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Common fixed point for three pairs of self-maps satisfying weakly commuting and weakly compatible condition in generalized metric spaces

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Abstract

In this paper, we use weakly commuting and weakly compatible conditions of self-mapping pairs, prove some new common fixed point theorems for three pairs of self-maps in the framework of generalized metric spaces. The results presented in this paper generalize the well known comparable results in the literature due to Abbas et al. [M. Abbas, T. Nazir, R. Saadati, Adv. Difference Equ., **2011** (2011), 20 pages]. We also provide illustrative examples in support of our new results.

Keywords: Generalized metric space, weakly commuting mapping pairs, weakly compatible mapping pairs, common fixed point.
2010 MSC: 47H10, 54H25, 54E50.

1. Introduction and Preliminaries

The study of fixed points of mappings satisfying certain conditions has been at the center of vigorous research activity. In 2006, Mustafa and Sims [31] introduced a new structure of generalized metric spaces, which are called G-metric spaces as the following.

Definition 1.1 ([31])**.** Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;

(G_2) $0 < G(x, x, y)$, for all $x, y \in X$with $x \neq y$;

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(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

It is known that the function $G(x, y, z)$ on G-metric space X is jointly continuous in all three of its variables, and $G(x, y, z) = 0$ if and only if $x = y = z$; see [31] and the reference therein for more details.

Based on the notion of generalized metric spaces, Mustafa et al. [29, 30, 32] obtained some fixed point results for mappings satisfying different contractive conditions. Aydi [8] obtained a fixed point result for a self-mapping on a G-metric space satisfying (ψ,φ)-weakly contractive conditions. Shatanawi [34] proved some fixed point results for self-maps in a complete G-metric space under some contractive conditions related to a nondecreasing map $\phi : R^+ \to R^+$ with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$. Chugh et al. [11] obtained some fixed point results for maps satisfying property P in G-metric spaces. Hussain et al. [20] introduced the notion of G^m-Meir-Keeler contractive, G_c^m-Meir-Keeler contractive and G-(α, ψ)-Meir-Keeler contractive mapping and prove some fixed point theorems for the several class of mappings in the setting of G-metric spaces. Abbas and Rhoades [7] initiated the study of a common fixed point theory in generalized metric spaces. Kaewcharoen [22] obtained some common fixed point results for contractive mappings satisfying Φ-maps in G-metric spaces. Abbas et al. [4] obtained some periodic point results in generalized metric spaces. Aydi et al. [10] obtained some common fixed point results for generalized weakly G-contraction mapping in G-metric spaces. Ye and Gu [38] obtained some common fixed point theorems for a class of twice power type contraction maps in G-metric spaces. In [16], Gu and Ye introduce the concept of φ-weakly commuting self-mapping pairs in G-metric space, and used this concept, they establish a new common fixed point theorem of Altman integral type mappings in G-metric space. Aydi [9] obtained a common fixed point theorem of integral type contraction in generalized metric spaces. Tahat et al. [37] obtained some common fixed point theorems for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces. Manro et al. [25] obtained some common fixed point theorems for expansion mappings in G-metric spaces. Abbas et al. [1] and Manro et al. [26] gives some common fixed point theorems for R-weakly commuting maps in G-metric spaces. In [27], the authors proved some common fixed point theorems of weakly compatible mappings in G-metric spaces. In [5, 6, 12, 15, 23, 33, 39], the authors proved some common fixed point results of three (or four, or six) mappings in G-metric spaces. Recently, Abbas et al. [3] and Mustafa et al. [28] obtained some common fixed point results for a pair of mappings satisfying $(E.A)$ property under certain generalized strict contractive conditions in G-metric spaces. Long et al. [24] obtained some common fixed points results of two pairs of mappings when only one pair satisfies $(E.A)$ property G-metric spaces. Gu and Yin [17] obtained some common fixed points results of three pairs of mappings for which only two pairs need to satisfy common $(E.A)$ property in the framework of a generalized metric space. Very recently, Gu and Shatanawi [14] used the concept of common $(E.A)$ property, proved some common fixed point theorems for three pairs of weakly compatible self-maps satisfying a generalized weakly G-contraction condition in generalized metric spaces. In [18, 19, 35], some coupled fixed point and common coupled fixed point results are obtained in generalized metric spaces. In [2, 13, 36], the authors proved some coupled fixed point results for mappings satisfying different contractive conditions in two generalized metric spaces. In 2014, Hussain et al. [21] introduced a new concept of generalized partial b-metric space using the concepts of G-metric, partial metric, and b-metric spaces and obtained some fixed point results for contractive mappings in such spaces.

The purpose of this paper is to use the concept of weakly commuting mappings and weakly compatible mappings to discuss some new common fixed point problem for three pairs of self-maps in G-metric spaces. The results presented in this paper extend and improve the corresponding results of Abbas, Nazir and Saadati [5].

We now recall some of the basic concepts and results in G-metric spaces.

Definition 1.2 ([31]). Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence (x_n), if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, and we say that the sequence (x_n) is G-convergent to x or (x_n) $G-$ convergent to x.

Thus, $x_n \to x$ in a G-metric space (X, G) if for any $\epsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq k$.

Proposition 1.3 ([31]). *Let (X, G) be a G-metric space, then the following are equivalent*

1. *(x_n) is G-convergent to x.*
2. *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$.*
3. *$G(x_n, x, x) \to 0$ as $n \to +\infty$.*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.4 ([31]). Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy if for every $\epsilon > 0$, there is $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq k$; that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.5 ([31]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

1. *The sequence (x_n) is G-Cauchy.*
2. *For every $\epsilon > 0$, there is $k \in N$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $m, n \geq k$.*

Definition 1.6 ([31]). Let (X, G) and (X', G') be G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if only if it is G-continuous for all $a \in X$.

Proposition 1.7 ([31]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.8 ([31]). A G-metric space (X, G) is G-complete if every G-cauchy sequence in (X, G) is G-convergent in X.

Definition 1.9 ([37]). Two self mappings f and g of a G-metric space (X, G) is said to be weakly commuting if $G(fgx, gfx, gfx) \leq G(fx, gx, gx)$ for all x in X.

Definition 1.10 ([37]). Let f and g be two self mappings from a G-metric space (X, G) into itself. Then the mappings f and g are said to be weakly compatible if $G(fgx, gfx, gfx) = 0$ whenever $G(fx, gx, gx) = 0$.

