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Positive solutions for nonlinear fractional semipositone differential equation with nonlocal boundary conditions

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Abstract

In this paper, we study the existence of positive solutions to the nonlinear fractional order singular and semipositone nonlocal boundary value problem

$$\begin{cases} \mathscr{D}_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \mu \int_0^1 u(s) ds \end{cases}$$

by using the Leray-Schauder nonlinear alternative and a fixed-point theorem on cones, where $0 < \mu < \alpha$, $2 \leq n-1 < \alpha \leq n$, $\mathscr{D}_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative, and $f(t, u)$ is semipositone and may be singular at $u = 0$.

Keywords: Singular fractional differential equation, semipositone, positive solutions, nonlocal boundary conditions.
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1. Introduction

The purpose of this paper is to establish the existence and multiplicity of positive solutions to following singular semipositone fractional differential equation with nonlocal conditions

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$$\begin{cases} \mathscr{D}_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \mu \int_0^1 u(s)ds, \end{cases} \tag{1.1}$$

where $0 < \mu < \alpha$, $2 \leq n - 1 < \alpha \leq n$, $\mathscr{D}_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f(t, u) : [0, 1] \times (0, +\infty) \to \mathbb{R}$ is continuous and $f(t, u)$ may also have singularity at $u = 0$.

Fractional differential equations have been of great interest in the last few decades. This is due to the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in rheology, dynamical processes in selfsimilar and porous structures, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science. For more details on this theory and its applications, we refer to references [1–4, 6, 8–12, 14–20]. Recently, many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using the techniques of nonlinear analysis. In [16], the authors investigated the properties of Green's function for the nonlinear fractional differential equation with three-point boundary condition

$$\begin{cases} \mathscr{D}_{0^+}^{\alpha} u(t) + f(t, u(t)) + e(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad \mathscr{D}_{0^+}^{\beta} u(1) = a\mathscr{D}_{0^+}^{\beta} u(\xi), \end{cases} \tag{1.2}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 \leq a \leq 1$, $0 < \xi < 1$, $\alpha - \beta - 1 \geq 0$, $e \in L^1[0, 1]$, $f : (0, 1) \times (0, +\infty) \to (0, +\infty)$ satisfies the Caratheodory conditions, $\mathscr{D}_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative. The authors present some existence results of positive solutions for singular boundary value problems (BVPs) (1.2) by means of the Schauder fixed-point Theorem. By using the Leray-Schauder nonlinear alternative and a fixed-point theorem on cones, Xu, Jiang and Yuan [17] investigated the existence of multiple positive solutions to positone and semipositone Dirichlet-type BVPs of the nonlinear fractional differential equations:

$$\begin{cases} \mathscr{D}_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $\mathscr{D}_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f(t, u)$ may be singular at $u = 0$.

Motivated by the above papers and [7], the aim of this paper is to establish the existence and multiplicity of positive solutions of BVP (1.1). We obtain the existence of positive solutions by means of the Leray-Schauder nonlinear alternative and a fixed-point theorem on cones. Our work presented in this paper has the following features. First of all, BVP (1.1) possesses singularity, that is, $f(t, u)$ may be singular at $u = 0$. And the nonlinearity f is semipositone. The second new feature is that we consider the general integral boundary conditions, which include two-point, three-point, multi-point and some nonlocal conditions as special cases. Thirdly, we consider the high order nonlinear fractional differential equation and we obtain the existence and multiplicity of positive solutions of BVP (1.1). Moreover, in this paper, it is possible to replace the Riemann integrals in the boundary conditions by Riemann-Stieltjes integrals with minor modifications.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. We also develop some properties of Green's function. In Section 3, we discuss the existence and multiplicity of positive solutions of BVP (1.1).

2. Preliminaries and lemmas

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results. For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in recent literature.

Definition 2.1 ([8])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s)ds,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([8])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathscr{D}_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 ([8])**.** *Let $\alpha \geq 0$. Then the following equality holds for $u \in L^1(0,1)$ and $D_{0+}^{\alpha} u \in L^1(0,1)$:*

$$I_{0+}^{\alpha} \mathscr{D}_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, $n-1 < \alpha \leq n$.

