Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4772–4779
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A hybrid algorithm with Meir-Keeler contraction for asymptotically pseudocontractive mappings

Youli Yu[a], Ching-Feng Wen[b,*], Xiaoyin Wang[c]

[a]*School of Mathematics and Information Engineering, Taizhou University, Linhai 317000, China.*
[b]*Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung, 807, Taiwan.*
[c]*Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China.*

Communicated by Y. Yao

Abstract

A hybrid algorithm with Meir-Keeler contraction for finding the fixed points of the asymptotically pseudocontractive mappings is presented. Some strong convergence results are given.

Keywords: Asymptotically pseudocontractive mapping, hybrid algorithm, fixed point.
2010 MSC: 47H05, 47H10, 47H17.

1. Introduction

Throughout this paper, let H be a real Hilbert space with inner $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\emptyset \neq C \subset H$ be a closed convex set. Let $T : C \to C$ be a nonlinear operator with nonempty fixed points set $Fix(T)$.

Definition 1.1. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.

*Corresponding author
Email addresses: `yuyouli@tzc.edu.cn` (Youli Yu), `cfwen@kmu.edu.tw` (Ching-Feng Wen), `wxywxq@163.com` (Xiaoyin Wang)

Received 2016-07-03

Definition 1.2. T is called asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$ and for all $n \geq 1$.

Remark 1.3. Note that (1.1) is equivalent to the following inequality

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(x - T^n x) - (y - T^n y)\|^2 \tag{1.2}$$

for all $x, y \in C$ and for all $n \geq 1$.

The class of asymptotic pseudocontractions was introduced by Schu [11] in 1991. In order to compute the fixed point of asymptotic pseudocontractions, Schu [11] designed the following convergence result.

Theorem 1.4. *Let H be a real Hilbert space. Let $\emptyset \neq C \subset H$ be a closed and convex set. Let $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with $\{k_n\}_{n=1}^{\infty} \subset [1, \infty)$. Let $\{x_n\}$ be a sequence generated by the following algorithm: for initial guess $x_0 \in C$, compute the sequence x_n by the form*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1. \end{cases} \tag{1.3}$$

Suppose the following conditions are satisfied:

(i) *C is bounded and T is completely continuous;*
(ii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*;*
(iii) $0 < \kappa_1 \leq \alpha_n \leq \beta_n \leq \kappa_2 < \frac{\sqrt{1+L^2}-1}{L^2}$ *for all $n \geq 1$.*

Then the sequence $\{x_n\}$ generated by (1.3) *converges strongly to some fixed point of T.*

But we observe that the assumption T is completely continuous, that is, $T(C)$ is relatively compact, is severe restriction. This brings us to the following question.

Question 1.5. *Can we construct an iterative algorithm for finding the fixed points of asymptotically pseudocontractive mappings without the assumption that T be completely continuous?*

There are a large number of works for finding the fixed points of the pseudocontractive mappings and asymptotically pseudocontractive mappings. We refer the reader to [1, 3, 5, 8, 14, 20, 21, 23–25] and [2, 4, 6, 10, 15–19, 22].

The main purpose of this paper is to introduce a hybrid algorithm with Meir-Keeler contraction for finding the fixed points of the asymptotically pseudocontractive mappings. We prove that the presented algorithm converges strongly to the fixed point of the asymptotically pseudocontractive mapping in Hilbert spaces.

2. Preliminaries

Recall that the metric projection $proj_C : H \to C$ satisfies

$$\|u - proj_C(u)\| = \inf\{\|u - u^\dagger\| : u^\dagger \in C\}.$$

The metric projection $proj$ is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is

$$\langle u - proj_C(u), u^\dagger - proj_C(u)\rangle \leq 0$$

for all $u \in H$, $u^\dagger \in C$.

Recall that a mapping T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, and if the sequence $\{T(x_n)\}$ strongly converges to $x^\dagger$, then $T(\tilde{x}) = x^\dagger$.

It is well-known that in a real Hilbert space H, the following equality holds:

$$\|\xi u + (1-\xi)u^\dagger\|^2 = \xi\|u\|^2 + (1-\xi)\|u^\dagger\|^2 - \xi(1-\xi)\|u-u^\dagger\|^2 \tag{2.1}$$

for all $u, u^\dagger \in H$ and $\xi \in [0,1]$.