Proposition 1.11 ([31]). *Let (X, G) be a G-metric space. Then, for all x, y, z in X, it follows that $G(x, y, y) \leq 2G(y, x, x)$.*

2. Main Results

Theorem 2.1. *Let (X, G) be a complete G-metric space and let f, g, h, A, B, and C are six mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) $\forall x, y, z \in X$,

$$G(fx, gy, hz) \leq k \max \left\{ \begin{array}{c} G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx), \\ G(Ax, gy, gy) + G(By, gy, gy) + G(Cz, gy, gy), \\ G(Ax, hz, hz) + G(By, hz, hz) + G(Cz, hz, hz) \end{array} \right\} \tag{2.1}$$

or

$$G(fx, gy, hz) \leq k \max \left\{ \begin{array}{c} G(Ax, Ax, fx) + G(By, By, fx) + G(Cz, Cz, fx), \\ G(Ax, Ax, gy) + G(By, By, gy) + G(Cz, Cz, gy), \\ G(Ax, Ax, hz) + G(By, By, hz) + G(Cz, Cz, hz) \end{array} \right\}, \tag{2.2}$$

where $k \in [0, \frac{1}{6})$. Then one of the pairs (f, A), (g, B), and (h, C) has a coincidence point in X. Moreover, if one of the following conditions is satisfied:

(a) Either f or A is G-continuous, the pair (f, A) is weakly commuting, the pairs (g, B) and (h, C) are weakly compatible;

(b) Either g or B is G-continuous, the pair (g, B) is weakly commuting, the pairs (f, A) and (h, C) are weakly compatible;

(c) Either h or C is G-continuous, the pair (h, C) is weakly commuting,the pairs (f, A) and (g, B) are weakly compatible.

Then the mappings f, g, h, A, B, and C have a unique common fixed point in X.

Proof. Let us first assume that mappings f, g, h, A, B, and C satisfy condition (2.1).

Let x_0 in X be an arbitrary point, since $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$ there exists the sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$y_{3n} = fx_{3n} = Bx_{3n+1},\ y_{3n+1} = gx_{3n+1} = Cx_{3n+2},\ y_{3n+2} = hx_{3n+2} = Ax_{3n+3}$$

for $n = 0, 1, 2, \cdots$.

If $y_{3n} = y_{3n+1}$, then $gp = Bp$ where $p = x_{3n+1}$; If $y_{3n+1} = y_{3n+2}$, then $hp = Cp$ where $p = x_{3n+2}$; If $y_{3n+2} = y_{3n+3}$, then $fp = Ap$ where $p = x_{3n+3}$. Without loss of generality, we can assume that $y_n \neq y_{n+1}$, for all $n = 0, 1, 2, \cdots$.

Now we prove that $\{y_n\}$ is a G-Cauchy sequence in X.

Actually, using the condition (2.1) and (G_3) we have

$$\begin{aligned}
&G(y_{3n-1}, y_{3n}, y_{3n+1})\\
&\quad = G(fx_{3n}, gx_{3n+1}, hx_{3n-1})\\
&\quad \leq k \max\left\{\begin{array}{c} G(Ax_{3n}, fx_{3n}, fx_{3n}) + G(Bx_{3n+1}, fx_{3n}, fx_{3n}) + G(Cx_{3n-1}, fx_{3n}, fx_{3n}),\\ G(Ax_{3n}, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(Cx_{3n-1}, gx_{3n+1}, gx_{3n+1}),\\ G(Ax_{3n}, hx_{3n-1}, hx_{3n-1}) + G(Bx_{3n+1}, hx_{3n-1}, hx_{3n-1}) + G(Cx_{3n-1}, hx_{3n-1}, hx_{3n-1}) \end{array}\right\}\\
&\quad = k \max\left\{\begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n}) + G(y_{3n}, y_{3n}, y_{3n}) + G(y_{3n-2}, y_{3n}, y_{3n}),\\ G(y_{3n-1}, y_{3n+1}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+1}) + G(y_{3n-2}, y_{3n+1}, y_{3n+1}),\\ G(y_{3n-1}, y_{3n-1}, y_{3n-1}) + G(y_{3n}, y_{3n-1}, y_{3n-1}) + G(y_{3n-2}, y_{3n-1}, y_{3n-1}) \end{array}\right\}\\
&\quad \leq k \max\left\{\begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-2}, y_{3n-1}, y_{3n}),\\ G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-2}, y_{3n}, y_{3n+1})\\ G(y_{3n-2}, y_{3n-1}, y_{3n}) + G(y_{3n-2}, y_{3n-1}, y_{3n}) \end{array}\right\}\\
&\quad \leq k\left[2G(y_{3n-1}, y_{3n}, y_{3n+1}) + 2G(y_{3n-2}, y_{3n-1}, y_{3n})\right],
\end{aligned}$$

which further implies that

$$(1 - 2k)G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq 2kG(y_{3n-2}, y_{3n-1}, y_{3n}).$$

Thus

$$G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq \lambda G(y_{3n-2}, y_{3n-1}, y_{3n}), \tag{2.3}$$

where $\lambda = \frac{2k}{1-2k}$. Obviously $0 \leq \lambda < 1$.

Similary it can be shown that

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq \lambda G(y_{3n-1}, y_{3n}, y_{3n+1}) \tag{2.4}$$

and

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq \lambda G(y_{3n}, y_{3n+1}, y_{3n+2}). \tag{2.5}$$

It follows from (2.3), (2.4), and (2.5) that, for all $n \in \mathbb{N}$,

$$G(y_n, y_{n+1}, y_{n+2}) \leq \lambda G(y_{n-1}, y_n, y_{n+1}) \leq \lambda^2 G(y_{n-2}, y_{n-1}, y_n) \leq \cdots \leq \lambda^n G(y_0, y_1, y_2).$$

Therefore, for all $n, m \in \mathbb{N}$, $n < m$, by (G_3) and (G_5) we have

$$\begin{aligned} G(y_n, y_m, y_m) &\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) \\ &\quad + \cdots + G(y_{m-1}, y_m, y_m) \\ &\le G(y_n, y_{n+1}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_{m+1}) \\ &\le (\lambda^n + \lambda^{n+1} + k^{n+2} + \cdots + \lambda^{m-1}) G(y_0, y_1, y_2) \\ &\le \frac{\lambda^n}{1-\lambda} G(y_0, y_1, y_2) \to 0, as\ n \to \infty. \end{aligned}$$

Hence, $\{y_n\}$ is a G-Cauchy sequence in X, since X is a complete G-metric space, there exists a point $u \in X$ such that $y_n \to u$ as $n \to \infty$.

Since the sequences $\{fx_{3n}\} = \{Bx_{3n+1}\}$,$\{gx_{3n+1}\} = \{Cx_{3n+2}\}$ and $\{hx_{3n-1}\} = \{Ax_{3n}\}$ are all subsequences of $\{y_n\}$, then they all converge to u.

$$y_{3n} = fx_{3n} = Bx_{3n+1} \to u, y_{3n+1} = gx_{3n+1} = Cx_{3n+2} \to u, y_{3n-1} = hx_{3n-1} = Ax_{3n} \to u \ \text{ as } \ n \to \infty. \tag{2.6}$$

Now we prove that u is a common fixed point of f, g, h, A, B, and C under the condition (a).

First, we suppose that A is continuous, the pair (f, A) is weakly commuting, the pairs (g, B) and (h, C) are weakly compatible.

Step 1. We prove that $u = fu = Au$.