Lemma 2.4 ([18])**.** *Let $y \in C[0,1]$ be a given function, then the boundary value problem*

$$\begin{cases} \mathscr{D}_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \ 2 \leq n-1 < \alpha \leq n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \mu \displaystyle\int_0^1 u(s) ds, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1}(\alpha-\mu+\mu s) - (\alpha-\mu)(t-s)^{\alpha-1}}{(\alpha-\mu)\Gamma(\alpha)}, \quad 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{\alpha-1}(\alpha-\mu+\mu s)}{(\alpha-\mu)\Gamma(\alpha)}, \quad 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Here $G(t,s)$ is called Green's function of boundary value problem (2.1)*.*

Lemma 2.5 ([18])**.** *The function $G(t,s)$ defined by* (2.2) *has the following properties:*

$$\mu t^{\alpha-1}\Phi(s) \leq G(t,s) \leq \frac{M_0}{(\alpha-\mu)\Gamma(\alpha)} t^{\alpha-1}, \quad G(t,s) \leq M_0 \Phi(s), \quad t, s \in [0,1],$$

where $M_0 = (\alpha-\mu)(\alpha-1) + \alpha + \mu$ and $\Phi(s) = \frac{1}{(\alpha-\mu)\Gamma(\alpha)} s(1-s)^{\alpha-1}$.

Lemma 2.6. *Suppose $e \in C[0,1]$, then*

$$\begin{cases} \mathscr{D}_{0+}^{\alpha} x(t) + e(t) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \mu \displaystyle\int_0^1 x(s) ds \end{cases}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s) e(s) ds, \quad and \quad x(t) \leq C_0 t^{\alpha-1}, \quad t \in [0,1],$$

where $C_0 = \frac{M_0}{(\alpha-\mu)\Gamma(\alpha)} \int_0^1 e(s) ds$.

Proof. From Lemma 2.4, we have

$$x(t)=\int_0^1 G(t,s)e(s)ds,\quad t\in[0,1].$$

According to Lemma 2.5, we get

$$x(t)=\int_0^1 G(t,s)e(s)ds\le t^{\alpha-1}\frac{M_0}{(\alpha-\mu)\Gamma(\alpha)}\int_0^1 e(s)ds=C_0t^{\alpha-1},\quad t\in[0,1].$$

□

Theorem 2.7 ([13])**.** *Let $1\le p\le\infty$ be a constant and q be such that $\frac{1}{p}+\frac{1}{q}=1$. Assume $f\in C[0,1]$,*

$$\begin{cases} g:[0,1]\times\mathbb{R}\to\mathbb{R} \text{ is a } L^q\text{-Caratheodory function. By this we mean:}\\ (i) \text{ the map } y\mapsto g(t,y) \text{ is continuous for almost all } t\in[0,1],\\ (ii) \text{ the map } t\mapsto g(t,y) \text{ is measurable for all } y\in\mathbb{R},\\ (iii) \text{ for any } r>0 \text{ there exists } \mu_r\in L^q[0,1] \text{ such that } |y|\le r \text{ implies}\\ |g(t,y)|\le\mu_r(t) \text{ for almost all } t\in[0,1]. \end{cases}$$

$$K_t(s)=K(t,s)\in L^p[0,1] \text{ for each } t\in[0,1]$$

and

$$\text{the map } t\mapsto K_t \text{ is continuous from } [0,1] \text{ to } L^p[0,1]$$

hold. In addition, suppose there is a constant $M>|f|_0=\sup_{[0,1]}|f(t)|$, independent of λ, with $|y|_0=\sup_{[0,1]}|y(t)|\ne M$ for any solution $y\in C[0,1]$ to

$$y(t)=f(t)+\lambda\int_0^1 K(t,s)g(s,y(s))ds,\quad t\in[0,1]$$

for each $\lambda\in(0,1]$. Then

$$y(t)=f(t)+\int_0^1 K(t,s)g(s,y(s))ds,\quad t\in[0,1]$$

has at least one solution $y\in C[0,1]$ with $|y|_0<M$.

Theorem 2.8 ([5])**.** *Let X be a Banach space, and let $P\subset X$ be a cone in X. Assume Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let $S:P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that, either*

(1) $\|Sw\|\le\|w\|,\ w\in P\cap\partial\Omega_1,\ \|Sw\|\ge\|w\|,\ w\in P\cap\partial\Omega_2$, *or*

(2) $\|Sw\|\ge\|w\|,\ w\in P\cap\partial\Omega_1,\ \|Sw\|\le\|w\|,\ w\in P\cap\partial\Omega_2$.

