Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3948–3961
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Sliding bifurcation analysis and global dynamics for a Filippov predator-prey system

Xuewen Tan[a,b,*], Wenjie Qin[c], Xinzhi Liu[d], Jin Yang[e], Shaoping Jiang[b]

[a] *College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, P. R. China.*
[b] *School of Mathematics and Computer Science, Yunnan Minzu University, Kunming 650500, P. R. China.*
[c] *College of Science, China Three Gorges University, Yichang 443002, P. R. China.*
[d] *Department of Applied Mathematics, University of Waterloo, Ontario N2L 3G1, Canada.*
[e] *Department of Mathematics, Chongqing Jiaotong University, Chongqing 400074, P. R. China.*

Communicated by R. Saadati

Abstract

This paper studies a Filippov predator-prey system, where chemical control strategies are proposed and analyzed. Initially, the exact sliding segment and its domains are addressed. Then the existence and stability of the regular, virtual, pseudo-equilibria and tangent points are discussed. It shows that two regular equilibria and a pseudo-equilibrium can coexist. By employing theoretical and numerical techniques several kinds of bifurcations are investigated, such as sliding bifurcations related to the boundary node (focus) bifurcations, touching bifurcations, sliding crossing bifurcation and buckling bifurcations (or sliding switching). Furthermore, it makes comparison of the obtained results with previous studies for the Filippov predator-prey system without control strategies. Some biological implications of our results with respect to pest control are also given. ©2016 All rights reserved.

Keywords: Filippov predator-prey system, control strategy, economic threshold, sliding bifurcation analysis.
2010 MSC: 34H20, 34K20.

1. Introduction

Non-smooth Filippov system models have been widely used in many fields of science and engineering in recent years [3, 7–9, 12, 13, 20–22, 25, 27, 29–33]. Among other things, many types of codimension one sliding bifurcations are extensively discussed in generic planar Filippov systems [4, 5, 10, 11, 19]. It is now

*Corresponding author
Email address: tanxw520@163.com (Xuewen Tan)

Received 2016-04-06

recognized that the theory of Filippov systems provides a natural framework for the mathematical modeling of many real-world phenomena.

In the 1930's, Gause carried out three experiments in order to provide evidence for stable population limit cycles in a microcosm experiment [15, 16]. Recently the classical Lotka-Volterra model was extended by using a piecewise saturating function to replace the linear consumption rate for considering the observed experimental results theoretically [17, 24]. For simplicity, denote $H(Z) = x - ET$ with $Z = (x, y)^{\mathrm{T}} \in R_+^2$, where ET describes the critical prey population threshold, and the parameter ε can be defined as follows

$$\varepsilon = \begin{cases} 0, & H(Z) = x - ET < 0, \\ 1, & H(Z) = x - ET > 0. \end{cases} \tag{1.1}$$

Therefore, if the density of the prey population falls below the threshold ET, that is, $H(Z) < 0$, then $\varepsilon = 0$, which indicates that the prey may avoid the predator via a habitat shift by moving to the refuge and the density of the predator will decrease [2]; if the density of the prey population increases and exceeds the threshold ET, that is, $H(Z) > 0$, then $\varepsilon = 1$, which indicates that the prey population may re-appear and once again become accessible to predators [2, 16, 17, 24].

According to the above definition, the extended classical Lotka-Volterra model with a piecewise saturating function can be defined as the following Filippov system

$$\begin{cases} \dfrac{dx(t)}{dt} = & rx(t) - \dfrac{\varepsilon bx(t)y(t)}{1 + bhx(t)}, \\ \dfrac{dy(t)}{dt} = & \dfrac{\varepsilon kbx(t)y(t)}{1 + bhx(t)} - \delta y(t), \end{cases} \tag{1.2}$$

where $x(t)$ and $y(t)$ are the densities of the prey and predator populations at time t, respectively. r is the intrinsic growth rate of the prey, b describes the search rate of a predator and h is the handling time, k denotes the conversion rate, δ is the death rate of the predator. System (1.2) has been firstly investigated by Gause et al. [16], then Křivan [17] has showed that system (1.2) could present six different qualitative dynamics, latter Tang et al. [24] provided the exact conditions for the above six different qualitative dynamics.

Note that the solutions of system (1.2) may approach infinity under some conditions, which is unrealistic. Therefore, Yang et al. [32] considered the effect of the carrying capacity on the prey population in system (1.2), which can be described as follows

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)\left[1 - \dfrac{x(t)}{K}\right] - \dfrac{\varepsilon bx(t)y(t)}{1 + bhx(t)}, \\ \dfrac{dy(t)}{dt} = \dfrac{\varepsilon kbx(t)y(t)}{1 + bhx(t)} - \delta y(t), \end{cases} \tag{1.3}$$

where K denotes the carrying capacity of the prey, and all other parameters are the same as those in model (1.2). It should be pointed out that the global dynamics of system (1.3) is not as complex as system (1.2) presented. The effect of carrying capacity can stabilize the non-smooth Gause predator-prey system (1.3) and cause the nonexistence of the infinity singularity point [32].

It is interesting to note that none of the models proposed above incorporate the control strategies into the non-smooth Gause model [16, 17, 24, 32]. In reality, however, if the variable $x(t)$ in system (1.3) represents the pest population, then the pest population may cause harms to crops once the density of the pest reaches and exceeds the economic threshold [23, 26]. Therefore, control strategies such as chemical control, biological control or their combinations should be implemented when the density of the pest reaches the economic threshold. Thus, in this paper, a novel Filippov predator-prey system with control strategies is proposed, which can be defined as follows

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)\left[1 - \dfrac{x(t)}{K}\right] - \dfrac{bx(t)y(t)}{1 + bhx(t)} - \varepsilon q_1 x(t), \\ \dfrac{dy(t)}{dt} = \dfrac{\varepsilon kbx(t)y(t)}{1 + bhx(t)} - \delta y(t) - \varepsilon q_2 y(t), \end{cases} \tag{1.4}$$

where q_1 and q_2 describes the killing rates of the prey and predator due to control tactics, and $0 < q_2 < q_1 < 1$. If the density of the prey population falls below the threshold ET, that is, $H(Z) < 0$, then $\varepsilon = 0$, which indicates that the prey do not cause any harms to crops and no actions should be taken; if the density of the prey population increases and exceeds the threshold ET, that is, $H(Z) > 0$, then $\varepsilon = 1$, which indicates that the prey may cause harms to crops and the control tactics should be implemented in order to control the density of prey under the threshold ET.

To investigate the dynamics of system (1.4), some useful definitions and lemmas about the non-smooth Filippov dynamic systems will be presented in Section 2. In Section 3, the dynamics of the subsystems are addressed, then the sliding domains are provided. Furthermore, the conditions for the existence of several types of equilibria are given and the relations between the existence of regular equilibria and a pseudo-equilibrium are also discussed. Moreover, the bifurcation set of the equilibria with respect to key parameters are addressed, and then the sliding bifurcations related to boundary node (focus) bifurcations, touching bifurcations, sliding crossing bifurcation and buckling bifurcations (or sliding switching) have been investigated by employing theoretical and numerical techniques. Finally, the main results obtained for model (1.4) with those obtained for model (1.3) are compared and the biological implications of the results are discussed.