By (2.6) and weakly commuting of mapping pair (f, A) we have

$$G(fAx_{3n}, Afx_{3n}, Afx_{3n}) \le G(fx_{3n}, Ax_{3n}, Ax_{3n}) \to 0 \ \text{ as } \ n \to \infty. \tag{2.7}$$

Since A is continuous, then $A^2x_{3n} \to Au$ as $n \to \infty$, $Afx_{3n} \to Au$ as $n \to \infty$. By (2.7) we know $fAx_{3n} \to Au$ as $n \to \infty$.

From the condition (2.1) we know:

$$\begin{aligned} &G(fAx_{3n}, gx_{3n+1}, hx_{3n+2}) \\ &\le k \max \left\{ \begin{array}{c} G(A^2x_{3n}, fAx_{3n}, fAx_{3n}) + G(Bx_{3n+1}, fAx_{3n}, fAx_{3n}) + G(Cx_{3n+2}, fAx_{3n}, fAx_{3n}), \\ G(A^2x_{3n}, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(Cx_{3n+2}, gx_{3n+1}, gx_{3n+1}), \\ G(A^2x_{3n}, hx_{3n+2}, hx_{3n+2}) + G(Bx_{3n+1}, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, hx_{3n+2}, hx_{3n+2}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, and using the Proposition 1.11 we have

$$\begin{aligned} G(Au, u, u) &\le k \max \left\{ \begin{array}{c} G(Au, Au, Au) + G(u, Au, Au) + G(u, Au, Au), \\ G(Au, u, u) + G(u, u, u) + G(u, u, u), \\ G(Au, u, u) + G(u, u, u) + G(u, u, u) \end{array} \right\} \\ &= k \max \{2G(u, Au, Au), G(Au, u, u)), G(Au, u, u)\} \\ &\le 4kG(Au, u, u). \end{aligned}$$

Hence, $G(Au, u, u)$=0 and $Au = u$, since $0 \le k < \frac{1}{6}$.

Again by use of the condition (2.1) we have

$$\begin{aligned} &G(fu, gx_{3n+1}, hx_{3n+2}) \\ &\le k \max \left\{ \begin{array}{c} G(Au, fu, fu) + G(Bx_{3n+1}, fu, fu) + G(Cx_{3n+2}, fu, fu), \\ G(Au, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(Cx_{3n+2}, gx_{3n+1}, gx_{3n+1}), \\ G(Au, hx_{3n+2}, hx_{3n+2}) + G(Bx_{3n+1}, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, hx_{3n+2}, hx_{3n+2}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, using $Au = u$ and the Proposition 1.11, we have

$$\begin{aligned} G(fu,u,u) &\le k \max \left\{ \begin{array}{c} G(u,fu,fu) + G(u,fu,fu) + G(u,fu,fu), \\ G(u,u,u) + G(u,u,u) + G(u,u,u), \\ G(u,u,u) + G(u,u,u) + G(u,u,u) \end{array} \right\} \\ &= 3kG(u,fu,fu) \\ &\le 6kG(fu,u,u), \end{aligned}$$

which implies that $G(fu,u,u)$=0 and so $fu = u$, since $0 \le k < \frac{1}{6}$. Thus we have $u = Au = fu$.

Step 2. We prove that $u = gu = Bu$.

Since $f(X) \subset B(X)$ and $u = fu \in f(X)$, there is a point $v \in X$ such that $u = fu = Bv$. Again by use of condition (2.1), we have

$$\begin{aligned} &G(fu,gv,hx_{3n+2}) \\ &\le k \max \left\{ \begin{array}{c} G(Au,fu,fu) + G(Bv,fu,fu) + G(Cx_{3n+2},fu,fu), \\ G(Au,gv,gv) + G(Bv,gv,gv) + G(Cx_{3n+2},gv,gv), \\ G(Au,hx_{3n+2},hx_{3n+2}) + G(Bv,hx_{3n+2},hx_{3n+2}) + G(Cx_{3n+2},hx_{3n+2},hx_{3n+2}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, using $u = Au = fu = Bv$ and the Proposition 1.11, we obtain

$$\begin{aligned} G(u,gv,u) &\le k \max \left\{ \begin{array}{c} G(u,u,u) + G(u,u,u) + G(u,u,u), \\ G(u,gv,gv) + G(u,gv,gv) + G(u,gv,gv), \\ G(u,u,u) + G(u,u,u) + G(u,u,u) \end{array} \right\} \\ &= 3kG(u,gv,gv) \\ &\le 6kG(u,gv,u), \end{aligned}$$

which gives that $G(u,gv,u) = 0$ because $0 \le k < \frac{1}{6}$, and so $gv = u = Bv$.

Since the pair (g,B) is weakly compatible, we have

$$gu = gBv = Bgv = Bu.$$

Again by use of condition (2.1), we have

$$\begin{aligned} &G(fu,gu,hx_{3n+2}) \\ &\le k \max \left\{ \begin{array}{c} G(Au,fu,fu) + G(Bu,fu,fu) + G(Cx_{3n+2},fu,fu), \\ G(Au,gu,gu) + G(Bu,gu,gu) + G(Cx_{3n+2},gu,gu), \\ G(Au,hx_{3n+2},hx_{3n+2}) + G(Bu,hx_{3n+2},hx_{3n+2}) + G(Cx_{3n+2},hx_{3n+2},hx_{3n+2}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$, using $u = Au = fu$, $gu = Bu$ and the Proposition 1.11, we have

$$\begin{aligned} G(u,gu,u) &\le k \max \left\{ \begin{array}{c} G(u,u,u) + G(gu,u,u) + G(u,u,u), \\ G(u,gu,gu) + G(gu,gu,gu) + G(u,gu,gu), \\ G(u,u,u) + G(gu,u,u) + G(u,u,u) \end{array} \right\} \\ &= k \max \{G(gu,u,u), 2G(u,gu,gu)\} \\ &\le 4kG(u,gu,u). \end{aligned}$$

This implies that $G(u,gu,u) = 0$ and so $u = gu = Bu$.

Step 3. We prove that $u = hu = Cu$.

Since $g(X) \subset C(X)$ and $u = gu \in g(X)$, there is a point $w \in X$ such that $u = gu = Cw$. Again by use of condition (2.1), we have

$$G(fu, gu, hw) \leq k \max \left\{ \begin{array}{c} G(Au, fu, fu) + G(Bu, fu, fu) + G(Cw, fu, fu), \\ G(Au, gu, gu) + G(Bu, gu, gu) + G(Cw, gu, gu), \\ G(Au, hw, hw) + G(Bu, hw, hw) + G(Cw, hw, hw) \end{array} \right\}.$$

Using $u = Au = fu$, $u = gu = Bu = Cw$ and the Proposition 1.11, we obtain

$$G(u, u, hw) \leq 3kG(u, hw, hw) \leq 6kG(u, u, hw),$$

which implies that $G(u, u, hw) = 0$ and so $hw = u = Cw$.