Lemma 2.1 ([25])**.** *Let C be a nonempty bounded and closed convex subset of a real Hilbert space H. Let $T\colon C \to C$ be a uniformly L-Lipschtzian and asymptotically pseudocontractive mapping. Then $I - T$ is demiclosed at zero.*

For convenient, in the sequel we shall use the following expressions:

- $x_n \rightharpoonup x^\dagger$ denotes the weak convergence of x_n to $x^\dagger$;
- $x_n \to x^\dagger$ denotes the strong convergence of x_n to $x^\dagger$.

Let the sequence $\{C_n\}$ be a nonempty closed convex subset of a Hilbert space H. We define $s - Li_nC_n$ and $w - Ls_nC_n$ as follows.

- $x \in s - Li_nC_n$ if and only if there exists $\{x_n\} \subset C_n$ such that $x_n \to x$.
- $x \in w - Ls_nC_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset C_{n_i}$ such that $y_i \rightharpoonup y$.

If C_0 satisfies

$$C_0 = s - Li_nC_n = w - Ls_nC_n,$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [9] and we write $C_0 = M - \lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Tsukada [13] proved the following theorem for the metric projection.

Lemma 2.2 ([13])**.** *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{proj_{C_n}(x)\}$ converges strongly to $proj_{C_0}(x)$, where $proj_{C_n}$ and $proj_{C_0}$ are the metric projections of H onto C_n and C_0, respectively.*

Let (X, d) be a complete metric space. A mapping $f : X \to X$ is called a Meir-Keeler contraction [7] if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \epsilon + \delta \ \text{ implies } \ d(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. It is well-known that the Meir-Keeler contraction is a generalization of the contraction.

Lemma 2.3 ([7])**.** *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 2.4 ([12])**.** *Let f be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then, for every $\epsilon > 0$, there exists $r \in (0,1)$ such that*

$$\|x - y\| \geq \epsilon \ \textit{implies} \ \|f(x) - f(y)\| \leq r\|x - y\|$$

for all $x, y \in C$.

Lemma 2.5 ([12])**.** *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C, and let f be a Meir-Keeler contraction on C. Then the following hold.*

(i) *$T \cdot f$ is a Meir-Keeler contraction on C;*

(ii) *for each $\alpha \in (0,1)$, $(1-\alpha)T + \alpha f$ is a Meir-Keeler contraction on C.*

3. Main results

In this section, we firstly introduce a projected fixed point algorithm with Meir-Keeler contraction for asymptotically pseudocontractive mappings in Hilbert spaces. Consequently, we show the strong convergence of our presented algorithm.

In the sequel, we assume that H is a real Hilbert space and $\emptyset \neq C \subset H$ is a bounded closed convex set. Let $T : C \to C$ be an $L(>1)$-Lipschitzian asymptotically pseudocontractive mapping with $Fix(T) \neq \emptyset$. Let $f : C \to C$ be a Meir-Keeler contractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$.

Algorithm 3.1. For $x_0 \in C_0 = C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \\ C_{n+1} = \{z \in C_n : \|(1-\alpha_n)x_n + \alpha_n T^n y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = proj_{C_{n+1}} f(x_n), \forall n \geq 0, \end{cases} \tag{3.1}$$

where $proj$ is the metric projection and $\theta_n = 2\alpha_n[1 + (2k_n - 1)\beta_n](k_n - 1)(diam\ C)^2 \to 0$ as $n \to \infty$.

Theorem 3.2. *If* $0 < a < \alpha_n \leq \beta_n < b < \frac{1}{\sqrt{k_n^2+L^2}+k_n}$ *for all* $n \geq 1$*, then the sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to* $x^\dagger = proj_{Fix(T)} f(x^\dagger)$.

Remark 3.3. Since f is a Meir-Keeler contraction of C, we get $proj_{Fix(T)} f$ is a Meir-Keeler contraction of C by Lemma 2.5. According to Lemma 2.3, there exists a unique fixed point $x^\dagger \in C$ such that $x^\dagger = proj_{Fix(T)} f(x^\dagger)$.

Proof. We first show by induction that $Fix(T) \subset C_n$ for all $n \geq 0$.

(i) $Fix(T) \subset C_0$ is obvious.