2. Preliminaries

Let

$$F_{S_1}(Z) = \left(rx(t)\big[1 - \frac{x(t)}{K}\big] - \frac{bx(t)y(t)}{1+bhx(t)}, \frac{kbx(t)y(t)}{1+bhx(t)} - \delta y(t)\right)^{\mathrm{T}},$$

$$F_{S_2}(Z) = \left(rx(t)\big[1 - \frac{x(t)}{K}\big] - \frac{bx(t)y(t)}{1+bhx(t)} - q_1x(t), \frac{kbx(t)y(t)}{1+bhx(t)} - \delta y(t) - q_2y(t)\right)^{\mathrm{T}},$$

then Filippov system (1.4) can be written as the following generalized Filippov system

$$\dot{Z}(t) = \begin{cases} \dfrac{dZ(t)}{dt} = F_{S_1}(Z), & Z \in S_1, \\ \dfrac{dZ(t)}{dt} = F_{S_2}(Z), & Z \in S_2, \end{cases} \tag{2.1}$$

where $F_{S_1}, F_{S_2} : R^2 \to R^2$ are sufficiently smooth in R^2 and $H : R^2 \to R$ is a sufficiently smooth scalar function of the system states, and

$$S_1 = \{Z \in R_2^+ | H(Z) < 0\}, \qquad S_2 = \{Z \in R_2^+ | H(Z) > 0\},$$

with $H(Z) = x - ET$ and $R_+^2 = \{Z = (x, y)^{\mathrm{T}} | x \geq 0, y \geq 0\}$.

Furthermore, the discontinuity boundary (or manifold) Σ which separates the two regions S_1 and S_2 is described as $\Sigma = \{Z \in R_+^2 | H(Z) = 0\}$. From now on, we call Fillipov system (2.1) defined in region S_1 as system S_1 and defined in region S_2 as system S_2.

For Filippov system (2.1), the dynamics can be determined not only by vector field F_{S_1} alone or F_{S_1} alone [1], but also by sliding dynamics of Filippov system (2.1). Therefore, in order to investigate Filippov system (2.1) we need to determine the sliding mode dynamics or sliding solutions on Σ for Filippov system (2.1), which can be realized by employing the well-known Filippov′s convex method [14] or Utkin′s equivalent control method [28].

Let

$$\sigma(Z) = \big\langle H_Z(Z), F_{S_1}(Z)\big\rangle \big\langle H_Z(Z), F_{S_2}(Z)\big\rangle,$$

where $\langle\cdot\rangle$ denotes the standard scalar product. Then the interior of the sliding mode domain can be defined as

$$\Sigma_S = \{Z \in \Sigma | \sigma(Z) < 0\}.$$

The sliding mode domain Σ_S can be distinguished by the following regions [4, 5, 19]:

(i) Escaping region: if $\langle H_Z(Z), F_{S_1}(Z)\rangle < 0$ and $\langle H_Z(Z), F_{S_2}(Z)\rangle > 0$;

(ii) Sliding region: if $\big\langle H_Z(Z), F_{S_1}(Z)\big\rangle > 0$ and $\big\langle H_Z(Z), F_{S_2}(Z)\big\rangle < 0$.

Now we introduce the definitions of several type of equilibria, which are very useful in the following parts [4, 5, 10, 19].

Definition 2.1. A point Z^* is called a regular equilibrium of system (2.1) if $F_{S_1}(Z^*) = 0$, $H(Z^*) < 0$ or $F_{S_2}(Z^*) = 0$, $H(Z^*) > 0$. A point Z^* is called a virtual equilibrium of system (2.1) if $F_{S_1}(Z^*) = 0$, $H(Z^*) > 0$ or $F_{S_2}(Z^*) = 0$, $H(Z^*) < 0$.

Definition 2.2. A point Z^* is called a pseudo-equilibrium if it is an equilibrium of the sliding mode of system (2.1), that is $\lambda F_{S_1}(Z^*) + (1-\lambda)F_{S_2}(Z^*) = 0$ and $0 < \lambda < 1$, where

$$\lambda = \frac{\big\langle H_Z(Z), F_{S_2}(Z)\big\rangle}{\big\langle H_Z(Z), F_{S_2}(Z) - F_{S_1}(Z)\big\rangle}.$$

Defining the vector field of Filippov system (2.1) on the sliding mode Σ_S as follows

$$\frac{dZ(t)}{dt} = F_S(Z), Z \in \Sigma_S,$$

where $F_S(Z) = \lambda F_{S_1}(Z) + (1-\lambda)F_{S_2}(Z)$ with $H(Z) = 0$.

Definition 2.3. A point Z^* is called a boundary equilibrium of system (2.1) if $F_{S_1}(Z^*) = 0$, $H(Z^*) = 0$ or $F_{S_2}(Z^*) = 0$, $H(Z^*) = 0$.

Definition 2.4. A point Z^* is called a tangency point of Filippov system (2.1) if $Z^* \in \Sigma_S$ and $F_{S_1}H(Z^*) = 0$ or $F_{S_2}H(Z^*) = 0$.

3. Mathematical analysis of system (1.4)

3.1. Qualitative analysis of subsystems

If $x < ET$, then the following system plays a key role in analyzing the Filippov system (1.4)

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)\left[1 - \dfrac{x(t)}{K}\right] - \dfrac{bx(t)y(t)}{1+bhx(t)}, \\ \dfrac{dy(t)}{dt} = \dfrac{kbx(t)y(t)}{1+bhx(t)} - \delta y(t). \end{cases} \tag{3.1}$$

Obviously, the subsystem (3.1) have three equilibria such as $(0,0)$, $(K,0)$ and $E_1^*(x_1^*, y_1^*)$, and E_1^* is the unique interior equilibrium of subsystem (3.1), where

$$x_1^* = \frac{\delta}{b(k-\delta h)}, \qquad y_1^* = \frac{rk[bK(k-\delta h) - \delta]}{b^2(k-\delta h)^2 K}.$$

The global dynamic behavior of subsystem (3.1) has been studied in details by Chen and Jing [6], but for readers interested in the technical aspects we give the details here. Obviously, the unique interior equilibrium (x_1^*, y_1^*) is positive provided that $k-\delta h > 0$ and $bK(k-\delta h)-\delta > 0$. Furthermore, assume that the subsystem (3.1) always has a unique positive equilibrium E_1^*, that is inequalities $k-\delta h > 0$ and $bK(k-\delta h) - \delta > 0$ hold true.

Meanwhile, the characteristic polynomial the subsystem (3.1) about the unique positive equilibrium E_1^* is

$$\lambda^2 - p\lambda - q = 0,$$

where

$$p = \frac{r\delta(k - hkbK + \delta h + \delta bh^2K)}{(k-\delta h)Kbk}, \qquad q = \frac{r\delta(kbK - \delta - \delta bhK)}{bKk},$$

then it is easy to see that $q > 0$ provided $bK(k - \delta h) - \delta > 0$. By simple calculation, if

$$x_1^* < \frac{Kbh-1}{2bh},$$

then $p < 0$; if

$$x_1^* > \frac{Kbh-1}{2bh},$$

then $p > 0$.