Since the pair (h, C) is weakly compatible, we have

$$hu = hCw = Chw = Cu.$$

Again by use of condition (2.1), we have

$$G(fu, gu, hu) \leq k \max \left\{ \begin{array}{c} G(Au, fu, fu) + G(Bu, fu, fu) + G(Cu, fu, fu), \\ G(Au, gu, gu) + G(Bu, gu, gu) + G(Cu, gu, gu), \\ G(Au, hu, hu) + G(Bu, hu, hu) + G(Cu, hu, hu) \end{array} \right\}.$$

Using $u = Au = fu$, $u = gu = Bu$, $Cu = hu$ and the Proposition 1.11, we have

$$G(u, u, hu) \leq k \max \{G(hu, u, u), 2G(u, hu, hu)\} \leq 4kG(u, u, hu),$$

which gives that $G(u, u, hu) = 0$ and so $u = hu = Cu$.

Therefore u is the common fixed point of f, g, h, A, B, and C when A is continuous and the pair (f, A) is weakly commuting, the pairs (g, B) and (h, C) are weakly compatible.

Next, we suppose that f is continuous, the pair (f, A) is weakly commuting, the pair (g, B) and (h, C) are weakly compatible.

Step 1. We prove that $u = fu$.

By (2.6) and weakly commuting of mapping pair (f, A) we have

$$G(fAx_{3n}, Afx_{3n}, Afx_{3n}) \leq G(fx_{3n}, Ax_{3n}, Ax_{3n}) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Since f is continuous, then $f^2x_{3n} \to fu$ as $n \to \infty$, $fAx_{3n} \to fu$ as $n \to \infty$. By (2.6) we know $Afx_{3n} \to fu$ as $n \to \infty$.

From the condition (2.1) we know

$$\begin{aligned} &G(f^2x_{3n}, gx_{3n+1}, hx_{3n+2}) \\ &\leq k \max \left\{ \begin{array}{c} G(Afx_{3n}, f^2x_{3n}, f^2x_{3n}) + G(Bx_{3n+1}, f^2x_{3n}, f^2x_{3n}) + G(Cx_{3n+2}, f^2x_{3n}, f^2x_{3n}), \\ G(Afx_{3n}, gx_{3n+1}, gx_{3n+1}) + G(Bx_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(Cx_{3n+2}, gx_{3n+1}, gx_{3n+1}), \\ G(Afx_{3n}, hx_{3n+2}, hx_{3n+2}) + G(Bx_{3n+1}, hx_{3n+2}, hx_{3n+2}) + G(Cx_{3n+2}, hx_{3n+2}, hx_{3n+2}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$ and the Proposition 1.11 we have

$$\begin{aligned} G(fu, u, u) &\leq k \max \left\{ \begin{array}{c} G(fu, fu, fu) + G(u, fu, fu) + G(u, fu, fu), \\ G(fu, u, u) + G(u, u, u) + G(u, u, u), \\ G(fu, u, u) + G(u, u, u) + G(u, u, u) \end{array} \right\} \\ &= k \max \{2G(u, fu, fu), G(fu, u, u)\} \end{aligned}$$

$$\leq 4kG(fu,u,u),$$

which implies that $G(fu,u,u)=0$ and so $fu=u$.

Step 2. We prove that $u=gu=Bu$.

Since $f(X)\subset B(X)$ and $u=fu\in f(X)$, there is a point $z\in X$ such that $u=fu=Bz$, again by use of condition (2.1), we have

$$\begin{aligned}
&G(f^2x_{3n},gz,hx_{3n+2})\\
&\leq k\max\left\{\begin{array}{c}G(Afx_{3n},f^2x_{3n},f^2x_{3n})+G(Bz,f^2x_{3n},f^2x_{3n})+G(Cx_{3n+2},f^2x_{3n},f^2x_{3n}),\\ G(Afx_{3n},gz,gz)+G(Bz,gz,gz)+G(Cx_{3n+2},gz,gz),\\ G(Afx_{3n},hx_{3n+2},hx_{3n+2})+G(Bz,hx_{3n+2},hx_{3n+2})+G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})\end{array}\right\}.
\end{aligned}$$

Letting $n\to\infty$, using $u=fu=Bz$ and the Proposition 1.11 we have

$$\begin{aligned}
G(u,gz,u)&\leq k\max\left\{\begin{array}{c}G(u,u,u)+G(u,u,u)+G(u,u,u),\\ G(u,gz,gz)+G(u,gz,gz)+G(u,gz,gz),\\ G(u,u,u)+G(u,u,u)+G(u,u,u)\end{array}\right\}\\
&=3kG(u,gz,gz)\\
&\leq 6kG(u,gz,gz),
\end{aligned}$$

which implies that $G(u,gz,u)=0$ and so $gz=u=Bz$.

Since the pair (g,B) is weakly compatible, we have

$$gu=gBz=Bgz=Bu.$$

Again by use of condition (2.1), we have

$$\begin{aligned}
&G(fx_{3n},gu,hx_{3n+2})\\
&\leq k\max\left\{\begin{array}{c}G(Ax_{3n},fx_{3n},fx_{3n})+G(Bu,fx_{3n},fx_{3n})+G(Cx_{3n+2},fx_{3n},fx_{3n}),\\ G(Ax_{3n},gu,gu)+G(Bu,gu,gu)+G(Cx_{3n+2},gu,gu),\\ G(Ax_{3n},hx_{3n+2},hx_{3n+2})+G(Bu,hx_{3n+2},hx_{3n+2})+G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})\end{array}\right\}.
\end{aligned}$$

Letting $n\to\infty$, using $u=fu$, $gu=Bu$ and the Proposition 1.11 we have

$$\begin{aligned}
G(u,gu,u)&\leq k\max\left\{\begin{array}{c}G(u,u,u)+G(gu,u,u)+G(u,u,u),\\ G(u,gu,gu)+G(gu,gu,gu)+G(u,gu,gu),\\ G(u,u,u)+G(gu,u,u)+G(u,u,u)\end{array}\right\}\\
&=k\max\{G(gu,u,u),2G(u,gu,gu)\}\\
&\leq 4kG(u,gu,u).
\end{aligned}$$

This implies that $G(u,gu,u)=0$ and so $gu=u=Bu$.

Step 3. We prove that $u=hu=Cu$.

Since $g(X)\subset C(X)$ and $u=gu\in g(X)$, there is a point $t\in X$ such that $u=gu=Ct$. Again by use of condition (2.1), we have

$$G(fx_{3n},gu,ht)\leq k\max\left\{\begin{array}{c}G(Ax_{3n},fx_{3n},fx_{3n})+G(Bu,fx_{3n},fx_{3n})+G(Ct,fx_{3n},fx_{3n}),\\ G(Ax_{3n},gu,gu)+G(Bu,gu,gu)+G(Ct,gu,gu),\\ G(Ax_{3n},ht,ht)+G(Bu,ht,ht)+G(Ct,ht,ht)\end{array}\right\}.$$

Letting $n \to \infty$, using $u = gu = Bu = Ct$ and the Proposition 1.11, we obtain

$$\begin{aligned} G(u,u,ht) &\leq k\max\left\{\begin{array}{c} G(u,u,u)+G(u,u,u)+G(u,u,u), \\ G(u,u,u)+G(u,u,u)+G(u,u,u), \\ G(u,ht,ht)+G(u,ht,ht)+G(u,ht,ht) \end{array}\right\} \\ &= 3kG(u,ht,ht) \\ &\leq 6kG(u,u,ht). \end{aligned}$$

Hence $G(u,u,ht) = 0$ and so $ht = u = Ct$.