(ii) Suppose that $Fix(T) \subset C_k$ for some $k \in \mathbb{N}$. Then, for $x^\dagger \in Fix(T) \subset C_k$, we have from (1.2) that

$$\|T^n x_n - x^\dagger\|^2 \leq (2k_n - 1)\|x_n - x^\dagger\|^2 + \|T^n x_n - x_n\|^2, \tag{3.2}$$

and

$$\begin{aligned} \|T^n y_n - x^\dagger\|^2 &= \|T^n((1-\beta_n)x_n + \beta_n T^n x_n) - x^\dagger\|^2 \\ &\leq (2k_n - 1)\|(1-\beta_n)(x_n - x^\dagger) + \beta_n(T^n x_n - x^\dagger)\|^2 \\ &\quad + \|(1-\beta_n)x_n + \beta_n T^n x_n - T^n y_n\|^2. \end{aligned} \tag{3.3}$$

From (2.1), we have

$$\begin{aligned} \|(1-\beta_n)x_n + \beta_n T^n x_n - T^n y_n\|^2 &= \|(1-\beta_n)(x_n - T^n y_n) + \beta_n(T^n x_n - T^n y_n)\|^2 \\ &= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|T^n x_n - T^n y_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2. \end{aligned} \tag{3.4}$$

Since T is uniformly L-Lipschitzian and $x_n - y_n = \beta_n(x_n - T^n x_n)$, by (3.4), we get

$$\begin{aligned} \|(1-\beta_n)x_n + \beta_n T^n x_n - T^n y_n\|^2 &\leq (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2\|x_n - T^n x_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &= (1-\beta_n)\|x_n - T^n y_n\|^2 + (\beta_n^3 L^2 + \beta_n^2 - \beta_n)\|x_n - T^n x_n\|^2. \end{aligned} \tag{3.5}$$

By (2.1) and (3.2), we have

$$\begin{aligned} \|(1-\beta_n)(x_n - x^\dagger) + \beta_n(T^n x_n - x^\dagger)\|^2 &= (1-\beta_n)\|x_n - x^\dagger\|^2 + \beta_n\|T^n x_n - x^\dagger\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\leq (1-\beta_n)\|x_n - x^\dagger\|^2 + \beta_n[(2k_n - 1)\|x_n - x^\dagger\|^2 \\ &\quad + \|x_n - T^n x_n\|^2] - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &= (1 - 2\beta_n + 2k_n\beta_n)\|x_n - x^\dagger\|^2 + \beta_n^2\|x_n - T^n x_n\|^2. \end{aligned} \tag{3.6}$$

By (3.3), (3.5) and (3.6), we obtain

$$\begin{aligned}\|T^n y_n - x^\dagger\|^2 \le (2k_n-1)(1-2\beta_n+2k_n\beta_n)\|x-x^\dagger\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2 \\ - \beta_n(1-2k_n\beta_n - \beta_n^2 L^2)\|x_n - T^n x_n\|^2.\end{aligned} \tag{3.7}$$

Since $\beta_n < b < \frac{1}{\sqrt{k_n^2+L^2}+k_n}$, we derive that

$$1 - 2k_n\beta_n - \beta_n^2 L^2 > 0, \quad \forall n \ge 1.$$

This together with (3.7) implies that

$$\|T^n y_n - x^\dagger\|^2 \le (2k_n-1)(1-2\beta_n+2k_n\beta_n)\|x_n - x^\dagger\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2. \tag{3.8}$$

By (2.1) and (3.8) and noting that $\alpha_n \le \beta_n$, we have

$$\begin{aligned}\|(1-\alpha_n)x_n + \alpha_n T^n y_n - x^\dagger\|^2 &= (1-\alpha_n)\|x_n - x^\dagger\|^2 + \alpha_n\|T^n y_n - x^\dagger\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - T^n y_n\|^2 \\ &\le [(1-\alpha_n) + \alpha_n(2k_n-1)(1-2\beta_n+2k_n\beta_n)]\|x_n - x^\dagger\|^2 \\ &\quad - \alpha_n(\beta_n - \alpha_n)\|x_n - T^n y_n\|^2 \\ &\le [(1-\alpha_n) + \alpha_n(2k_n-1)(1-2\beta_n+2k_n\beta_n)]\|x_n - x^\dagger\|^2 \\ &= \|x_n - x^\dagger\|^2 + 2\alpha_n[1+(2k_n-1)\beta_n](k_n-1)\|x_n - x^\dagger\|^2 \\ &\le \|x_n - x^\dagger\|^2 + \theta_n,\end{aligned}$$

and hence $x^\dagger \in C_{k+1}$. This indicates that

$$Fix(T) \subset C_n$$

for all $n \ge 0$. Next, we show that C_n is closed and convex for all $n \ge 0$.