Denote $\Delta = p^2 - 4q$, if

$$x_1^* > \frac{Kbh-1}{2bh}, \qquad \Delta < 0,$$

then E_1^* is a stable focus; if

$$x_1^* < \frac{Kbh-1}{2bh}, \qquad \Delta < 0,$$

then E_1^* is an unstable focus; if

$$x_1^* > \frac{Kbh-1}{2bh}, \qquad \Delta \geq 0,$$

then E_1^* is a stable node; if

$$x_1^* < \frac{Kbh-1}{2bh}, \qquad \Delta \geq 0,$$

then E_1^* is an unstable node.

When $x_1^* < \frac{Kbh-1}{2bh}$ and $\Delta < 0$, then E_1^* is an unstable focus and we have the following Lemma.

Lemma 3.1. *Assume that $x_1^* \leq \frac{Kbh-1}{2bh}$ and $\Delta < 0$, then E_1^* is an unstable focus and there exists a unique globally asymptotically stable limit cycle in subsystem* (3.1) *in the first quadrant.*

The proof of Lemma 3.1 was provided by many authors, for details see [18, 32].

If $x > ET$, then system (1.4) becomes

$$\begin{cases} \dfrac{dx(t)}{dt} = rx(t)\left[1 - \dfrac{x(t)}{K}\right] - \dfrac{bx(t)y(t)}{1+bhx(t)} - q_1x(t), \\ \dfrac{dy(t)}{dt} = \dfrac{kbx(t)y(t)}{1+bhx(t)} - \delta y(t) - q_2y(t), \end{cases} \tag{3.2}$$

which has three equilibria such as $(0,0)$, $\big(K(1-\frac{q_1}{r}),0\big)$ and $E_2^*(x_2^*, y_2^*)$ with

$$x_2^* = \frac{\delta + q_2}{b[k - h(\delta+q_2)]},$$
$$y_2^* = \frac{k\big\{brK[k-h(\delta+q_2)] - r(\delta+q_2) - q_1Kbk + q_1Kb\delta h + q_1q_2Kbh\big\}}{b^2[k-h(\delta+q_2)]^2K}.$$

Two isoclines of system (3.2) are

$$L_3 : x = x_2^*, \qquad L_4 : y = \frac{[r(1-\frac{x}{K}) - q_1](1+bhx)}{b}.$$

The unique interior equilibrium (x_2^*, y_2^*) is positive provided

$$k - h(\delta + q_2) > 0,$$

$$brK[k - h(\delta+q_2)] + q_1Kb\delta h + q_1q_2Kbh > r(\delta+q_2) + q_1Kbk.$$

Let $r_1 = r - q_1$ and $\delta_1 = \delta + q_2$, then the conditions for the stability of E_2^* of system (3.2) can be obtained similarly based on the analysis of system (3.1), and we do not address it here for simplicity.

3.2. Sliding segments and its domains

In view of the definitions provided in Section 2, the interior of the sliding mode domain can be defined as

$$\Sigma_S = \{Z \in \Sigma | \sigma(Z) < 0\}.$$

According to the definition of $\sigma(Z)$, we have

$$\sigma(Z) = \left\{ rET\left(1 - \frac{ET}{K}\right) - \frac{bETy}{1 + bhET} \right\} \cdot \left\{ rET\left(1 - \frac{ET}{K}\right) - \frac{bETy}{1 + bhET} - q_1 ET \right\},$$

solving the inequality $\sigma(Z) < 0$ yields

$$y_{c1} < y < y_{c2},$$

and

$$y_{c1} = \frac{r\left(1 - \frac{ET}{K}\right)(1 + bhET)}{b}, \qquad y_{c2} = \frac{\left[r\left(1 - \frac{ET}{K}\right) - q_1\right](1 + bhET)}{b}.$$

Therefore, the sliding segment of Filippov system (2.1) can be defined as

$$\Sigma_S = \{(x, y)^T \in R_+^2 | x = ET, y_{c2} < y < y_{c1}\}.$$

3.3. Sliding mode dynamics

The differential equation for sliding mode dynamics in the region Σ_S can be determined by using Utkin′s equivalent control method which is introduced in [28]. It follows from $H = 0$ that

$$\frac{dH}{dt} = rET\left(1 - \frac{ET}{K}\right) - \frac{bETy}{1 + bhET} - \epsilon q_1 ET = 0,$$

and solving the equation with respect to ϵ yields

$$\epsilon = \frac{r\left(1 - \frac{ET}{K}\right) - \frac{by}{1+bhET}}{q_1}.$$

According to the Utkin's equivalent control method, the dynamics on the sliding mode Σ_S can be determined by the following equation

$$\frac{dy}{dt} = \frac{kbETy}{1 + bhET} - \delta y - q_2 y \frac{r\left(1 - \frac{ET}{K}\right) - \frac{by}{1+bhET}}{q_1} = \phi(y).$$

3.4. Existence of the equilibria

There may be four types of equilibria for system (2.1), that is, regular equilibrium (E_R), virtual equilibrium (E_V), pseudo-equilibrium (E_p), and boundary equilibrium (E_B). The tangent point is denoted as E_T. From the analyses of Subsection 3.3, the existence of all types of equilibria will be discussed briefly in the following:

(1). If $k > \delta h$ and $b > \frac{\delta}{ET(k-\delta h)}$, then there exists a unique regular equilibrium for system S_1, that is, $E_R^1 = E_1^* = (x_1^*, y_1^*)$. If $k > \delta h$ and $b < \frac{\delta}{ET(k-\delta h)}$, then E_1^* becomes a virtual equilibrium denoted by E_V^1.

(2). If $k > (\delta + q_2)h$, $brK[k - h(\delta + q_2)] + q_1 Kb\delta h + q_1 q_2 Kbh > r(\delta + q_2) + q_1 Kbk$, and $b < \frac{\delta+q_2}{ET[k-h(\delta+q_2)]}$, then there exists a unique regular equilibrium for system S_2, that is, $E_R^2 = E_2^*$. If $k > (\delta + q_2)h$ and $b > \frac{\delta+q_2}{ET[k-h(\delta+q_2)]}$, then E_2^* becomes a virtual equilibrium denoted by E_V^2.

(3). If $k > (\delta + q_2)h$, $brK[k - h(\delta + q_2)] + q_1 Kb\delta h + q_1 q_2 Kbh > r(\delta + q_2) + q_1 Kbk$, and $\frac{\delta}{ET(k-\delta h)} < b < \frac{\delta+q_2}{ET[k-h(\delta+q_2)]}$, then E_R^1 and E_R^2 can coexist. Now we investigate the pseudo-equilibrium.

