alpha(s) < 1$ for any $t \in [0,+\infty)$, such that

$$H(T(x),T(y)) \leq \alpha(d(x,y))\ d(x,y)$$

for any different $x,y \in X$. Using Lemma 2.1, we can easily prove that for any different $x,y \in X$ and $a \in T(x)$, there exists $b \in T(y)$ such that

$$d(a,b) \leq \beta(d(x,y))\ d(x,y),$$

where $\beta = \frac{1}{2}(1+\alpha)$ which satisfies $\limsup\limits_{s \to t+} \beta(s) < 1$ for any $t \in [0,+\infty)$.

The following definition is more appropriate than Definition 2.2.

Definition 2.5. Let (X, d, G) be a metric space endowed with a reflexive directed graph G with no-parallel edges. The multivalued map $T : X \to \mathcal{C}(\mathcal{X})$ is called a Reich G-contraction if there exists $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t+} k(s) < 1$ for any $t \in [0, +\infty)$, such that for any different $x, y \in X$ with $(x, y) \in E(G)$ and any $a \in T(x)$, there exists $b \in T(y)$ such that $(a, b) \in E(G)$ and

$$d(a, b) \leq k(d(x, y))\ d(x, y).$$

A point $x \in X$ is a fixed point of T if $x \in T(x)$.

3. Main Results

In [22], Reich raised the following problem:

Problem 3.1. Let (X, d) be a complete metric space. Consider the multivalued map $T : X \to \mathcal{CB}(\mathcal{X})$ for which there exists $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t+}\ k(s) < 1$ for any $t \in (0, +\infty)$, such that for any different $x, y \in X$, we have

$$H(T(x), T(y)) \leq k(d(x, y))\ d(x, y).$$

Does T have a fixed point?

In [21], Reich proved that such mappings have a fixed point provided they have compact values. Clearly, if $k(t)$ is a constant function, Nadler gives a positive answer to Reich's problem. In [18], Mizoguchi and Takahashi gave a positive answer when the function $k(t)$ is defined on $[0, +\infty)$. It is still unclear whether Reich's problem has a positive answer. In this section, we will discuss the graph version of the fixed point theorem of Mizoguchi and Takahashi.

First, we state and give a simpler proof to the original theorem of [18] without boundedness.

Theorem 3.2. *Let (X, d) be a complete metric space. Then any Reich contraction mapping $T : X \to \mathcal{C}(\mathcal{X})$ has a fixed point.*

Proof. Since $T : X \to \mathcal{C}(\mathcal{X})$ is a Reich contraction mapping, then there exists $k : (0, +\infty) \to [0, 1)$ with $\limsup_{s\to t+} k(s) < 1$, for any $t \in [0, +\infty)$, such that for any $x, y \in X$ and $a \in T(x)$, there exists $b \in T(y)$ such that

$$d(a, b) \leq k(d(x, y))\ d(x, y).$$

Fix $y_0 \in X$. If y_0 is a fixed point of T, then we have nothing to prove. Otherwise, choose $y_1 \in T(y_0)$ different from y_0. Using the contractive assumption of T, there exists $y_2 \in T(y_1)$ such that

$$d(y_1, y_2) \leq k(d(y_0, y_1))\ d(y_0, y_1).$$

By induction, we construct a sequence $\{y_n\}$ in X such that $y_{n+1} \in T(y_n)$ and $y_n \neq y_{n+1}$ with

$$d(y_n, y_{n+1}) \leq k(d(y_{n-1}, y_n))\ d(y_{n-1}, y_n)$$

for any $n \geq 1$. Since $k(t) < 1$, for any $t \in [0, +\inf)$, we conclude that $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of positive numbers. Let

$$t_0 = \lim_{n\to+\infty} d(y_n, y_{n+1}) = \inf_{n\in\mathbb{N}} d(y_n, y_{n+1}).$$