Since the pair (h, C) is weakly compatible, we have

$$hu = hCt = Cht = Cu.$$

Again by use of condition (2.1), we have

$$G(fx_{3n},gu,hu) \leq k\max\left\{\begin{array}{c} G(Ax_{3n},fx_{3n},fx_{3n})+G(Bu,fx_{3n},fx_{3n})+G(Cu,fx_{3n},fx_{3n}), \\ G(Ax_{3n},gu,gu)+G(Bu,gu,gu)+G(Cu,gu,gu), \\ G(Ax_{3n},hu,hu)+G(Bu,hu,hu)+G(Cu,hu,hu) \end{array}\right\}.$$

Letting $n \to \infty$, using $u = gu = Bu$, $Cu = hu$ and the Proposition 1.11, we have

$$\begin{aligned} G(u,u,hu) &\leq k\max\left\{\begin{array}{c} G(u,u,u)+G(u,u,u)+G(hu,u,u), \\ G(u,u,u)+G(u,u,u)+G(hu,u,u), \\ G(u,hu,hu)+G(u,hu,hu)+G(hu,hu,hu) \end{array}\right\} \\ &= k\max\{G(hu,u,u), 2G(u,hu,hu)\} \\ &\leq 4kG(u,u,hu), \end{aligned}$$

which gives that $G(u,u,hu) = 0$ and so $hu = u = Cu$.

Step 4. We prove that $u = Au$.

Since $h(X) \subset A(X)$ and $u = hu \in h(X)$, there is a point $p \in X$ such that $u = hu = Ap$. Again by use of condition (2.1), we have

$$G(fp,gu,hu) \leq k\max\left\{\begin{array}{c} G(Ap,fp,fp)+G(Bu,fp,fp)+G(Cu,fp,fp), \\ G(Ap,gu,gu)+G(Bu,gu,gu)+G(Cu,gu,gu), \\ G(Ap,hu,hu)+G(Bu,hu,hu)+G(Cu,hu,hu) \end{array}\right\}.$$

Using $u = gu = Bu$, $u = hu = Cu = Ap$ and the Proposition 1.11, we obtain

$$\begin{aligned} G(fp,u,u) &\leq k\max\left\{\begin{array}{c} G(u,fp,fp)+G(u,fp,fp)+G(u,fp,fp), \\ G(u,u,u)+G(u,u,u)+G(u,u,u), \\ G(u,u,u)+G(u,u,u)+G(u,u,u) \end{array}\right\} \\ &= 3kG(u,fp,fp) \\ &\leq 6kG(fp,u,u), \end{aligned}$$

which implies that $G(fp,u,u) = 0$ and so $fp = u = Ap$.

Since the pair (f, A) is weakly compatible, we have

$$fu = fAp = Afp = Au = u.$$

Therefore u is the common fixed point of f, g, h, A, B, and C when S is continuous and the pair (f, A) is weakly commuting, the pair (g, B) and (h, C) are weakly compatible.

Similarly we can prove the result that u is a common fixed point of f, g, h, A, B, and C when under the condition of (b) or (c).

Finally we prove uniqueness of common fixed point u.

Let u and q are two common fixed point of f, g, h, A, B, and C, by use of condition (2.1), we have

$$\begin{aligned} G(q,u,u) &= G(fq,gu,hu) \\ &\leq k\max\left\{\begin{array}{c} G(Aq,fq,fq)+G(Bu,fq,fq)+G(Cu,fq,fq), \\ G(Aq,gu,gu)+G(Bu,gu,gu)+G(Cu,gu,gu), \\ G(Aq,hu,hu)+G(Bu,hu,hu)+G(Cu,hu,hu) \end{array}\right\} \\ &= k\max\left\{\begin{array}{c} G(q,q,q)+G(u,q,q)+G(u,q,q), \\ G(q,u,u)+G(u,u,u)+G(u,u,u), \\ G(q,u,u)+G(u,u,u)+G(u,u,u) \end{array}\right\} \\ &= k\max\{2G(u,q,q),G(q,u,u)\} \\ &\leq 6kG(q,u,u), \end{aligned}$$

which implies that $G(q,u,u)=0$ and so $q=u$. Thus common fixed point is unique.

The proof using (2.2) is similar. This completes the proof. □

Remark 2.2. Theorem 2.1 improves and extends the corresponding results of Abbas, Nazir and Saadati [5, Theorem 2.6] from three self-mappings to six self-mappings.

Remark 2.3. In Theorem 2.1, if we take: 1) $f=g=h$; 2) $A=B=C$; 3) $f=g=h$ and $A=B=C$; 4) $g=h$ and $B=C$; 5) $g=h$ and $B=C=I$, several new results can be obtained.

In Theorem 2.1, if we take $A=B=C=I$ (I is identity mapping, the same below), then we have the following corollary.

Corollary 2.4 ([36],Theorem 2.6)**.** *Let (X,G) be a complete G-metric space and let f, g and h are three mappings of X into itself satisfying the following conditions*

$$G(fx,gy,hz) \leq k\max\left\{\begin{array}{c} G(x,fx,fx)+G(y,fx,fx)+G(z,fx,fx), \\ G(x,gy,gy)+G(y,gy,gy)+G(z,gy,gy), \\ G(x,hz,hz)+G(y,hz,hz)+G(z,hz,hz) \end{array}\right\} \tag{2.9}$$

or

$$G(fx,gy,hz) \leq k\max\left\{\begin{array}{c} G(x,x,fx)+G(y,y,fx)+G(z,z,fx), \\ G(x,x,gy)+G(y,y,gy)+G(z,z,gy), \\ G(z,z,hz)+G(y,y,hz)+G(z,z,hz) \end{array}\right\} \tag{2.10}$$

$\forall x,y,z\in X$, where $k\in[0,\frac{1}{6})$. Then f, g, and h have a unique common fixed point in X.

Also, if we take $f=g=h$ and $A=B=C=I$ in Theorem 2.1, then we get the following.

Corollary 2.5. *Let (X,G) be a complete G-metric space and let f be a mapping of X into itself satisfying the following conditions*

$$G(fx,fy,fz) \leq k\max\left\{\begin{array}{c} G(x,fx,fx)+G(y,fx,fx)+G(z,fx,fx), \\ G(x,gy,gy)+G(y,gy,gy)+G(z,gy,gy), \\ G(x,hz,hz)+G(y,hz,hz)+G(z,hz,hz) \end{array}\right\} \tag{2.11}$$

or

$$G(fx,fy,fz) \leq k\max\left\{\begin{array}{c} G(x,x,fx)+G(y,y,fx)+G(z,z,fx), \\ G(x,x,fy)+G(y,y,fy)+G(z,z,fy), \\ G(z,z,fz)+G(y,y,fz)+G(z,z,fz) \end{array}\right\} \tag{2.12}$$

$\forall x,y,z\in X$, where $\alpha\in[0,\frac{1}{6})$. Then f has a unique fixed point in X.