(i) It is obvious from the assumption that $C_0 = C$ is closed convex.

(ii) Suppose that C_k is closed and convex for some $k \in N$. For $z \in C_k$, we know that $\|(1-\alpha_k)x_k + \alpha_k T y_k - z\|^2 \le \|x_k - z\|^2 + \theta_k$ is equivalent to

$$2\alpha_k\langle x_k - T^k y_k, z\rangle \le \|x_k\|^2 - \|(1-\alpha_k)x_k + \alpha_k T y_k\|^2 + \theta_k.$$

So, C_{k+1} is closed and convex. By the induction, we deduce that C_n is closed and convex for all $n \ge 0$. This implies that $\{x_n\}$ is well-defined. Next, we prove that

$$\lim_{n\to\infty}\|x_n - u\| = 0$$

for some $u \in \cap_{n=1}^\infty C_n$ and

$$\langle f(u) - u, u - y\rangle \ge 0$$

for all $y \in Fix(T)$.

Since $\bigcap_{n=1}^\infty C_n$ is closed and convex, we also have that $proj_{\bigcap_{n=1}^\infty C_n}$ is well-defined and so $proj_{\bigcap_{n=1}^\infty C_n} f$ is a Meir-Keeler contraction on C. By Lemma 2.3, there exists a unique fixed point $u \in \bigcap_{n=1}^\infty C_n$ of $proj_{\bigcap_{n=1}^\infty C_n} f$. Since C_n is a nonincreasing sequence of nonempty closed convex subset of H with respect to inclusion, it follows that

$$\emptyset \ne Fix(T) \subset \bigcap_{n=1}^\infty C_n = M - \lim_{n\to\infty} C_n.$$

Setting $u_n := proj_{C_n} f(u)$ and applying Lemma 2.2, we can conclude

$$\lim_{n\to\infty} u_n = proj_{\bigcap_{n=1}^{\infty} C_n} f(u) = u.$$

Now we show that $\lim_{n\to\infty} \|x_n - u\| = 0$. Assume $d = \overline{\lim}_n \|x_n - u\| > 0$, then for all ϵ, $0 < \epsilon < d$, we can choose a $\delta_1 > 0$ such that

$$\overline{\lim_n} \|x_n - u\| > \epsilon + \delta_1. \tag{3.9}$$

Since f is a Meir-Keeler contraction, for above ϵ, there exists another $\delta_2 > 0$ such that

$$\|x - y\| < \epsilon + \delta_2 \text{ implies } \|f(x) - f(y)\| < \epsilon \tag{3.10}$$

for all $x, y \in C$. In fact, we can choose a common $\delta > 0$ such that (3.9) and (3.10) hold. If $\delta_1 > \delta_2$, then

$$\overline{\lim_n} \|x_n - u\| > \epsilon + \delta_1 > \epsilon + \delta_2.$$

If $\delta_1 \le \delta_2$, then from (3.10), we deduce that

$$\|x - y\| < \epsilon + \delta_1 \text{ implies } \|f(x) - f(y)\| < \epsilon$$

for all $x, y \in C$. Thus, we have

$$\overline{\lim_n} \|x_n - u\| > \epsilon + \delta, \tag{3.11}$$

and

$$\|x - y\| < \epsilon + \delta \text{ implies } \|f(x) - f(y)\| < \epsilon \text{ for all } x, y \in C. \tag{3.12}$$

Since $u_n \to u$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n - u\| < \delta, \quad \forall n \ge n_0. \tag{3.13}$$

We now consider two possible cases.