Solving $\phi(y) = 0$ with $y \in [y_{c2}, y_{c1}]$, we get two pseudo-equilibria $E_p = (ET, y_1)$ and $E_p^1 = (ET, 0)$ (here we omit it for $y = 0$) where

$$y_1 = \frac{\left[q_1\delta + q_2 r(1 - \frac{ET}{K})\right](1 + bhET) - q_1 kbET}{bq_2}.$$

The tangent points of Σ_S satisfy $x = ET$ and

$$rET\left(1 - \frac{ET}{K}\right) - \frac{bETy}{1 + bhET} - \varepsilon q_1 ET = 0.$$

Then there are two tangent points denoted as $E_T^1 = (ET, y_{c1})$ for system S_1 and $E_T^2 = (ET, y_{c2})$ for system S_2, respectively.

The boundary equilibrium of Filippov system (2.1) satisfies equations

$$rx\left(1 - \frac{x}{K}\right) - \frac{bxy}{1 + bhx} - q_1\varepsilon x = 0, \qquad \frac{kbxy}{1 + bhx} - \delta y - q_2\varepsilon y = 0,$$

where $x = ET$. From the second equation, we get $ET = \frac{\delta}{b(k-\delta h)}$ or $ET = \frac{\delta+q_2}{b[k-h(\delta+q_2)]}$ with $y > 0$. So the boundary equilibria of system (2.1) are as follows

$$E_B^1 = \left(ET, \frac{r\left(1 - \frac{ET}{K}\right)(1 + bhET)}{b}\right), \text{ or } E_B^2 = \left(ET, \frac{\left[r\left(1 - \frac{ET}{K}\right) - q_1\right](1 + bhET)}{b}\right).$$

Lemma 3.2. *The two regular equilibria E_R^1, E_R^2 and the pseudo-equilibrium E_p can coexist in system* (2.1). *Further, E_p is unstable in the sliding domain.*

Proof. E_p is existed when $y \in [y_{c2}, y_{c1}]$, that is,

$$y_{c2} \le \frac{\left[q_1\delta + q_2 r(1 - \frac{ET}{K})\right](1 + bhET) - q_1 kbET}{bq_2} \le y_{c1},$$

where $y_{c1} = \frac{r\left(1-\frac{ET}{K}\right)(1+bhET)}{b}$ and $y_{c2} = \frac{\left[r\left(1-\frac{ET}{K}\right)-q_1\right](1+bhET)}{b}$. By direct calculation, we get $\frac{\delta}{ET(k-\delta h)} \le b \le \frac{\delta+q_2}{ET[k-h(\delta+q_2)]}$, this inequality is equivalent to the conditions for the coexistence of two regular equilibria E_R^1 and E_R^2. In order to prove that E_p is unstable in the sliding domain, we only need to show that the inequality $\phi'(y_1) > 0$ holds true,

$$\phi'(y_1) = y_1 \frac{q_2 b}{q_1(1 + bhET)} > 0,$$

therefore, E_p is unstable in the sliding domain if it exists. This completes the proof. □

According to the above analyses, if $x_1^* < ET < x_2^*$, then there exist two regular equilibria (E_1^*, E_2^*) and a pseudo-equilibrium (E_P), E_P is unstable. If $ET < x_1^*$, then E_1^* becomes a virtual equilibrium, E_2^* is a regular equilibrium. In this case, there exists no pseudo-equilibrium for system (2.1). If $ET > x_2^*$, then E_1^* becomes a regular equilibrium, E_2^* is a virtual equilibrium. Meanwhile, system (2.1) does not exist the pseudo-equilibrium.

4. Sliding bifurcation analysis

4.1. Bifurcation set of equilibria

Based on the above analysis, the rich dynamics of Filippov system (2.1) mainly depend on the positions between the threshold ET and the equilibria of subsystem S_1 or S_2. Therefore, it is necessary to investigate the bifurcation set of equilibria and sliding mode of Filippov system (2.1). To do this, as the search rate b of predator and the economic threshold ET are two key parameters for system (2.1), we choose b and ET

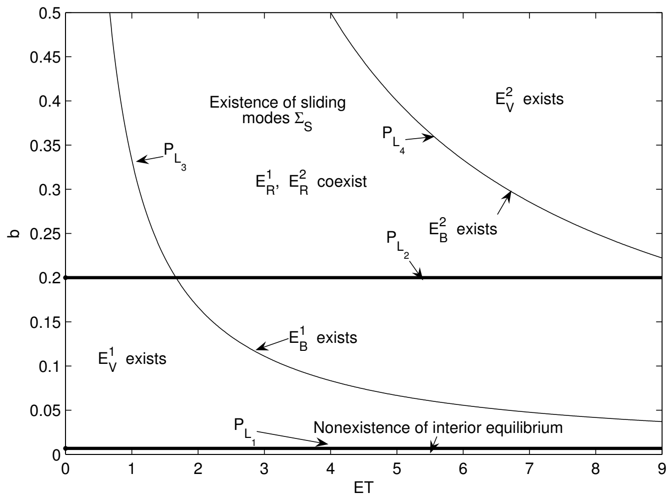

Figure 1: The bifurcation diagram of Filippov system (2.1) with respect to search rate b and the economic threshold ET. All other parameters are fixed as follows: $r = 1$, $K = 50$, $\delta = 0.1$, $k = 0.8$, $h = 2$, $q_1 = 0.8$, $q_2 = 0.1$.

to address the richness of the possible equilibria and sliding modes that Filippov system (2.1) can exhibit. Defining four lines with respect to b and ET as follows:

$$
\begin{aligned}
P_{L_1} &= \left\{ (b, ET) \middle| b = \frac{\delta}{K(k - \delta h)} \right\}, \\
P_{L_2} &= \left\{ (b, ET) \middle| b = \frac{r(\delta + q_2)}{rK[k - h(\delta + q_2)] + q_1 K \delta h + q_1 q_2 K h - q_1 K k} \right\}, \\
P_{L_3} &= \left\{ (b, ET) \middle| b = \frac{\delta}{ET(k - \delta h)} \right\}, \\
P_{L_4} &= \left\{ (b, ET) \middle| b = \frac{\delta + q_2}{ET[k - h(\delta + q_2)]} \right\}.
\end{aligned}
$$

Four lines P_{L_1}, P_{L_2}, P_{L_3}, and P_{L_4} divide $b - ET$ parameter space into six regions, and the existence or coexistence of the virtual or regular equilibria are shown in each region. In particular, the boundary equilibria E_B^1 and E_B^2 can only exist on the line P_{L_3} and P_{L_4}, respectively. Moreover, the existence of sliding mode and coexistence of regular equilibrium with pseudo-equilibrium are indicated in the region bounded by P_{L_2}, P_{L_3}, and P_{L_4}. As an example, if we fix $b = 0.3$, then E_V^1 and E_R^2 coexist $\rightarrow$ E_B^1 exists $\rightarrow$ E_R^1, E_R^2 and E_P coexist $\rightarrow$ E_B^2 exists $\rightarrow$ E_R^1 and E_V^2 coexist as ET increases (for details see Figure 1). It is found that bifurcations can occur when ET varies, which will be addressed in detail in the following parts.

4.2. Local sliding bifurcations

In this subsection, we investigate two types of local sliding bifurcations such as boundary focus bifurcation and boundary node bifurcation for Filippov system (2.1).

