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Splitting methods for monotone operators and bifunctions

Yan Hao[a,b], Zhisong Liu[a,b], Sun Young Cho[c,*]

[a]School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan, Zhejiang 316022, China.
[b]Key Laboratory of Oceanographic Big Data Mining and Application of Zhejiang Province, Zhoushan, Zhejiang 316022, China.
[c]School of Mathematics, Gyeongsang National University, Jinju 660-701, Korea.

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Abstract

The purpose of this article is to investigate fixed point problems of a nonexpansive mapping, solutions of quasi variational inclusion problem, and solutions of a generalized equilibrium problem based on a splitting method. Our convergence theorems are established under mild restrictions imposed on the control sequences. The main results improve and extend the recent corresponding results.

Keywords: Variational inclusion, monotone operator, operator equation, bifunction, convergence.
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1. Introduction and Preliminaries

Monotone variational inequalities have played a significant and fundamental role in the development of new and innovative techniques for solving complex and complicated problems arising in pure and applied sciences. Variational inequalities have recently been extended and generalized in various directions using novel and innovative techniques; see, for example, [1, 4, 7, 10, 11, 19–22] and the references therein. A useful and important generalization is called the general variational inclusion involving the sum of two nonlinear operators A and B. Recently, much attention has been given to develop iterative algorithms for solving the variational inclusions. Resolvent methods and its variants forms including the resolvent equations represent important tools for finding the approximate solution of variational inclusions. The main idea in this technique is to establish the equivalence between the variational inclusions and the fixed-point

*Corresponding author
Email addresses: `zjhaoyan@aliyun.com` (Yan Hao), `ooly61@hotmail.com` (Sun Young Cho)

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problem by using the concept of resolvent operator. It is known that such techniques require an evaluation of the resolvent operator of the type $(I - r(A + B))^{-1}$. The main difficulty with such problems is that the resolvent operator may be hard to invert. This difficulty has been overcome by using the resolvent operators $(I - rA)^{-1}$ and $(I - rB)^{-1}$ separately rather than $(I - r(A + B))^{-1}$. Such a technique is called the splitting method. These methods for solving variational inclusions have been studied extensively, see, for example, [1, 3, 6, 9, 14–17, 20, 24] and the references therein.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we also call A is an α-strongly monotone mapping. A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we also call A is an α-inverse-strongly monotone mapping. We remark here that every α-inverse-strongly monotone mapping is strongly monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and let $M : C \to H$ be a monotone operator. We consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle y - x, Mx \rangle \geq 0, \forall y \in C. \tag{1.1}$$

In this paper, the set of such an $x \in C$ is denoted by $Sol(F, M)$.

If $M = 0$, then generalized equilibrium problem (1.1) is reduced to the following equilibrium problem in the terminology of Blum and Oettli [4]:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \forall y \in C. \tag{1.2}$$

In this paper, the set of such an $x \in C$ is denoted by $Sol(F)$.

If $F = 0$, then generalized equilibrium problem (1.1) is reduced to the following variational inequality:

$$\text{Find } x \in C \text{ such that } \langle y - x, Mx \rangle \geq 0, \forall y \in C. \tag{1.3}$$

In this paper, the set of such an $x \in C$ is denoted by $VI(C, A)$.

To study the equilibrium problems, we assume that F satisfies the following conditions:

(R1) $F(x, x) = 0$ for all $x \in C$;
(R2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(R3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(R4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

The equilibrium problems provide us a unified framework to study many problems arise in engineering areas. The equilibrium problems are general which include saddle point problems, variational inequality problems and complementarity problem as special cases. Recently, convergence theorems of solutions to the equilibrium problems were established; see [2, 8, 12, 13] and the references therein.

Recall that a set-valued mapping $B : H \to 2^H$ is said to be monotone iff for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $B : H \to 2^H$ is maximal iff the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal iff, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(B)$ implies $f \in Bx$. Let A be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\},$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ iff $\langle Av, u - v\rangle \geq 0$ for all $u \in C$; see [22] and the references therein.