Case 1. There exists $n_1 \ge n_0$ such that

$$\|x_{n_1} - u\| \le \epsilon + \delta.$$

By (3.12) and (3.13), we get

$$\begin{aligned}
\|x_{n_1+1} - u\| &\le \|x_{n_1+1} - u_{n_1+1}\| + \|u_{n_1+1} - u\| \\
&= \|proj_{C_{n_1+1}} f(x_{n_1}) - proj_{C_{n_1+1}} f(u)\| + \|u_{n_1+1} - u\| \\
&\le \|f(x_{n_1}) - f(u)\| + \|u_{n_1+1} - u\| \\
&\le \epsilon + \delta.
\end{aligned}$$

By the induction, we can obtain

$$\|x_{n_1+m} - u\| \le \epsilon + \delta$$

for all $m \ge 1$, which implies that

$$\overline{\lim_n} \|x_n - u\| \le \epsilon + \delta,$$

and this contradicts to (3.11). Therefore, we conclude that $\|x_n - u\| \to 0$ as $n \to \infty$.

Case 2. $\|x_n - u\| > \epsilon + \delta$ for all $n \ge n_0$.

We shall prove that case 2 is impossible. Suppose case 2 holds true. By Lemma 2.4, there exists $r \in (0, 1)$ such that

$$\|f(x_n) - f(u)\| \le r\|x_n - u\|, \quad \forall n \ge n_0.$$

Thus, we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \|proj_{C_{n+1}}f(x_n)-proj_{C_{n+1}}f(u)\|\\ &\le \|f(x_n)-f(u)\|\\ &\le r\|x_n-u\|\end{aligned}$$

for every $n \ge n_0$. It follows that

$$\begin{aligned}\overline{\lim_n}\|x_{n+1}-u\| &= \overline{\lim_n}\|x_{n+1}-u_{n+1}\|\\ &\le r\overline{\lim_n}\|x_n-u\|\\ &< \overline{\lim_n}\|x_n-u\|,\end{aligned}$$

which gives a contradiction. Hence, we obtain

$$\lim_{n\to\infty}\|x_n-u\|=0.$$

Finally, we prove $u \in Fix(T)$. Observe that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|x_n-u\|+\|u-u_{n+1}\|+\|u_{n+1}-x_{n+1}\|\\ &= \|x_n-u\|+\|u-u_{n+1}\|+\|proj_{C_{n+1}}f(x_n)-proj_{C_{n+1}}f(u)\|\\ &\le \|x_n-u\|+\|u-u_{n+1}\|+\|f(x_n)-f(u)\|.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.14}$$

From $x_{n+1} \in C_{n+1}$, we have

$$\|(1-\alpha_n)x_n+\alpha_n T^n y_n - x_{n+1}\|^2 \le \|x_n-x_{n+1}\|^2+\theta_n.$$

This together with (3.14) implies that

$$\lim_{n\to\infty}\|T^n y_n - x_n\|=0.$$

Note that

$$\begin{aligned}\|x_n-T^n x_n\| &\le \|x_n-T^n y_n\|+\|T^n y_n-T^n x_n\|\\ &\le \|x_n-T^n y_n\|+L\|x_n-y_n\|\\ &\le \|x_n-T^n y_n\|+L\beta_n\|x_n-T^n x_n\|.\end{aligned}$$

It follows that

$$\|x_n-T^n x_n\| \le \frac{1}{1-\beta_n L}\|x_n-T^n y_n\| \le \frac{1}{1-aL}\|x_n-T^n y_n\| \to 0. \tag{3.15}$$

Observe that

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|+\|T^{n+1}x_{n+1}-T^{n+1}x_n\|+\|T^{n+1}x_n-Tx_n\|\\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|+L\|x_{n+1}-x_n\|+L\|T^n x_n-x_n\|\\ &= (1+L)\|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|+L\|T^n x_n-x_n\|.\end{aligned} \tag{3.16}$$

By Lemma 2.1, (3.14), (3.15) and (3.16), we have $u \in Fix(T)$. Since $x_{n+1} = proj_{C_{n+1}}f(x_n)$, we have

$$\langle f(x_n)-x_{n+1}, x_{n+1}-y\rangle \ge 0, \quad \forall y \in C_{n+1}.$$

Since $Fix(T) \subset C_{n+1}$, we get

$$\langle f(x_n)-x_{n+1}, x_{n+1}-y\rangle \ge 0, \quad \forall y \in Fix(T).$$

We have from $x_n \to u \in Fix(T)$ that

$$\langle f(u)-u, u-y\rangle \ge 0, \quad \forall y \in Fix(T).$$

Thus, $u = proj_{Fix(T)}f(u) = x^\dagger$. This completes the proof. □

Acknowledgment

The corresponding author was partially supported by the grant from Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Taiwan.