Boundary focus bifurcation for Filippov system (2.1) may occur when the regular equilibrium E_R^1 (a focus), pseudo-equilibrium E_P and the tangent point E_T^1 collide together simultaneously once parameter ET passes through a critical value [10, 19].

If we choose $ET = 0.4167$, then the virtual equilibrium E_V^1, the invisible tangent point E_T^1 and the pseudo-equilibriumE_P can collide together at the boundary equilibrium E_B^1, as shown in Figure 2 (b), where $ET = \delta/[b(k - \delta h)]$. Further, this boundary equilibrium E_B^1 is unstable, and the solution passes through E_B^1 will reach the visible tangent point E_T^2 and then tends to the regular equilibrium E_R^2 finally. This indicates that system (2.1) will stabilize at E_R^2, and the pest population can not be controlled below the threshold ET.

If $0 < ET < \delta/[b(k - \delta h)]$, then the invisible tangent point E_T^1, the visible tangent point E_T^2, the virtual equilibrium E_V^1 and the regular equilibrium E_R^2 can coexist (as shown in Figure 2 (a)), and the pseudo-equilibrium E_P is disappeared.

If $\delta/[b(k - \delta h)] < ET < (\delta + q_2)/\{b[k - h(\delta + q_2)]\}$, the virtual equilibrium E_V^1 becomes regular, so the virtual/regular equilibrium bifurcation occurs. Moreover, the two tangent points E_T^1 and E_T^2 becomes visible, E_P lies in the sliding domain and is unstable, E_R^1, E_R^2, E_P, E_T^1, and E_T^2 can coexist. It means that any solution may tend to E_R^1 or E_R^2 with different initial values, which suggests that the successful control for pest depends on the initial conditions, in particular, the solution stabilizes at E_R^1 indicates the pest population can be controlled below ET while the solution stabilizes at E_R^2 implies pest outbreaks, see Figure 2 (c). This boundary focus bifurcation shows how a virtual equilibrium becomes a regular equilibrium.

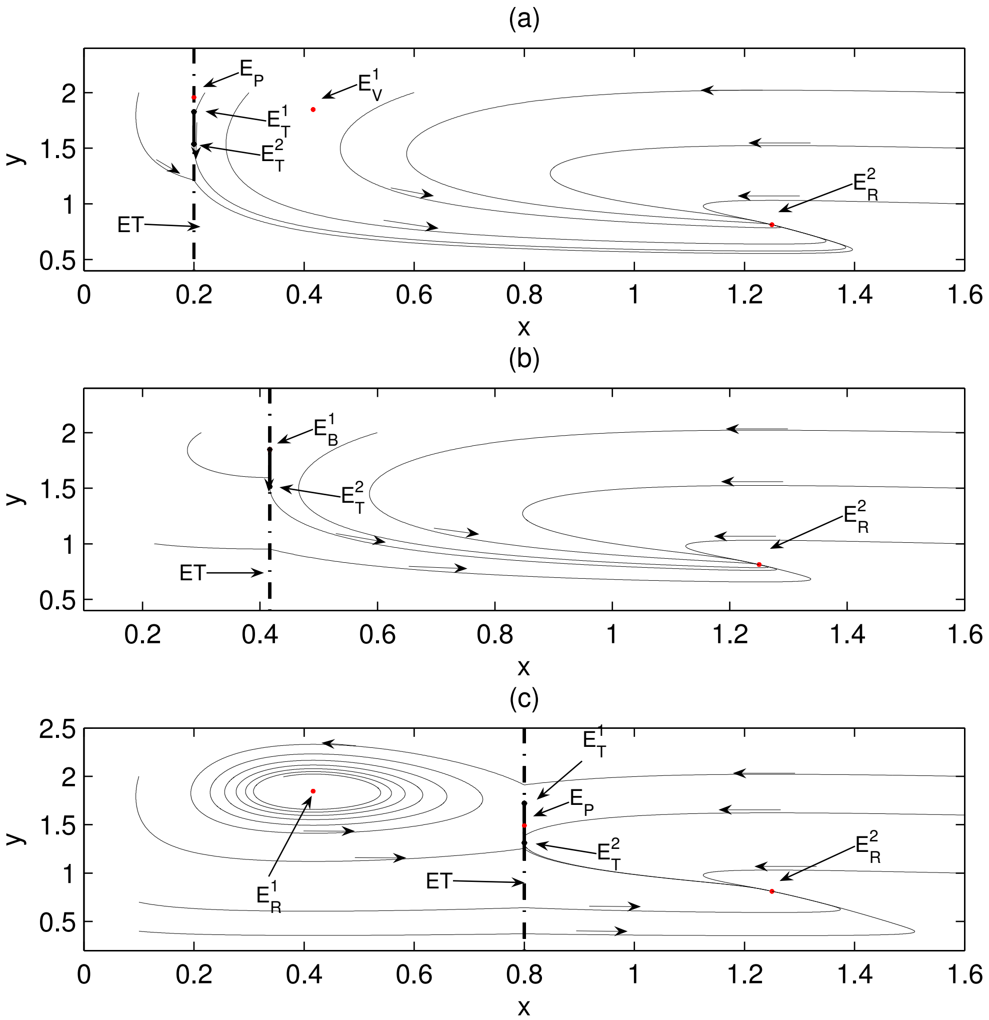

Figure 2: Boundary focus bifurcation for Filippov system (2.1). Here we choose ET as a bifurcation parameter and fix all other parameters as follows: $r = 0.7$, $b = 0.4$, $K = 2$, $\delta = 0.1$, $k = 0.8$, $h = 2$, $q_1 = 0.1$, $q_2 = 0.1$: (a) $ET = 0.2$; (b) $ET = 0.4167$; (c) $ET = 0.8$.

Boundary node bifurcation for Filippov system (2.1) may occur when the virtual equilibrium E_V^2 (a node), pseudo-equilibrium E_P and the tangent point E_T^2 collide together simultaneously once parameter ET passes through a critical value [10, 19].

If we choose $ET = 1.25$, then E_R^2, E_P and E_T^2 can collide together at E_B^2, as shown in Figure 3 (a), where ET is obtained by $ET = (\delta + q_2)/\{b[k - h(\delta + q_2)]\}d$. Furthermore, this boundary equilibrium E_B^2 is also unstable, and all solutions of Filippov system (2.1) will finally tend to the stable equilibrium E_R^2. It is revealed that the pest population is controlled below ET. When $ET > (\delta + q_2)/\{b[k - h(\delta + q_2)]\}$, then the tangent point E_T^2 becomes invisible, the pseudo-equilibriumE_P lies below x axis, and the regular equilibrium E_R^2 becomes virtual and the regular/virtual equilibrium bifurcation occurs, as shown in Figure 3 (b). This boundary node bifurcation shows how a regular equilibrium becomes a virtual equilibrium.