Let I denotes the identity operator on H and $B : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r^B : H \to H$ by $J_r^B = (I + rB)^{-1}$. It is called the resolvent of B.

Let S be a mapping on C. $Fix(S)$ stands for the fixed point set of S. Recall that S is said to be firmly nonexpansive iff

$$\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle, \quad \forall x, y \in C.$$

S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Let I denote the identity operator on H and $B : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r^B : H \to H$ by $J_r^B = (I + rB)^{-1}$. It is called the resolvent of B. We know that $B^{-1}0 = Fix(J_r^B)$ for all $r > 0$ and J_r^B is firmly nonexpansive.

Moreover, we need the following lemmas to prove our main results.

Lemma 1.1 ([2])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping and let $B : H \rightrightarrows H$ be a maximal monotone operator. Then $Fix(J_r(I - rA)) = (A + B)^{-1}(0)$.*

Lemma 1.2 ([18])**.** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following relation:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 1.3 ([4])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (R1)-(R4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that $rF(z, y) + \langle y - z, z - x\rangle \geq 0$, $\forall y \in C$. Further, define*

$$T_r x = \{z \in C : rF(z, y) + \langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued firmly nonexpansive;*

(b) *$Fix(T_r) = Sol(F)$ is closed and convex.*

Lemma 1.4 ([5])**.** *Let C be a nonempty closed and convex subset of H and $S : C \to C$ a nonexpansive mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$, and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, then $x = Sx$.*

Lemma 1.5 ([23]). *Let $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$, and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \quad \limsup_{n\to\infty} \|y_n\| \leq d,$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d,$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $M : C \to H$ a κ-inverse-strongly monotone mapping and $B : H \rightrightarrows H$ a maximal monotone mapping such that its domain in C. Let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S) \cap Sol(F, M) \cap (A + B)^{-1}(0)$ is nonempty. Let $\{r_n\}$ and $\{t_n\}$ be positive real number sequences. Let $\{\alpha_n\}$ be a real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} t_n F(z_n, z) + t_n\langle Mx_n, z - z_n\rangle + \langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)J_{r_n}\big(z_n - r_n Az_n + e_n\big), \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the control sequences satisfy the following restrictions: $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, $0 < t \leq t_n \leq t' < 2\kappa$, $0 < r \leq r_n \leq r' < 2\alpha$, where α, α', t, t', r and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap Sol(F, M) \cap (A + B)^{-1}(0)$.

Proof. From the restrictions on $\{r_n\}$ and $\{t_n\}$, we have

$$\|(I - r_n A)x - (I - r_n A)y\|^2 \leq \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2,$$

and

$$\|(I - t_n M)x - (I - t_n M)y\|^2 \leq \|x - y\|^2 - t_n(2\kappa - t_n)\|Mx - My\|^2.$$

Let $p \in Fix(S) \cap Sol(F, M) \cap (A + B)^{-1}(0)$ be fixed arbitrarily. It follows from (1.1) and (1.3) that $p = T_{t_n}(p - t_n Mp) = J_{r_n}(p - r_n Ap)$. Putting $y_n = J_{r_n}\big(z_n - r_n Az_n + e_n\big)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|Sx_n - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|J_{r_n}\big(z_n - r_n Az_n + e_n\big) - J_{r_n}\big(p - r_n Ap\big)\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|\big(z_n - r_n Az_n + e_n\big) - \big(p - r_n Ap\big)\| \\ &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|z_n - p\| + (1-\alpha_n)\|e_n\| \\ &\leq \|x_n - p\| + (1-\alpha_n)\|T_{t_n}(x_n - t_n Mx_n) - T_{t_n}(p - t_n Mp)\| + \|e_n\| \\ &\leq \|x_n - p\| + e_n. \end{aligned}$$