Since $\limsup_{s\to t_0+} k(s) < 1$, there exist $\alpha < 1$ and $n_0 \geq 1$ such that $k(d(y_n, y_{n+1})) \leq \alpha$ for any $n \geq n_0$. Then, we have

$$d(y_n, y_{n+1}) \leq \prod_{k=n_0}^{k=n} k(d(y_k, y_{k+1}))\ d(y_{n_0}, y_{n_0+1}) \leq \alpha^{n-n_0}\ d(y_{n_0}, y_{n_0+1})$$

for any $n \geq n_0$. This will imply that $\sum d(y_n, y_{n+1})$ is convergent. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, then $\{y_n\}$ converges to some point $x \in X$. Let us prove that x is a fixed point of T. Using the contractive assumption of T, there exists $z_n \in T(x)$ such that

$$d(y_{n+1}, z_n) \leq k(d(y_n, x))\ d(y_n, x) < d(y_n, x),$$

for any $n \in \mathbb{N}$. This will force $\{z_n\}$ to also converge to x. Since $T(x)$ is closed, we conclude that $x \in T(x)$, that is, x is a fixed point of T as claimed. □

Next, we discuss the extension of Theorem 3.2 to metric spaces endowed with a graph. We will need the following property introduced by Jachymski [10]:

Property 3.3. Let (X, d, G) be a metric space endowed with a reflexive directed graph G with no-parallel edges. (X, d, G) is said to satisfy Property 3.3 if and only if for any sequence $\{x_n\}$ in X such that $(x_n, x_{n+1}) \in E(G)$ and $\{x_n\}$ converges to x, then $(x_{\varphi(n)}, x) \in E(G)$ for any $n \in \mathbb{N}$, where $\{x_{\varphi(n)}\}$ is a subsequence of $\{x_n\}$.

Theorem 3.4. *Let (X, d) be a complete metric space. Let G be a reflexive graph with no-parallel edges such that $E(G) = X$. Assume that (X, d, G) satisfies Property 3.3. Let $T : X \to \mathcal{C}(\mathcal{X})$ be a Reich G-contraction. Then there exists $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s \to t+} k(s) < 1$ for any $t \in [0, +\infty)$, such that for any different $x, y \in X$ with $(x, y) \in E(G)$ and any $a \in T(x)$, there exists $b \in T(y)$ such that $(a, b) \in E(G)$ and*

$$d(a, b) \leq k(d(x, y))\ d(x, y).$$

Set $X_T = \{x \in X;$ there exists $y \in T(x)$ such that $(x, y) \in E(G)\}$. If $X_T \neq \emptyset$, then T has a fixed point.

Proof. Assume $X_T \neq \emptyset$. Let $y_0 \in X_T$. Then there exists $y_1 \in T(y_0)$ such that $(y_0, y_1) \in E(G)$. If $y_1 = y_0$, then y_0 is a fixed point of T. Assume $y_0 \neq y_1$. then there exists $y_2 \in T(y_1)$ such that

$$d(y_1, y_2) \leq k(d(y_0, y_1))\ d(y_0, y_1).$$

By induction, we construct a sequence $\{y_n\}$ such that $y_n \neq y_{n+1}$, $y_{n+1} \in T(y_n)$, $(y_n, y_{n+1}) \in E(G)$ and

$$d(y_n, y_{n+1}) \leq k(d(y_{n-1}, y_n))\ d(y_{n-1}, y_n),$$

for any $n \geq 1$. As we did in the proof of Theorem 3.2, we will show that $\{y_n\}$ converges to some point $x \in X$. Let us prove that x is a fixed point of T. Since (X, d, G) satisfies Property 3.3, there exists a subsequence $\{y_{\varphi(n)}\}$ of $\{y_n\}$ such that $(y_{\varphi(n)}, x) \in E(G)$ for any $n \in \mathbb{N}$. Using the contractive assumption of T, there exists $z_n \in T(x)$ such that

$$d(y_{\varphi(n)+1}, z_n) \leq k(d(y_{\varphi(n)}, x))\ d(y_{\varphi(n)}, x) < d(y_{\varphi(n)}, x)$$

for any $n \in \mathbb{N}$. This will force $\{z_n\}$ to also converge to x. Since $T(x)$ is closed, we conclude that $x \in T(x)$, that is, x is a fixed point of T as claimed. □

Remark 3.5. Once Theorem 3.2 and Theorem 3.4 are established, it is easy to extend them to the case of uniformly locally contractive mappings in the sense of Edelstein [5] with or without a graph.