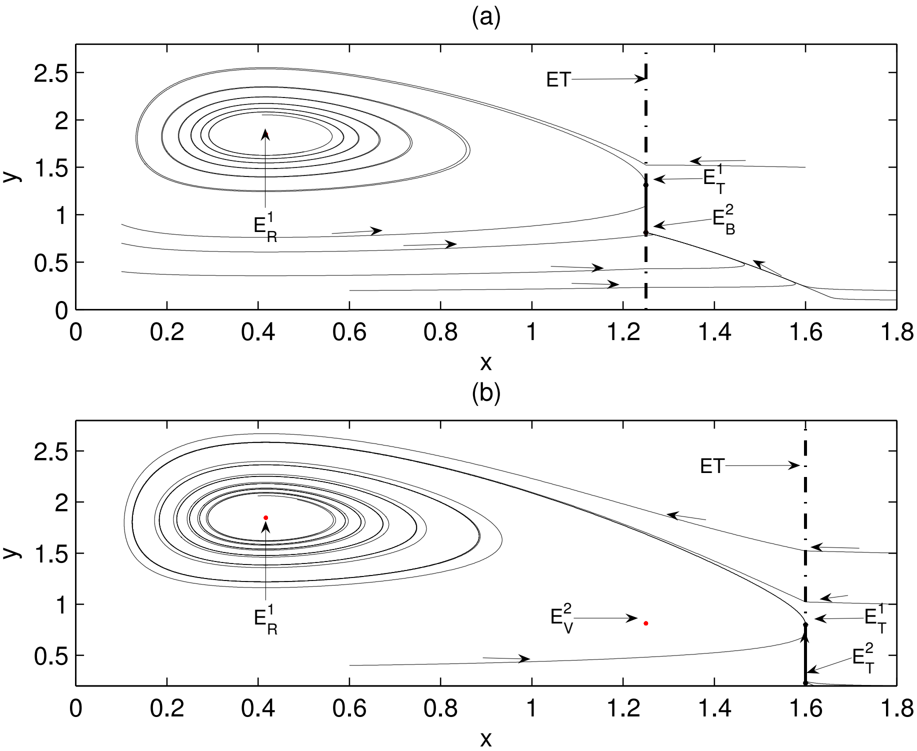

Figure 3: Boundary node bifurcation for Filippov system (2.1). Here we choose ET as a bifurcation parameter and fix all other parameters as follows: $r = 0.7$, $b = 0.4$, $K = 2$, $\delta = 0.1$, $k = 0.8$, $h = 2$, $q_1 = 0.1$, $q_2 = 0.1$: (a) $ET = 1.25$; (b) $ET = 1.6$.

4.3. Global sliding bifurcations

Note that the two subsystems S_1 and S_2 of Filippov system (2.1) could have standard periodic solutions through a Hopf bifurcation, denote the limit cycle that lies in subsystem S_1 (or S_2) completely as Ω_1 (or Ω_2), and let the leftmost (rightmost) coordinate of the limit cycle Ω_1 of subsystem S_1 as $L_1(x_{L_1}, y_{L_1})$ $(R_1(x_{R_1}, y_{R_1}))$, and let the leftmost (rightmost) coordinate of the limit cycle Ω_2 of subsystem S_2 as $L_2(x_{L_2}, y_{L_2})$ $(R_2(x_{R_2}, y_{R_2}))$. Therefore, there may be many complex dynamics for Filippov system (2.1) when ET varies.

Touching bifurcation of the sliding cycle: A typical touching bifurcation of Filippov system (2.1) may occur once the limit cycle Ω_2 can be tangent to the sliding segment Σ_S at the tangent point E_T^2 when $ET = x_{L_2}$ [19]. For example, if we choose ET as a bifurcation parameter and fix all other parameter values as shown in Figure 4, then a touching bifurcation occurs at $ET = 0.397$, in this case, the stable limit cycle Ω_2 becomes a touching cycle of system (2.1) and is tangent to the switching line $x = ET$ at the visible tangent point E_T^2, as shown in Figure 4 (b). Further, this stable touching cycle is an attractor with an incoming stable sliding orbit. When $ET < x_{L_2}$, then Ω_2 is a limit cycle of system (2.1), as shown in Figure 4 (a) with $ET = 0.1$. If $ET > x_{L_2}$, then the touching cycle becomes a canard cycle [4], as shown in Figure 4 (c).

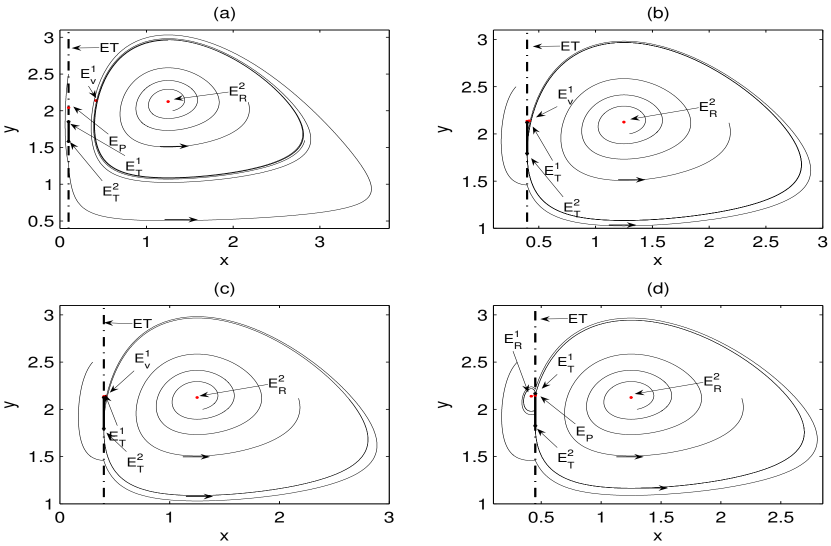

Figure 4: The phase portrait of Filippov system (2.1). We choose ET as a bifurcation parameter and fix all other parameters as follows: $r = 0.7$, $b = 0.4$, $K = 5$, $\delta = 0.1$, $k = 0.8$, $h = 2$, $q_1 = 0.1$, $q_2 = 0.1$: (a) $ET = 0.1$; (b) $ET = 0.397$; (c) $ET = 0.4$; (d) $ET = 0.45$.

Sliding crossing bifurcation: If we choose ET as a bifurcation parameter and fix all other parameters as those in Figure 5, then a sliding crossing bifurcation can be observed. If $ET = 2.7$, then there exists a sliding cycle with a single sliding segment ending at the visible tangent point E_T^1 (as shown in Figure 5 (d)). As ET decreases, then the sliding cycle only passes the tangent point E_T^1 of the sliding segment with $ET = 2.2$ and thus a sliding crossing bifurcation (or crossing critical cycle) occurs (Figure 5 (c)). If ET further decreases, the sliding segment shrinks as $ET \to 0.8$ and the cycle becomes a crossing cycle when $ET = 0.8$ (Figure 5 (b)), which is stable from both inside and outside. The sliding crossing bifurcation shows how a stable sliding cycle of system (2.1) becomes a stable crossing cycle as parameter ET varies.