This implies from Lemma 1.2 that the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence, we have $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Since A is inverse-strongly monotone, we find that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|(z_n - r_n Az_n) - (p - r_n Ap) + e_n\|^2 \\ &\leq \|(z_n - p) - r_n(Az_n - Ap)\|^2 + \|e_n\|(\|e_n\| + 2\|e_n\|\|z_n - p\|) \\ &\leq \|z_n - p\|^2 - r_n(2\alpha - r_n)\|Az_n - Ap\|^2 + \|e_n\|(\|e_n\| + 2\|e_n\|\|z_n - p\|) \\ &\leq \|x_n - p\|^2 - t_n(2\kappa - t_n)\|Mx_n - Mp\|^2 - r_n(2\alpha - r_n)\|Az_n - Ap\|^2 \\ &\quad + \|e_n\|(\|e_n\| + 2\|e_n\|\|z_n - p\|). \end{aligned}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\leq \|x_n-p\|^2-(1-\alpha_n)t_n(2\kappa-t_n)\|Mx_n-Mp\|^2\\ &\quad -(1-\alpha_n)r_n(2\alpha-r_n)\|Az_n-Ap\|^2+\|e_n\|(\|e_n\|+2\|e_n\|\|z_n-p\|)).\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)t_n(2\kappa-t_n)\|Mx_n-Mp\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2-(1-\alpha_n)r_n(2\alpha-r_n)\|Az_n-Ap\|^2\\ +\|e_n\|(\|e_n\|+2\|e_n\|\|z_n-p\|)),\end{aligned}$$

and

$$\begin{aligned}(1-\alpha_n)r_n(2\alpha-r_n)\|Az_n-Ap\|^2 \leq \|x_n-p\|^2-(1-\alpha_n)t_n(2\kappa-t_n)\|Mx_n-Mp\|^2\\ -\|x_{n+1}-p\|^2+\|e_n\|(\|e_n\|+2\|e_n\|\|z_n-p\|)).\end{aligned}$$

Using the restrictions on $\{r_n\}$ and $\{t_n\}$, we find

$$\lim_{n\to\infty}\|Az_n-Ap\|=\lim_{n\to\infty}\|Mx_n-Mp\|=0. \tag{2.1}$$

Since J_{r_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|y_n-p\|^2 &\leq \langle\big(z_n-r_nAz_n+e_n\big)-\big(p-r_nAp\big),y_n-p\rangle\\ &=\frac{1}{2}\big(\|\big(z_n-r_nAz_n+e_n\big)-\big(p-r_nAp\big)\|^2+\|y_n-p\|^2\\ &\quad -\|\big((z_n-r_nAz_n+e_n)-\big(p-r_nAp\big)\big)-(y_n-p)\|^2\\ &\leq \frac{1}{2}\big(\|z_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)+\|y_n-p\|^2\\ &\quad -\|z_n-y_n-r_n(Az_n-Ap)+e_n\|^2\big)\\ &\leq \frac{1}{2}\big(\|z_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)+\|y_n-p\|^2-\|y_n-z_n\|^2\\ &\quad -\|r_n(Az_n-Ap)-e_n\|^2+2\|z_n-y_n\|\|r_n(Az_n-Ap)-e_n\|\big),\end{aligned}$$

that is,

$$\begin{aligned}\|y_n-p\|^2 \leq \|z_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)-\|z_n-y_n\|^2\\ +2r_n\|z_n-y_n\|\|Az_n-Ap\|+2\|z_n-y_n\|\|e_n\|.\end{aligned} \tag{2.2}$$

It follows from (2.2) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)\\ &\quad -(1-\alpha_n)\|z_n-y_n\|^2+2r_n(1-\alpha_n)\|z_n-y_n\|\|Az_n-Ap\|+2\|z_n-y_n\|\|e_n\|\\ &\leq \|x_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)\\ &\quad -(1-\alpha_n)\|z_n-y_n\|^2+2r_n(1-\alpha_n)\|z_n-y_n\|\|Az_n-Ap\|+2\|z_n-y_n\|\|e_n\|,\end{aligned}$$

that is,

$$\begin{aligned}(1-\alpha_n)\|z_n-y_n\|^2 \leq \|x_n-p\|^2+\|e_n\|(\|e_n\|+2\|z_n-p\|)\\ -\|x_{n+1}-p\|^2+2r_n\|z_n-y_n\|\|Az_n-Ap\|+2\|z_n-y_n\|\|e_n\|.\end{aligned}$$