4. Application: A generalized Bernstein operator

In [11], Kelisky and Rivlin investigated the behavior of the iterates of the Bernstein polynomial of degree $n \geq 1$ defined by

$$B_n(f)(t) = \sum_{k=0}^{k=n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}$$

for any $f \in C([0,1])$ and $t \in [0,1]$, where $C([0,1])$ is the space of continuous functions defined on $[0,1]$. In particular, they proved that for any $f \in C([0,1])$, we have

$$\lim_{j\to+\infty} B_n^j(f)(t) = f(0)(1-t) + f(1)t, \ \ 0 \le t \le 1. \tag{KRB}$$

Their proof uses the techniques of matrix algebra. Rus [24] was the first one to notice that a proof of (KRB) exists which is metric in nature. In fact, his proof inspired Jachymski [10] to rephrase it in the identical using the graph language. In [26], Sultana and Vetrivel modified the Bernstein operator to obtain a nonlinear version which will not be suitable for the technique used by Kelisky and Rivlin. Indeed, Sultana and Vetrivel introduced the operator:

$$B_n'(f)(t) = \sum_{k=0}^{k=n} \left| f\left(\frac{k}{n}\right)\right| \begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k}$$

for any $f \in C([0,1])$. Then, they proved that

$$\lim_{j\to+\infty} (B_n')^j(f)(t) = f(0)(1-t) + f(1)t, \ \ 0 \le t \le 1$$

for any $f \in C([0,1])$ such that $f(0) \ge 0$ and $f(1) \ge 0$. In fact, we have a better conclusion. Indeed, it is easy to see that $B_n'(f) = B_n(|f|)$ for any $f \in C([0,1])$, which allows us to obtain the following more general result:

Proposition 4.1. *Let $f \in C([0,1])$. Then*

$$\lim_{j\to+\infty} (B_n')^j(f)(t) = |f(0)|(1-t) + |f(1)|t, \ \ 0 \le t \le 1.$$

Next we extend the classical Bernstein operator to the vector case. Indeed, let $(X, \|.\|_X)$ be a Banach space. Consider the Banach space $C([0,1],X)$, the space of all continuous functions defined on $[0,1]$ with values in X. The norm in $C([0,1],X)$ is given by

$$\|f\| = \sup\{\|f(t)\|_X;\ t \in [0,1]\}.$$

Fix $n \ge 1$ and define the generalized Bernstein operator $B_n : C([0,1],X) \to C([0,1],X)$ by

$$B_n(f)(t) = \sum_{k=0}^{k=n} \begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k}\ f\left(\frac{k}{n}\right)$$

for any $t \in [0,1]$. In this case, we have a similar conclusion to Kelisky and Rivlin's result.

Theorem 4.2. *For any $f \in C([0,1],X)$, we have*

$$\lim_{j\to+\infty} B_n^j(f)(t) = (1-t)f(0) + tf(1), \ \ 0 \le t \le 1.$$

Proof. We may use the graph language to prove this conclusion. Since the proof is identical to the one used by Rus [24], we prefer to give this proof instead. Indeed, let us first notice that

$$\sum_{k=0}^{k=n} \begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k} = 1 \text{ and } \sum_{k=0}^{k=n} \frac{k}{n} \begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k} = t$$

for any $t \in [0,1]$. Set $g(t) = (1-t)f(0) + tf(1)$ for $t \in [0,1]$. Obviously $g \in C([0,1],X)$. We have $B_n(g) = g$. Since $f(0) = g(0)$ and $f(1) = g(1)$, we have

$$B_n(f)(t) - B_n(g)(t) = \sum_{k=1}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k}\ \left(f\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right)\right)$$

for any $t \in [0,1]$. Hence

$$\|B_n(f)(t) - B_n(g)(t)\| \leq \sum_{k=1}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k} \left\| f\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right) \right\|$$

for any $t \in [0,1]$, which implies

$$\|B_n(f)(t) - B_n(g)(t)\| \leq \left(1 - \frac{1}{2^{n-1}}\right) \|f - g\|$$

for any $t \in [0,1]$. So

$$\|B_n(f) - g\| = \|B_n(f) - B_n(g)\| \leq \left(1 - \frac{1}{2^{n-1}}\right) \|f - g\|.$$

By induction, we obtain

$$\|B_n^j(f) - g\| \leq \left(1 - \frac{1}{2^{n-1}}\right)^j \|f - g\|$$

for any $j \in \mathbb{N}$. This clearly implies the conclusion of Theorem 4.2 as claimed. $\square$