References

[1] L. C. Ceng, A. Petrusel, J. C. Yao, *Strong convergence of modified implicit iterative algorithms with perturbed mappings for continuous pseudocontractive mappings*, Appl. Math. Comput., **209** (2009), 162–176. 1

[2] C. E. Chidume, *Strong convergence theorems for fixed points of asymptotically pseudocontractive semi-groups*, J. Math. Anal. Appl., **296** (2004), 410–421. 1

[3] C. E. Chidume, M. Abbas, B. Ali, *Convergence of the Mann iteration algorithm for a class of pseudocontractive mappings*, Appl. Math. Comput., **194** (2007), 1–6. 1

[4] R. Dewangan, B. S. Thakur, M. Postolache, *Strong convergence of asymptotically pseudocontractive semigroup by viscosity iteration*, Appl. Math. Comput., **248** (2014), 160–168. 1

[5] F. Gu, *Approximate fixed point sequence for a finite family of asymptotically pseudocontractive mappings*, Comput. Math. Appl., **53** (2007), 1792–1799. 1

[6] J. K. Kim, Y. M. Nam, *Modified Ishikawa iterative sequences with errors for asymptotically set-valued pseudocontractive mappings in Banach spaces*, Bull. Korean Math. Soc., **43** (2006), 847–860. 1

[7] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329. 2, 2.3

[8] C. Moore, B. V. C. Nnoli, *Strong convergence of averaged approximants for Lipschitz pseudocontractive maps*, J. Math. Anal. Appl., **260** (2001), 269–278. 1

[9] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Advances in Math., **3** (1969), 510–585. 2

[10] E. U. Ofoedu, *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space*, J. Math. Anal. Appl., **321** (2006), 722–728. 1

[11] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **158** (1991), 407–413. 1

[12] T. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl., **325** (2007), 342–352. 2.4, 2.5

[13] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory, **40** (1984), 301–309. 2, 2.2

[14] A. Udomene, *Path convergence, approximation of fixed points and variational solutions of Lipschitz pseudocontractions in Banach spaces*, Nonlinear Anal., **67** (2007), 2403–2414. 1

[15] Y. Yao, R. P. Agarwal, M. Postolache, Y. C. Liou, *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 1

[16] Y. Yao, W. Jigang, Y. C. Liou, *Regularized methods for the split feasibility problem*, Abstr. Appl. Anal., **2012** (2012), 15 pages. 1

[17] Y. Yao, Y. C. Liou, J. C. Yao, *Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction*, Fixed Point Theory Appl., **2015** (2015), 19 pages. 1

[18] Y. Yao, G. Marino, H. K. Xu, Y. C. Liou, *Construction of minimum-norm fixed points of pseudocontractions in Hilbert spaces*, J. Inequal. Appl., **2014** (2014), 14 pages. 1

[19] Y. Yao, M. Postolache, S. M. Kang, *Strong convergence of approximated iterations for asymptotically pseudocontractive mappings*, Fixed Point Theory Appl., **2014** (2014), 13 pages. 1

[20] Y. Yao, M. Postolache, Y. C. Liou, *Coupling Ishikawa algorithms with hybrid techniques for pseudocontractive mappings*, Fixed Point Theory Appl., **2013** (2013), 8 pages. 1

[21] Y. Yao, M. Postolache, Y. C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 1

[22] H. Zegeye, M. Robdera, B. Choudhary, *Convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense*, Comput. Math. Appl., **62** (2011), 326–332. 1

[23] H. Zegeye, N. Shahzad, *An algorithm for a common fixed point of a family of pseudocontractive mappings*, Fixed Point Theory Appl., **2013** (2013), 14 pages. 1

[24] H. Zegeye, N. Shahzad, T. Mekonen, *Viscosity approximation methods for pseudocontractive mappings in Banach spaces*, Appl. Math. Comput., **185** (2007), 538–546. 1

[25] H. Zhou, *Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70** (2009), 3140–3145. 1, 2.1