Theorem 2.6. *Let (X,G) be a complete G-metric space and let f, g, h, A, B, and C are six mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) The pairs (f,A), (g,B) and (h,C) are commuting mappings;

(iii) $\forall x,y,z \in X$,

$$G(f^px, g^qy, h^rz) \leq k \max \left\{ \begin{array}{c} G(Ax, f^px, f^px)+G(By, f^px, f^px)+G(Cz, f^px, f^px), \\ G(Ax, g^qy, g^qy)+G(By, g^qy, g^qy)+G(Cz, g^qy, g^qy), \\ G(Ax, h^rz, h^rz)+G(By, h^rz, h^rz)+G(Cz, h^rz, h^rz) \end{array} \right\} \tag{2.13}$$

or

$$G(f^px, g^qy, h^rz) \leq k \max \left\{ \begin{array}{c} G(Ax, Ax, f^px) + G(By, By, f^px) + G(Cz, Cz, f^px), \\ G(Ax, Ax, g^qy) + G(By, By, g^qy) + G(Cz, Cz, g^qy), \\ G(Ax, Ax, h^rz) + G(By, By, h^rz) + G(Cz, Cz, h^rz) \end{array} \right\}, \tag{2.14}$$

where $k \in [0, \frac{1}{6})$, $p,q,r \in \mathbb{N}$, then f, g, h, A, B, and C have a unique common fixed point in X.

Proof. Suppose that mappings f, g, h, A, B, and C satisfies condition (2.13). Since $f^pX \subset f^{p-1}X \subset \cdots \subset fX$, $fX \subset BX$, so that $f^pX \subset BX$. Similarly, we can show that $g^qX \subset CX$ and $h^rX \subset AX$. From the Theorem 2.1, we see that f^p, g^q, h^r, A, B and C have a unique common fixed point u.

Since $fu = f(f^pu) = f^{p+1}u = f^p(fu)$, so that

$$G(f^pfu, g^qu, h^ru) \leq k \max \left\{ \begin{array}{c} G(Afu, f^pfu, f^pfu)+G(Bu, f^pfu, f^pfu)+G(Cu, f^pfu, f^pfu), \\ G(Afu, g^qu, g^qu)+G(Bu, g^qu, g^qu)+G(Cu, g^qu, g^qu), \\ G(Afu, h^ru, h^ru)+G(Bu, h^ru, h^ru)+G(Cu, h^ru, h^ru) \end{array} \right\},$$

note that $Afu = fAu = fu$ and the Proposition 1.11, we obtain

$$\begin{aligned} G(fu,u,u) &\leq k \max \left\{ \begin{array}{c} G(fu,fu,fu) + G(u,fu,fu) + G(u,fu,fu), \\ G(fu,u,u) + G(u,u,u) + G(u,u,u), \\ G(fu,u,u) + G(u,u,u) + G(fu,u,u) \end{array} \right\} \\ &= k \max \{2G(u,fu,fu), G(fu,u,u)\} \\ &\leq 4kG(fu,u,u). \end{aligned}$$

This implies that $G(fu,u,u) = 0$ and so $fu = u$.

By the same argument, we can prove $gu = u$ and $hu = u$. Thus we have $u = fu = gu = hu = Au = Bu = Cu$, so that f, g, h, A, B and C have a common fixed point u in X. Let v be any other common fixed point of f, g, h, A, B and C, then use of condition (2.13), we have

$$\begin{aligned} G(u,u,v) &= G(f^pu, g^qu, h^rv) \\ &\leq k \max \left\{ \begin{array}{c} G(Au, f^pu, f^pu) + G(Bu, f^pu, f^pu) + G(Cv, f^pu, f^pu), \\ G(Au, g^qu, g^qu) + G(Bu, g^qu, g^qu) + G(Cv, g^qu, g^qu), \\ G(Au, h^rv, h^rv) + G(Bu, h^rv, h^rv) + G(Cv, h^rv, h^rv) \end{array} \right\} \\ &= k \max \left\{ \begin{array}{c} G(u,u,u) + G(u,u,u) + G(v,u,u), \\ G(u,u,u) + G(u,u,u) + G(v,u,u), \\ G(u,v,v) + G(u,v,v) + G(v,v,v) \end{array} \right\} \\ &\leq k \max \{G(v,u,u), 2G(u,v,v)\} \\ &\leq 4kG(u,u,v), \end{aligned}$$

which implies that $G(u,u,v) = 0$ and so $u = v$. Thus common fixed point is unique.

The proof using (2.14) is similar. This completes the proof. $\square$

Remark 2.7. Theorem 2.6 improves and extends the corresponding results in of Abbas, Nazir and Saadati [5, Corollary 2.8] from three self-mappings to six self-mappings.

Remark 2.8. In Theorem 2.6, if we take: 1) $f = g = h$; 2) $A = B = C$; 3) $f = g = h$ and $A = B = C$; 4) $g = h$ and $B = C$; 5) $g = h$ and $B = C = I$; 6) $p = q = r$, several new result can be obtained.

In Theorem 2.6, if we take $A = B = C = I$, then we have the following corollary.

Corollary 2.9. *Let (X, G) be a complete G-metric space and let f, g, and h are three mappings of X into itself satisfying the following conditions*

$$G(f^px, g^qy, h^rz) \le k \max \left\{ \begin{array}{c} G(x, f^px, f^px)+G(y, f^px, f^px)+G(z, f^px, f^px), \\ G(x, g^qy, g^qy)+G(By, g^qy, g^qy)+G(z, g^qy, g^qy), \\ G(, h^rz, h^rz)+G(y, h^rz, h^rz)+G(z, h^rz, h^rz) \end{array} \right\} \tag{2.15}$$

or

$$G(f^px, g^qy, h^rz) \le k \max \left\{ \begin{array}{c} G(x, x, f^px) + G(y, y, f^px) + G(z, z, f^px), \\ G(x, x, g^qy) + G(y, y, g^qy) + G(z, z, g^qy), \\ G(x, x, h^rz) + G(y, y, h^rz) + G(z, z, h^rz) \end{array} \right\} \tag{2.16}$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{6})$, $p, q, r \in \mathbb{N}$, then f, g and h have a unique common fixed point in X.

Remark 2.10. If $p = q = r = m$, the Corollary 2.9 is reduced to Corollary 2.8 of Abbas, Nazir and Saadati [5].

Also, if we take $f = g = h$ and $A = B = C = I$ in Theorem 2.6, then we get the following.

Corollary 2.11. *Let (X, G) be a complete G-metric space and let f be a mapping of X into itself satisfying the following conditions*

$$G(f^px, f^qy, f^rz) \le k \max \left\{ \begin{array}{c} G(x, f^px, f^px)+G(y, f^px, f^px)+G(z, f^px, f^px), \\ G(x, f^qy, f^qy)+G(By, f^qy, f^qy)+G(z, f^qy, f^qy), \\ G(z, f^rz, f^rz)+G(y, f^rz, f^rz)+G(z, f^rz, f^rz) \end{array} \right\} \tag{2.17}$$

or

$$G(f^px, f^qy, f^rz) \le k \max \left\{ \begin{array}{c} G(x, x, f^px) + G(y, y, f^px) + G(z, z, f^px), \\ G(x, x, f^qy) + G(y, y, f^qy) + G(z, z, f^qy), \\ G(x, x, f^rz) + G(y, y, f^rz) + G(z, z, f^rz) \end{array} \right\} \tag{2.18}$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{6})$, $p, q, r \in \mathbb{N}$, then f has a unique fixed point in X.