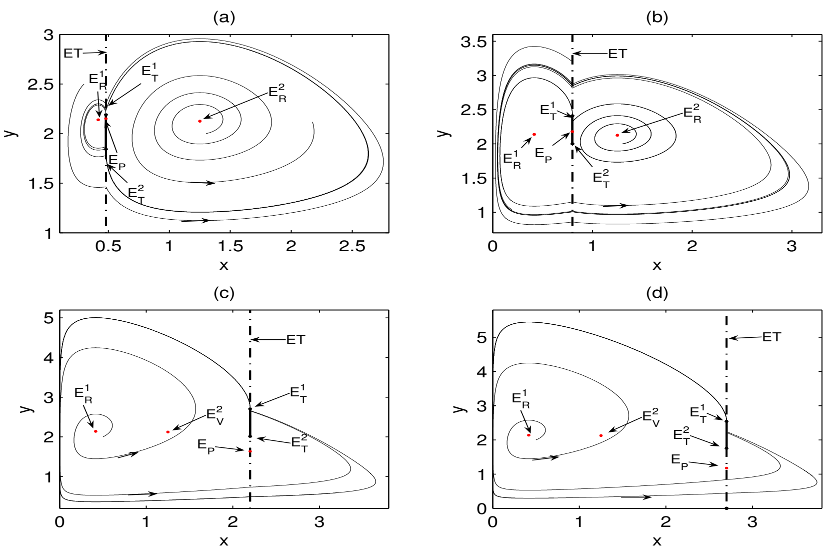

Figure 5: The phase portrait of Filippov system (2.1). We choose ET as a bifurcation parameter and fix all other parameters as follows: $r = 0.7$, $b = 0.4$, $K = 5$, $\delta = 0.1$, $k = 0.8$, $h = 2$, $q_1 = 0.1$, $q_2 = 0.1$: (a) $ET = 0.48$; (b) $ET = 0.8$; (c) $ET = 2.2$; (d) $ET = 2.7$.

Buckling bifurcation of the sliding cycle: If $ET < 3.3$, then there exists a sliding cycle with part of it enters into subsystem S_2 before returning back to the sliding segment when $ET = 2.7$ (Figure 5 (d)). If $ET = 3.3$, then a buckling bifurcation (or sliding switching) occurs [10, 19] (Figure 6 (a)), and this type of sliding cycle passes through the whole piece of sliding segment when $ET = 3.3$. As ET increases, there exists a sliding cycle (that is, canard cycle) as shown in (Figure 6 (b)). Moreover, another buckling bifurcation of the sliding cycle can also be observed From Figure 4 (c) to Figure 5 (a), and from Figure 6 (b) to Figure 6 (d) there also exist a touching bifurcation for system (2.1).

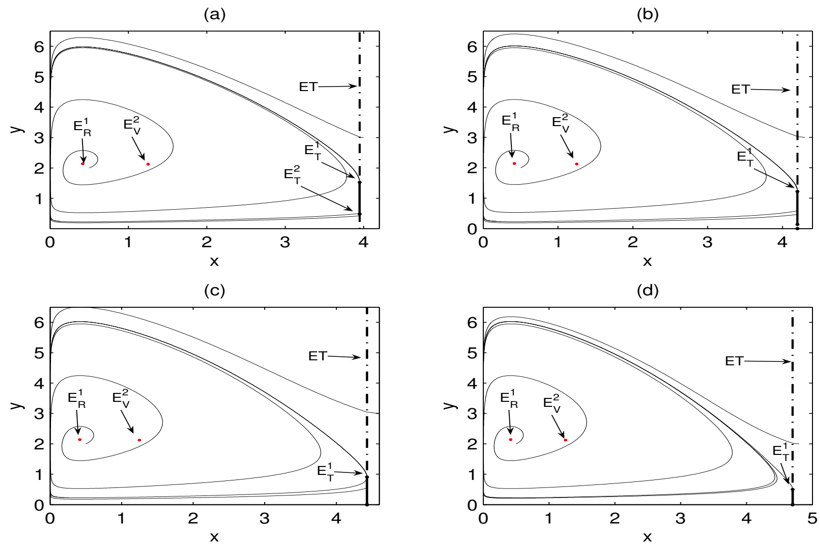

Figure 6: The phase portrait of Filippov system (2.1). We choose ET as a bifurcation parameter and fix all other parameters as follows: $r = 0.7$, $b = 0.4$, $K = 5$, $\delta = 0.1$, $k = 0.8$, $h = 2$, $q_1 = 0.1$, $q_2 = 0.1$: (a) $ET = 3.3$; (b) $ET = 4.2$; (c) $ET = 4.428$; (d) $ET = 4.5$.

5. Conclusion

There are many research papers [17, 24, 32] with numerical method for the non-smooth Filippov model proposed by Gause, which describes the relations between predator and prey interaction. However, those research papers have not considered that the effects of the control strategies as chemical control or biological control. In this paper, we proposed a Filippov predator-prey model considering the chemical control strategies, and the main purpose of this paper is to incorporate the control strategies into the non-smooth Gause model and show how it affects the dynamics of the Filippov Gause system. To do this, the dynamics of the proposed Filippov predator-prey system (1.4) with control strategies have been studied in detail by using qualitative analysis techniques of non-smooth Filippov dynamic systems and numerical techniques.

Especially, the sliding mode dynamics and existence of the several types of equilibria are discussed, and the results indicate that the two regular equilibria and the pseudo-equilibrium can coexist. Further, sliding bifurcations with respect to boundary node (focus) bifurcations, touching bifurcations, sliding crossing bifurcation and buckling bifurcations (or sliding switching) have been investigated.

In order to keep the prey population below the economic threshold effectively, the control strategies are taken once the prey population reaches and exceeds the economic threshold. As the economic threshold varies, the prey and predator can either stabilize at the regular equilibrium, or coexist and oscillate periodically along a limit cycle, a touching cycle, a canard cycle, a crossing cycle or a periodic solution with sliding segments. This indicates that if the proposed Filippov system (1.4) depicts the interaction between pest and natural enemy, then the existence of the periodic solutions including a limit cycle, crossing cycle and limit cycles with sliding segments show that the pest and natural enemy can coexist and oscillate periodically.

Moreover, it suggests that the economic threshold should be chosen carefully such that the density of the pest population can be stabilized at the threshold level.

Compared to the previous studies without considering the control strategies [17, 24, 32], the differences are listed as follows: (1) the two regular equilibria E_R^1, E_R^2, and the pseudo-equilibriumE_p can coexist, and the pseudo-equilibrium is unstable in the sliding domain, this implies that the prey and predator population may stabilize at E_R^1 and E_R^2 at the same time; (2) it is interesting to note that the boundary node bifurcation and boundary focus bifurcation can occur as the threshold increases; (3) more complex bifurcations such as sliding crossing bifurcation and buckling bifurcation of the sliding cycle are observed in addition to the touching bifurcations. All these observations confirm that the effects of the control strategies on the Filippov system (1.4) play an important role in determining the dynamical behaviors of the proposed Filippov system.