Motivated by Sultana and Vetrivel example, we introduce the following Bernstein operator $B_n' : C([0,1], X) \to C([0,1], X)$ defined by

$$B_n'(f)(t) = \sum_{k=0}^{k=n} \left(\begin{array}{c} n \\ k \end{array} \right) t^k (1-t)^{n-k}\, T\left(f\left(\frac{k}{n}\right)\right), \tag{4.1}$$

where $T : X \to X$ is continuous. Since $B_n'(f) = B_n(T \circ f)$, we obtain the following result:

Theorem 4.3. *For any $f \in C([0,1], X)$, we have*

$$\lim_{j \to +\infty} (B_n')^j(f)(t) = (1-t)T\Big(f(0)\Big) + tT\Big(f(1)\Big), \;\; 0 \leq t \leq 1.$$

Remark 4.4. Paying a careful attention to the definition of B_n, we see that it is a convex combination because $\sum\limits_{i=0}^{i=m} \left(\begin{array}{c} m \\ i \end{array} \right) t^i (1-t)^{m-i} = 1$ for any $m \geq 1$. Therefore, we may have similar conclusions as Theorems 4.2 and 4.3 if we take for X a hyperbolic metric space like $CAT(0)$ spaces. For more on hyperbolic spaces, we refer to [4, 8, 14, 15, 17, 23].

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References

[1] M. R. Alfuraidan, *Remarks on monotone multivalued mappings on a metric space with a graph,* J. Inequal. Appl., **2015** (2015), 7 pages. 2

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[3] I. Beg, A. R. Butt, *Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces*, Nonlinear Anal., **71** (2009), 3699–3704. 2

[4] H. Busemann, *Spaces with non-positive curvature*, Acta. Math., **80** (1948), 259–310. 4.4

[5] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc., **12** (1961), 7–10. 3.5

[6] Y. Feng, S. Liu, *Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings*, J. Math. Anal. Appl., **317** (2006), 103–112. 2

[7] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, (1990). 1

[8] K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, Inc., New York, (1984). 4.4

[9] C. A. R. Hoare, *A Model for communicating sequential processes*, Technical Report PRG-22, Programming Research Group, Oxford University Computing Lab, (1981). 2

[10] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373. 2, 3, 4

[11] R. P. Kelisky, T. J. Rivlin, *Iterates of Bernstein polynomials*, Pacific J. Math., **21** (1967), 511–520. 1, 4

[12] M. A. Khamsi, W. A. Kirk, *An introduction to metric spaces and fixed point theory*, Wiley-Interscience, New York, (2001). 1

[13] M. A. Khamsi, D. Misane, *Disjunctive Signed Logic Programs*, Fund. Inform., **32** (1997), 349–357. 2

[14] W. A. Kirk, *Fixed point theory for nonexpansive mappings*, Lecture Notes in Mathematics, Springer, Berlin, (1981), 485–505. 4.4

[15] W. A. Kirk, *A fixed point theorem in CAT(0) spaces and* $\mathbb{R}$*-trees*, Fixed Point Theory Appl., **2004** (2004), 309–316. 4.4

[16] D. Klim, D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **334** (2007), 132–139. 2

[17] L. Leustean, *A quadratic rate of asymptotic regularity for CAT (0)-spaces*, J. Math. Anal. Appl., **325** (2007), 386–399. 4.4

[18] N. Mizoguchi, W. Takahashi, *Fixed Point Theorems for Multivalued Mappings on Complete Metric Spaces*, J. Math. Anal. Appl., **141** (1989), 177–188. 1, 3

[19] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1, 2, 2.1

[20] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1, 2, 2

[21] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital., **5** (1972), 26–42. 3

[22] S. Reich, *Some fixed point problems*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **57** (1974), 194–198. 1, 3

[23] S. Reich, I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal., **15** (1990), 537–558. 4.4

[24] I. A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl., **292** (2004), 259–261. 4, 4

[25] M. B. Smyth, *Power domains*, J. Comput. System Sci., **16** (1978), 23–36. 2

[26] A. Sultana, V. Vetrivel, *Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications*, J. Math. Anal. Appl., **417** (2014), 336–344. 1, 2, 2.2, 4