Using (2.1), one finds that

$$\lim_{n\to\infty}\|y_n-z_n\|=0. \tag{2.3}$$

On the other hand, one has

$$\begin{aligned}
\|z_n - p\|^2 &\leq \langle (x_n - t_n M x_n) - (p - t_n A p), z_n - p\rangle \\
&= \frac{1}{2}\big(\|(x_n - t_n M x_n) - (p - t_n A p)\|^2 + \|z_n - p\|^2 \\
&\quad - \|\big((x_n - t_n M x_n) - (p - r_n A p)\big) - (z_n - p)\|^2 \\
&\leq \frac{1}{2}\big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n - t_n(Mx_n - Mp)\|^2\big) \\
&\leq \frac{1}{2}\big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 \\
&\quad - \|t_n(Mx_n - Mp)\|^2 + 2t_n\|x_n - z_n\|\|Mx_n - Mp\|\big),
\end{aligned}$$

that is,

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2t_n\|x_n - z_n\|\|Mx_n - Mp\|.$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|z_n - p\|) \\
&\quad + 2r_n(1 - \alpha_n)\|z_n - y_n\|\|Az_n - Ap\| + 2\|z_n - y_n\|\|e_n\| \\
&\leq \|x_n - p\|^2 - (1 - \alpha_n)\|x_n - z_n\|^2 + 2t_n\|x_n - z_n\|\|Mx_n - Mp\| \\
&\quad + \|e_n\|(\|e_n\| + 2\|z_n - p\|) + 2r_n\|z_n - y_n\|\|Az_n - Ap\| + 2\|z_n - y_n\|\|e_n\|.
\end{aligned}$$

This in turn implies from (2.1) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{2.4}$$

Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ. It follows that the subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to ξ. Notice that

$$\frac{z_n - y_n + e_n}{r_n} - Az_n \in By_n.$$

Let $\mu \in B\nu$. Since B is monotone, we find that

$$\left\langle \frac{z_n - y_n + e_n}{r_n} - Az_n - \mu, y_n - \nu \right\rangle \geq 0.$$

It follows from (2.3) that $\langle -A\xi - \mu, \xi - \nu\rangle \geq 0$. This implies that $-A\xi \in B\bar{x}$, that is, $\xi \in (A+B)^{-1}(0)$.

Now, we are in a position to show that $\xi \in Fix(S)$. Since $\lim_{n\to\infty}\|x_n - p\|$ exists, we put $\lim_{n\to\infty}\|x_n - p\| = d > 0$. It follows that $\lim_{n\to\infty}\|(1-\alpha_n)(y_n - p) + \alpha_n(Sx_n - p)\| = d$. Notice both $\limsup_{n\to\infty}\|Sx_n - p\| \leq d$ and $\limsup_{n\to\infty}\|y_n - p\| \leq d$. It follows from Lemma 1.5 that

$$\lim_{n\to\infty}\|Sx_n - y_n\| = 0. \tag{2.5}$$

In view of (2.3), (2.4), and (2.5), we find that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. Using Lemma 1.4, we have $\xi \in Fix(S)$.

Now, we are in a position to show that $\xi \in Sol(F, M)$. Notice that

$$t_n F(z_n, z) + t_n\langle Mx_n, z - z_n\rangle + \langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C.$$

By use of condition (R2), we see that

$$\langle Mx_n, z - z_n\rangle + \langle z - z_n, \frac{z_n - x_n}{t_n}\rangle \geq F(z, z_n), \quad \forall z \in C. \tag{2.6}$$