Note that the effect of the biological control on the prey and predator population is ignored in this paper. However, it is shown that biological control is more popular in reality, and it also affects the development and extinction of the prey population [23, 26]. The more comprehensive analysis for Filippov system (1.4) with the effects of both chemical and biological control will be studied in the future.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (No. 11461082, 11461083, 31460297), the Educational Commission of Hubei Province(Q20161212), the Educational Commission of Yunnan Province(2015Y223) and the Youth Foundation of China Three Gorges University(KJ2015A006). Moreover, this work is partly supported by Key Laboratory of IOT Application Technology of Universities in Yunnan Province.

References

[1] S. M. Baer, B. W. Kooi, Y. A. Kuznetsov, H. R. Thieme, *Multiparametric bifurcation analysis of a basic two-stage population model*, SIAM J. Appl. Math., **66** (2006), 1339–1365. 2

[2] A. A. Berryman, B. A. Hawkins, *The refuge as an integrating concept in ecology and evolution*, Oikos, **115** (2006), 192–196. 1

[3] B. Brogliato, *Nonsmooth Mechanics*, Springer-Verlag, London, (1999). 1

[4] C. A. Buzzi, T. D. Carvalho, P. R. da Silva, *Canard cycles and Poincar index of non-smooth vector fields on the plane*, J. Dyn. Control Syst., **2** (2010), 173–193. 1, 2, 4.3

[5] C. A. Buzzi, P. R. da Silva, M. A. Teixeira, *A singular approach to discontinuous vector fields on the plane*, J. Differential Equations, **231** (2006), 633–655. 1, 2

[6] L. S. Chen, Z. J. Jing, *Existence and uniqueness of limit cycles for the differential equations of predator-prey interaction*, Chinese Sci. Bull., **29** (1984), 521–523. 3.1

[7] M. I. da Silveira Costa, *Harvesting induced fluctuations: insights from a threshold management policy*, Math. Biosci., **205** (2007), 77–82. 1

[8] M. I. da Silveira Costa, E. Kaszkurewicz, A. Bhaya, L. Hsu, *Achieving global convergence to an equilibrium population in predator-prey systems by the use of a discontinuous harvesting policy*, Ecol. Model., **128** (2000), 89–99.

[9] M. I. da Silveira Costa, M. E. M. Meza, *Application of a threshold policy in the management of multispecies fisheries and predator culling*, Math. Med. Biol., **23** (2006), 63–75. 1

[10] M. D. de Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, A. B. Nordmark, G. Olivar Tost, P. T. Piiroinen, *Bifurcations in nonsmooth dynamical systems*, SIAM Rev., **50** (2008), 629–701. 1, 2, 4.2, 4.2, 4.3

[11] M. D. de Bernardo, P. Kowalczyk, A. Nordmark, *Bifurcations of dynamical systems with sliding: derivation of normal-form mappings,* Phys. D, **170** (2002), 175–205. 1

[12] F. Dercole, A. Gragnani, S. Rinaldi, *Bifurcation analysis of piecewise smooth ecological models*, Theoret. Popul. Biol., **72** (2007), 197–213. 1

[13] S. H. Doole, S. J. Hogan, *A piecewise linear suspension bridge model: nonlinear dynamics and orbit continuation*, Dynam. Stability Systems, **11** (1996), 19–47. 1

[14] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Translated from the Russian. Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, (1988). 2

[15] G. F. Gause, *The struggle for existence*, Soil Science, **41** (1934). 1

[16] G. F. Gause, N. P. Smaragdova, A. A. Witt, *Further studies of interaction between predators and prey*, J. Animal Ecol., **5** (1936), 1–18. 1, 1, 1, 1

[17] V. Křivan, *On the Gause predator-prey model with a refuge: a fresh look at the history*, J. Theoret. Biol., **274** (2011), 67–73. 1, 1, 1, 1, 5

[18] Y. Kuang, H. I. Freedman, *Uniqueness of limit cycles in Gause-type models of predator-prey systems*, Math. Biosci., **88** (1988), 67–84. 3.1

[19] Y. A. Kuznetsov, S. Rinaldi, A. Gragnani, *One-parameter bifurcations in planar Filippov systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **13** (2003), 2157–2188. 1, 2, 4.2, 4.2, 4.3, 4.3

[20] M. E. M. Meza, A. Bhaya, E. Kaszkurewicz, M. I. da Silveira Costa, *Threshold policies control for predatorprey systems using a control Liapunov function approach*, Theoret. Popul. Biol., **67** (2005), 273–284. 1

[21] W. Qin, S. Tang, *The selection pressures induced non-smooth infectious disease model and bifurcation analysis*, Chaos Solitons Fractals, **69** (2014), 160–171.

[22] W. Qin, S. Tang, C. Xiang, Y. Yang, *Effects of limited medical resource on a Filippov infectious disease model induced by selection pressure*, Appl. Math. Comput., **283** (2016), 339–354 1

[23] S. Tang, R. A. Cheke, *State-dependent impulsive models of integrated pest management (IPM) strategies and their dynamic consequences*, J. Math. Biol., **50** (2005), 257–292. 1, 5

[24] S. Tang, J. Liang, *Global qualitative analysis of a non-smooth Gause predator-prey model with a refuge*, Nonlinear Anal., **76** (2013), 165–180. 1, 1, 1, 1, 5

[25] S. Tang, J. Liang, Y. Xiao, R. A. Cheke, *Sliding bifurcations of Filippov two stage pest control models with economic thresholds*, SIAM J. Appl. Math., **72** (2012), 1061–1080. 1

[26] S. Tang, Y. Xiao, L. Chen, R. A. Cheke, *Integrated pest management models and their dynamical behaviour*, Bull. Math. Biol., **67** (2005), 115–135. 1, 5

[27] S. Tang, Y. Xiao, N. Wang, H. Wu, *Piecewise HIV virus dynamic model with* $\mathrm{CD4}^+$ *T cell count-guided therapy*, J. Theoret. Biol., **308** (2012), 123–134. 1

[28] V. I. Utkin, J. Guldner, J. X. Shi, *Sliding mode control in electro-mechanical systems*, second ed., CRC press, (2009). 2, 3.3

[29] B. L. Van de Vrande, D. H. Van Campen, A. De Kraker, *An approximate analysis of dry-friction-induced stick-slip vibrations by a smoothing procedure*, Nonlinear Dyn., **19** (1999), 157–169. 1

[30] Y. Xiao, H. Miao, S. Tang, H. Wu, *Modeling antiretroviral drug responses for HIV-1 infected patients using differential equation models*, Adv. Drug Deliv. Rev., **65** (2013), 940–953.

[31] Y. Xiao, S. Tang, J. Wu, *Media impact switching surface during an infectious disease outbreak*, Scientific Rep., **5** (2015), 7838–7846.

[32] J. Yang, S. Tang, R. A. Cheke, *Global stability and sliding bifurcations of a non-smooth Gause predator-prey system*, Appl. Math. Comput., **224** (2013), 9–20. 1, 1, 3.1, 5

[33] J. Yang, S. Tang, R. A. Cheke, *Modelling the regulatory system for diabetes mellitus with a threshold window*, Commun. Nonlinear Sci. Numer. Simul., **22** (2015), 478–491. 1