Corollary 2.12. *Let (X, G) be a complete G-metric space and let f, g, h, A, B and C are six mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) $\forall x, y, z \in X$,

$$\begin{aligned} G(fx, gy, hz) \le\ & a\{G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx)\} \\ & + b\{G(Ax, gy, gy) + G(By, gy, gy) + G(Cz, gy, gy)\} \\ & + c\{G(Ax, hz, hz) + G(By, hz, hz) + G(Cz, hz, hz)\} \end{aligned} \tag{2.19}$$

or

$$\begin{aligned} G(fx, gy, hz) \le\ & a\{G(Ax, Ax, fx) + G(By, By, fx) + G(Cz, Cz, fx)\} \\ & + b\{G(Ax, Ax, gy) + G(By, By, gy) + G(Cz, Cz, gy)\} \\ & + c\{G(Ax, Ax, hz) + G(By, By, hz) + G(Cz, Cz, hz)\}, \end{aligned} \tag{2.20}$$

where $0 \le a + b + c < \frac{1}{6}$. Then one of the pairs (f, A), (g, B) and (h, C) has a coincidence point in X. Moreover, if one of the following conditions is satisfied:

(a) Either f or A is G-continuous, the pair (f, A) is weakly commuting, the pairs (g, B) and (h, C) are weakly compatible;

(b) Either g or B is G-continuous, the pair (g, B) is weakly commuting, the pairs (f, A) and (h, C) are weakly compatible;

(c) Either h or C is G-continuous, the pair (h, C) is weakly commuting, the pairs (f, A) and (g, B) are weakly compatible.

Then The mappings f, g, h, A, B and C have a unique common fixed point in X.

Proof. Suppose that mappings f, g, h, A, B and C satisfies condition (2.19). For $x, y, z \in X$, let

$$M(x,y,z) = \max\left\{\begin{array}{c} G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx),\\ G(Ax,gy,gy)+G(By,gy,gy)+G(Cz,gy,gy),\\ G(Ax,hz,hz)+G(By,hz,hz)+G(Cz,hz,hz) \end{array}\right\}.$$

Then

$$\begin{aligned} &a\{G(Ax,Ax,fx)+G(By,By,fx)+G(Cz,Cz,fx)\}\\ &\quad +b\{G(Ax,Ax,gy)+G(By,By,gy)+G(Cz,Cz,gy)\}\\ &\quad +c\{G(Ax,Ax,hz)+G(By,By,hz)+G(Cz,Cz,hz)\}\\ &\le (a+b+c)M(x,y,z). \end{aligned}$$

So, if

$$\begin{aligned} G(fx,gy,hz) \le\, &a\{G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx)\}\\ &+b\{G(Ax,gy,gy)+G(By,gy,gy)+G(Cz,gy,gy)\}\\ &+c\{G(Ax,hz,hz)+G(By,hz,hz)+G(Cz,hz,hz)\} \end{aligned}$$

then $G(fx, gy, hz) \le (a+b+c)M(x, y, z)$. Taking $k = a+b+c$ in Theorem 2.1, the conclusion of Corollary 2.12 can be obtained from Theorem 2.1 immediately.

The proof using (2.20) is similar. This completes the proof. □

Remark 2.13. In Corollary 2.12, if we take: 1) $f = g = h$; 2) $A = B = C$; 3) $f = g = h$ and $A = B = C$; 4) $g = h$ and $B = C$; 5) $g = h$ and $B = C = I$, several new results can be obtained.

Corollary 2.14. *Let (X, G) be a complete G-metric space and let f, g, h, A, B and C are six mappings of X into itself satisfying the following conditions:*
(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;
(ii) The pairs (f, A), (g, B) and (h, C) are commuting mappings;
(iii) $\forall x, y, z \in X$,

$$\begin{aligned} G(f^px,g^qy,h^rz) \le\, &a\{G(Ax,f^px,f^px)+G(By,f^px,f^px)+G(Cz,f^px,f^px)\}\\ &+b\{G(Ax,g^qy,g^qy)+G(By,g^qy,g^qy)+G(Cz,g^qy,g^qy)\}\\ &+c\{G(Ax,h^rz,h^rz)+G(By,h^rz,h^rz)+G(Cz,h^rz,h^rz)\} \end{aligned} \tag{2.21}$$

or

$$\begin{aligned} G(f^px,g^qy,h^rz) \le\, &a\{G(Ax,Ax,f^px)+G(By,By,f^px)+G(Cz,Cz,f^px)\}\\ &+b\{G(Ax,Ax,g^qy)+G(By,By,g^qy)+G(Cz,Cz,g^qy)\}\\ &+c\{G(Ax,Ax,h^rz)+G(By,By,h^rz)+G(Cz,Cz,h^rz)\}, \end{aligned} \tag{2.22}$$

where $0 \le a+b+c < \frac{1}{6}$, $p, q, r \in \mathbb{N}$, then f, g, h, A, B, and C have a unique common fixed point in X.

Proof. The proof follows from Corollary 2.12, and from an argument similar to that used in Theorem 2.6. □

Remark 2.15. In Corollary 2.14, if we take: 1) $f = g = h$; 2) $A = B = C$; 3) $f = g = h$ and $A = B = C$; 4) $g = h$ and $B = C$; 5) $g = h$ and $B = C = I$; 6) $p = q = r$, several new results can be obtained.

Now we introduce an example to support Theorem 2.1.

Example 2.16. Let $X = [0, 1]$, and (X, G) be a G-metric space defined by $G(x, y, z) = |x-y|+|y-z|+|z-x|$ for all x, y, z in X. Let f, g, h, A, B and C be self mappings defined by

$$fx = \frac{18}{19}, \quad gx = \begin{cases} \frac{20}{21}, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1]. \end{cases}, \quad hx = \begin{cases} \frac{19}{20}, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

$$Ax = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1), \\ \frac{19}{20}, & x = 1. \end{cases}, \quad Bx = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1]. \end{cases}, \quad Cx = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{18}{19}, & x \in (\frac{1}{2}, 1), \\ \frac{20}{21}, & x = 1. \end{cases}$$

Note that f is G-continuous in X, and g, h, A, B and C are not G-continuous in X.

Clearly we can get $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$.

By the definition of the mappings of f and A, for all $x \in [0, 1]$, we have

$$G(fAx, Afx, Afx) = G\left(\frac{18}{19}, \frac{18}{19}, \frac{18}{19}\right) = 0 \leq G(fx, Ax, Ax),$$

so we can get the pair (f, A) is weakly commuting.