Then S has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Main results

Let $E=C[0,1]$, then E is a Banach space endowed with the norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$. Let $P=\{u\in E:u(t)\ge0,\ t\in[0,1]\}$, $K=\left\{u\in P:u(t)\ge\frac{\mu t^{\alpha-1}}{M_0}\|u\|,\ t\in[0,1]\right\}$. It is easy to see that P and K are cones in E. For any $0<r<+\infty$, let $\Omega=\{u\in K:\|u\|<r\}$, $\partial\Omega=\{u\in K:\|u\|=r\}$, $\overline{\Omega}=\{u\in K:\|u\|\le r\}$.

Theorem 3.1. *Suppose the following conditions are satisfied:*

$$\begin{cases} f:[0,1]\times(0,+\infty)\to\mathbb{R} \text{ is continuous and there exists a function } e\in C[0,1], \\ e(t)>0 \text{ for } t\in(0,1), \text{ with } f(t,u)+e(t)\geq 0 \text{ for } (t,u)\in[0,1]\times(0,+\infty); \end{cases} \tag{3.1}$$

$$\begin{cases} f^*(t,u)=f(t,u)+e(t)\leq q(t)[g(u)+h(u)] \text{ on } [0,1]\times(0,+\infty) \text{ with} \\ g>0 \text{ continuous and nonincreasing on } (0,+\infty), h\geq 0 \text{ continuous on } [0,+\infty), \\ \dfrac{h}{g} \text{ nondecreasing on } (0,+\infty), \text{ and } q\in L^1[0,1], \; q(t)>0 \text{ on } (0,1); \end{cases} \tag{3.2}$$

$$\exists K_0>0, \text{ with } g(xy)\leq K_0 g(x)g(y), \quad \forall x>0, \; y>0; \tag{3.3}$$

$$b_0=\int_0^1 q(s)g(s^{\alpha-1})ds<+\infty; \tag{3.4}$$

$$\exists r>\frac{M_0}{\mu}C_0 \quad \text{with} \qquad \frac{r}{g(\frac{\mu r}{M_0}-C_0)\left\{1+\frac{h(r)}{g(r)}\right\}}>M_0K_0a_0, \tag{3.5}$$

where

$$a_0=\int_0^1 \Phi(s)q(s)g(s^{\alpha-1})ds<\infty; \tag{3.6}$$

$$\begin{cases} \text{there exist } 0<\theta<1 \text{ and a continuous, nonincreasing function } g_1:(0,+\infty)\to(0,+\infty), \\ \text{and a continuous function } h_1:[0,+\infty)\to(0,+\infty) \text{with } \dfrac{h_1}{g_1} \text{ nondecreasing on } (0,+\infty) \\ \text{and } f^*(t,u)\geq q_1(t)[g_1(u)+h_1(u)] \text{ for } (t,u)\in[\theta,1-\theta]\times(0,+\infty), q_1\in C([0,1],[0,+\infty)), \end{cases} \tag{3.7}$$

and $\exists R>r$ *with*

$$\frac{Rg_1(\epsilon\theta^{\alpha-1}R)}{g_1(R)g_1(\epsilon\theta^{\alpha-1}R)+g_1(R)h_1(\epsilon\theta^{\alpha-1}R)}\leq\int_\theta^{1-\theta}G(\sigma,s)q_1(s)ds, \tag{3.8}$$

$$\text{here } \epsilon>0 \text{ is any constant so that } \frac{\mu}{M_0}-\frac{C_0}{R}\geq\epsilon \text{ and } \int_\theta^{1-\theta}G(\sigma,s)ds=\sup_{t\in[0,1]}\int_\theta^{1-\theta}G(t,s)ds.$$

Then BVP (1.1) *has a solution* u *with* $u(t)>0$ *for* $t\in(0,1)$.