For t with $0 < t \leq 1$, and $z \in C$, let $z_t = tz + (1-t)\xi$. Since $y \in C$, and $\xi \in C$, we have $z_t \in C$. Using (2.6), we find that

$$\begin{aligned}\langle z_t - z_n, Mz_t\rangle &\geq \langle z_t - z_n, Mz_t\rangle - \langle Mx_n, z_t - z_n\rangle - \langle z_t - z_n, \frac{z_n - x_n}{t_n}\rangle + F(z_t, z_n)\\ &\geq \langle z_t - z_n, Mz_t - Mz_n\rangle + \langle z_t - z_n, Mz_n - Mx_n\rangle - \langle z_t - z_n, \frac{z_n - x_n}{t_n}\rangle + F(z_t, z_n)\\ &\geq \langle z_t - z_n, Mz_n - Mx_n\rangle - \langle z_t - z_n, \frac{z_n - x_n}{t_n}\rangle + F(z_t, z_n).\end{aligned}$$

Since $\{z_{n_i}\}$ converges weakly to ξ, we find that

$$\langle z_t - \xi, Mz_t\rangle \geq F(z_t, \xi),$$

which implies that

$$\begin{aligned}0 =&F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, \xi)\\ &\leq tF(z_t, z) + (1-t)\langle z_t - \xi, Mz_t\rangle\\ &= tF(z_t, z) + (1-t)t\langle z - \xi, Mz_t\rangle,\end{aligned}$$

that is, $0 \leq F(z_t, z) + (1-t)\langle z - \xi, Mz_t\rangle$. Letting $t \to 0$, we have $0 \leq F(\xi, z) + \langle z - \xi, M\xi\rangle$. This implies that $\xi \in Sol(F, M)$.

Finally, we show that $\{x_n\}$ weakly converges to ξ. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ξ', where $\xi' \neq \xi$. In the same way, we can show that $\xi' \in (A+B)^{-1}(0) \cap Sol(F, M) \cap Fix(S)$. Since space H has the Opial's condition, we obtain that

$$\begin{aligned}d = &\liminf_{i\to\infty} \|x_{n_i} - \xi\| < \liminf_{i\to\infty} \|x_{n_i} - \xi'\|\\ &= \liminf_{j\to\infty} \|x_j - \xi'\| < \liminf_{j\to\infty} \|x_j - \xi\| = d.\end{aligned}$$

This is a contradiction. Hence $\xi = \xi'$. This proves that $\{x_n\}$ converges weakly to $\xi \in Fix(S) \cap EP(F, M) \cap (A+B)^{-1}(0)$. This completes the proof. □

From Theorem 2.1, the following results are not hard to derive.

Corollary 2.2. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping, and $B : H \rightrightarrows H$ a maximal monotone mapping such that its domain in C. Let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S) \cap Sol(F) \cap (A+B)^{-1}(0)$ is nonempty. Let $\{r_n\}$ and $\{t_n\}$ be positive real number sequences. Let $\{\alpha_n\}$ be a real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} t_n F(z_n, z) + \langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C,\\ x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)J_{r_n}\big(z_n - r_n Az_n + e_n\big), \quad n \geq 1,\end{cases}$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the control sequences satisfy the following restrictions: $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, $0 < t \leq t_n$, $0 < r \leq r_n \leq r' < 2\alpha$, where α, α', t, r, and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap Sol(F) \cap (A+B)^{-1}(0)$.

Corollary 2.3. *Let C be a nonempty closed convex subset of H, $A : C \to H$ be an α-inverse-strongly monotone mapping, and $B : H \rightrightarrows H$ a maximal monotone mapping such that its domain in C. Let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S) \cap (A+B)^{-1}(0)$ is nonempty. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$ be a real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$x_{n+1} = \alpha_n Sx_n + (1-\alpha_n)J_{r_n}\big(z_n - r_n Az_n + e_n\big), \quad n \geq 1,$$

where $\{e_n\}$ *is a bounded sequence in* H *such that* $\sum_{n=1}^{\infty}\|e_n\| < \infty$*. Assume that the control sequences satisfy the following restrictions:* $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, $0 < r \leq r_n \leq r' < 2\alpha$*, where* α, α', r *and* r' *are real constants. Then* $\{x_n\}$ *converges weakly to some point in* $Fix(S) \cap (A+B)^{-1}(0)$.