By the definition of the mappings of g and B, only for $x \in (\frac{1}{2}, 1]$, $gx = Bx$, at this time $gBx = g(\frac{18}{19}) = \frac{18}{19} = B(\frac{18}{19}) = Bgx$, so $gBx = Bgx$, so we can obtain the pair (g, B) is weakly compatible. Similarly we can proof the pair (h, C) is also weakly compatible.

Now we proof the mappings f, g, h, A, B and C are satisfying the condition (2.1) of Theorem 2.1 with $k = \frac{2}{21} \in [0, \frac{1}{6})$. Let

$$M(x, y, z) = \max \left\{ \begin{array}{c} G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx), \\ G(Ax, gy, gy) + G(By, gy, gy) + G(Cz, gy, gy), \\ G(Ax, hz, hz) + G(By, hz, hz) + G(Cz, hz, hz) \end{array} \right\}.$$

Case 1. If $x, y, z \in \left[0, \frac{1}{2}\right]$, then

$$G(fx, gy, hz) = G\left(\frac{18}{19}, \frac{20}{21}, \frac{19}{20}\right) = \frac{4}{399},$$

$$\begin{aligned} &G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx) \\ &= G\left(1, \frac{18}{19}, \frac{18}{19}\right) + G\left(1, \frac{18}{19}, \frac{18}{19}\right) + G\left(1, \frac{18}{19}, \frac{18}{19}\right) \\ &= \frac{6}{19}. \end{aligned}$$

Thus we have

$$\begin{aligned} G(fx, gy, hz) = \frac{4}{399} &< \frac{2}{21} \cdot \frac{6}{19} \\ &= \frac{2}{21}(G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx)) \\ &\leq \frac{2}{21} M(x, y, z). \end{aligned}$$

Case 2. If $x, y \in \left[0, \frac{1}{2}\right], z \in \left(\frac{1}{2}, 1\right]$, then

$$G(fx, gy, hz) = G\left(\frac{18}{19}, \frac{20}{21}, \frac{18}{19}\right) = \frac{4}{399},$$

$$\begin{aligned}
&G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx) \\
&\geq G(Ax, fx, fx) + G(By, fx, fx) \\
&= G\left(1, \frac{18}{19}, \frac{18}{19}\right) + G\left(1, \frac{18}{19}, \frac{18}{19}\right) \\
&= \frac{4}{19}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
G(fx, gy, hz) &= \frac{4}{399} < \frac{2}{21} \cdot \frac{4}{19} \\
&\leq \frac{2}{21}(G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx)) \\
&\leq \frac{2}{21} M(x, y, z).
\end{aligned}$$

Case 3. If $x, z \in \left[0, \frac{1}{2}\right], y \in \left(\frac{1}{2}, 1\right]$, then

$$G(fx, gy, hz) = G\left(\frac{18}{19}, \frac{18}{19}, \frac{19}{20}\right) = \frac{1}{190},$$

$$\begin{aligned}
&G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx) \\
&= G\left(1, \frac{18}{19}, \frac{18}{19}\right) + G\left(\frac{18}{19}, \frac{18}{19}, \frac{18}{19}\right) + G\left(1, \frac{18}{19}, \frac{18}{19}\right) \\
&= \frac{4}{19}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
G(fx, gy, hz) &= \frac{1}{190} < \frac{2}{21} \cdot \frac{4}{19} \\
&= \frac{2}{21}(G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx)) \\
&\leq \frac{2}{21} M(x, y, z).
\end{aligned}$$

Case 4. If $y, z \in \left[0, \frac{1}{2}\right], x \in \left(\frac{1}{2}, 1\right]$, then

$$G(fx, gy, hz) = G\left(\frac{18}{19}, \frac{20}{21}, \frac{19}{20}\right) = \frac{4}{399},$$

$$\begin{aligned}
&G(Ax, fx, fx) + G(By, fx, fx) + G(Cz, fx, fx) \\
&\geq G(By, fx, fx) + G(Cz, fx, fx) \\
&= G\left(1, \frac{18}{19}, \frac{18}{19}\right) + G\left(1, \frac{18}{19}, \frac{18}{19}\right) \\
&= \frac{4}{19}.
\end{aligned}$$

So we get

$$\begin{aligned}G(fx,gy,hz) &= \frac{4}{399} < \frac{2}{21}\cdot\frac{4}{19}\\ &\leq \frac{2}{21}(G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx))\\ &\leq \frac{2}{21}M(x,y,z).\end{aligned}$$

Case 5. If $x \in \left[0,\frac{1}{2}\right], y,z \in \left(\frac{1}{2},1\right]$, then

$$G(fx,gy,hz) = G\left(\frac{18}{19},\frac{18}{19},\frac{18}{19}\right) = 0 \leq \frac{2}{21}M(x,y,z).$$

Case 6. If $y \in \left[0,\frac{1}{2}\right], x,z \in \left(\frac{1}{2},1\right]$, then

$$G(fx,gy,hz) = G\left(\frac{18}{19},\frac{20}{21},\frac{18}{19}\right) = \frac{4}{399},$$

$$G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx) \geq G(By,fx,fx) = G\left(1,\frac{18}{19},\frac{18}{19}\right) = \frac{2}{19}.$$

Thus we have

$$\begin{aligned}G(fx,gy,hz) &= \frac{4}{399} = \frac{2}{21}\cdot\frac{2}{19}\\ &\leq \frac{2}{21}(G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx))\\ &\leq \frac{2}{21}M(x,y,z).\end{aligned}$$

Case 7. If $z \in \left[0,\frac{1}{2}\right], x,y \in \left(\frac{1}{2},1\right]$, then

$$G(fx,gy,hz) = G\left(\frac{18}{19},\frac{18}{19},\frac{19}{20}\right) = \frac{1}{190},$$

$$\begin{aligned}&G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx)\\ &\quad\geq G(By,fx,fx)+G(Cz,fx,fx)\\ &\quad= G\left(\frac{18}{19},\frac{18}{19},\frac{18}{19}\right)+G\left(1,\frac{18}{19},\frac{18}{19}\right)\\ &\quad= \frac{2}{19}.\end{aligned}$$

Hence we have

$$\begin{aligned}G(fx,gy,hz) &= \frac{1}{190} < \frac{2}{21}\cdot\frac{2}{19}\\ &\leq \frac{2}{21}(G(Ax,fx,fx)+G(By,fx,fx)+G(Cz,fx,fx))\\ &\leq \frac{2}{21}M(x,y,z).\end{aligned}$$

Case 8. If $x,y,z \in \left(\frac{1}{2},1\right]$, then

$$G(fx,gy,hz) = G\left(\frac{18}{19},\frac{18}{19},\frac{18}{19}\right) = 0 \leq \frac{2}{21}M(x,y,z).$$

Then in all the above cases, the mappings f,g,h,A,B, and C are satisfying the condition (2.1) of the Theorem 2.1 with $k=\frac{2}{21}$. So that all the conditions of Theorem 2.1 are satisfied. Moreover, $\frac{18}{19}$ is the unique common fixed point for all of the mappings f,g,h,A,B, and C.

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