Let i_C be a function defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

It is easy to see that i_C is a proper lower and semicontinuous convex function on H, and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent $J_r := (I + r\partial i_C)^{-1}$ of the subdifferential operator ∂i_C. Letting $x = J_r y$, we find that $y \in x + r\partial i_C x \iff x = Proj_C y$, where $N_C x := \{e \in H : \langle e, v - x\rangle, \forall v \in C\}$.

Putting $B = \partial i_C$ and $M = 0$ in Theorems 2.1, we find the following results immediately.

Corollary 2.4. *Let* C *be a nonempty closed convex subset of* H *and let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies (A1)–(A4). Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping, and* $B : H \rightrightarrows H$ *a maximal monotone mapping such that its domain in* C*. Let* $S : C \to C$ *be a nonexpansive mapping. Assume that* $Fix(S) \cap Sol(F) \cap VI(C, A)$ *is nonempty. Let* $\{r_n\}$ *and* $\{t_n\}$ *be positive real number sequences. Let* $\{\alpha_n\}$ *be a real number sequences in* $(0, 1)$*. Let* $\{x_n\}$ *be a sequence generated in the following process:* $x_1 \in C$ *and*

$$\begin{cases} t_n F(z_n, z) + \langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n S x_n + (1-\alpha_n) P_C\big(z_n - r_n A z_n + e_n\big), \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ *is a bounded sequence in* H *such that* $\sum_{n=1}^{\infty}\|e_n\| < \infty$*. Assume that the control sequences satisfy the following restrictions:* $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, $0 < t \leq t_n$, $0 < r \leq r_n \leq r' < 2\alpha$*, where* α, α', t, r *and* r' *are real constants. Then* $\{x_n\}$ *converges weakly to some point in* $Fix(S) \cap Sol(F) \cap VI(C, A)$.

Now, we are in a position to consider the problem of finding minimizers of proper lower semicontinuous convex functions. For a proper lower semicontinuous convex function $g : H \to (-\infty, \infty]$, the subdifferential mapping ∂g of g is defined by $\partial g(x) = \{x^* \in H : g(x) + \langle y - x, x^*\rangle \leq g(y), \forall y \in H\}$, $\forall x \in H$. Rockafellar [21] proved that ∂g is a maximal monotone operator. It is easy to verify that $0 \in \partial g(v)$ if and only if $g(v) = \min_{x\in H} g(x)$.

Theorem 2.5. *Let* $g : H \to (-\infty, \infty]$ *be a proper convex and lower semicontinuous function. Let* $\{r_n\}$ *be a positive real number sequence. Let* $\{\alpha_n\}$, $\{\beta_n\}$*, and* $\{\gamma_n\}$ *be real number sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$*. Let* $\{x_n\}$ *be a sequence generated in the following process:* $x_1 \in C$ *and* $x_{n+1} = \alpha_n S x_n + (1-\alpha_n) \arg\min_{z\in H}\{g(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}$, $n \geq 1$*, where* $\{e_n\}$ *is a bounded sequence in* H *such that* $\sum_{n=1}^{\infty}\|e_n\| < \infty$ *and* $\{f_n\}$ *is bounded sequence in* C*. Assume that the control sequences satisfy restrictions:* $0 < \beta \leq \beta_n \leq \beta' < 1$, $\sum_{n=1}^{\infty}\gamma_n < \infty$*, and* $0 < r \leq r_n \leq r' < 2\alpha$*, where* β, β', r *and* r' *are real constants. Then* $\{x_n\}$ *converges weakly to some point in* $(\partial g)^{-1}(0)$.

Proof. Since $g : H \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function, we see that subdifferential ∂g of g is maximal monotone. Putting $F(x, y) = M = A = 0$, $t_n = 1$, we have $y_n = J_{r_n}(x_n + e_n)$. It follows that $y_n = \arg\min_{z\in H}\{g(z) + \frac{\|z - x_n - e_n\|^2}{2r_n}\}$ is equivalent to $0 \in \partial g(y_n) + \frac{1}{r_n}(y_n - x_n - e_n)$. It follows that $x_n + e_n \in y_n + r_n\partial g(y_n)$. By use of Theorem 2.1, we find the desired conclusion immediately. □

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