Proof. To show BVP (1.1) has a nonnegative solution we will look at the boundary value problem

$$\begin{cases} \mathscr{D}_{0+}^{\alpha}y(t)+f^*\Big(t,y(t)-x(t)\Big)=0, \quad 0<t<1, \\ y(0)=y'(0)=\cdots=y^{(n-2)}(0)=0, \quad y(1)=\mu\displaystyle\int_0^1 y(s)ds, \end{cases} \tag{3.9}$$

where x is as in Lemma 2.6.

We will show, using Theorem 2.8, that there exists a solution $y_1(t)$ to BVP (3.9) with $y_1(t)>x(t)$ for $t\in(0,1)$. If this is true, then $u(t)=y_1(t)-x(t)$ is a positive solution of BVP (1.1), since

$$\begin{aligned} \mathscr{D}_{0+}^{\alpha}u(t)&=\mathscr{D}_{0+}^{\alpha}y_1(t)-\mathscr{D}_{0+}^{\alpha}x(t)=-f^*(t,y_1(t)-x(t))+e(t) \\ &=-[f(t,y_1(t)-x(t))+e(t)]+e(t)=-f(t,u(t)), \quad 0<t<1. \end{aligned}$$

As a result, we will concentrate our study on BVP (3.9). Suppose that $y(t)$ is a solution of BVP (3.9), then

$$y(t)=\int_0^1 G(t,s)f^*(s,y(s)-x(s))ds, \quad 0<t<1.$$

Let

$$\Omega_1 = \{u \in K : \|u\| < r\}, \quad \Omega_2 = \{u \in K : \|u\| < R\},$$

and let $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to E$ be defined by

$$(Ty)(t) = \int_0^1 G(t,s) f^*(s, y(s) - x(s)) ds, \quad 0 < t < 1.$$

First, we show that T is well defined. For each $y \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, we have $r \le \|y\| \le R$ and $y(t) \ge \frac{\mu t^{\alpha-1}}{M_0}\|y\| \ge \frac{\mu t^{\alpha-1}}{M_0} r$ for $0 \le t \le 1$. For each $t \in [0,1]$, we have

$$R \ge y(t) - x(t) \ge \frac{\mu t^{\alpha-1}}{M_0} r - t^{\alpha-1} C_0 = t^{\alpha-1}\left(\frac{\mu r}{M_0} - C_0\right),$$

and by (3.2), we see that

$$\begin{aligned} f^*(t, y(t) - x(t)) =& f(t, y(t) - x(t)) + e(t) \\ \le& q(t)[g(y(t) - x(t)) + h(y(t) - x(t))] \\ =& q(t) g(y(t) - x(t)) \left\{1 + \frac{h(y(t) - x(t))}{g(y(t) - x(t))}\right\} \\ \le& K_0 q(t) g\left(\frac{\mu r}{M_0} - C_0\right) g(t^{\alpha-1}) \left\{1 + \frac{h(R)}{g(R)}\right\}. \end{aligned}$$

Thus, using above inequalities together with (3.4), we deduce that $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to E$ is well defined. Note that $y \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$ also implies that

$$\|Ty\| \le M_0 \int_0^1 \Phi(s) f^*(s, y(s) - x(s)) ds,$$

and

$$(Ty)(t) \ge \mu t^{\alpha-1} \int_0^1 \Phi(s) f^*(s, y(s) - x(s)) ds \ge \frac{\mu t^{\alpha-1}}{M_0}\|Ty\|, \quad t \in [0,1].$$

Thus, we conclude that $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$.

Next, we show that $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ is continuous and compact. Let $y_n, y \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$ with $\|y_n - y\| \to 0$ as $n \to \infty$, then $r \le \|y_n\| \le R$, $r \le \|y\| \le R$, $y_n(t) \ge \frac{\mu t^{\alpha-1}}{M_0} r$, and $y(t) \ge \frac{\mu t^{\alpha-1}}{M_0} r$, for $0 \le t \le 1$. Notice also that $y_n(s) - x(s) \ge s^{\alpha-1}\left(\frac{\mu r}{M_0} - C_0\right)$ and $y(s) - x(s) \ge s^{\alpha-1}\left(\frac{\mu r}{M_0} - C_0\right)$ for $s \in [0,1]$, so

$$\rho_n(s) := |f^*(s, y_n(s) - x(s)) - f^*(s, y(s) - x(s))| \to 0, \quad \text{as } n \to \infty,$$

and

$$\rho_n(s) \le 2K_0 q(s) g\left(\frac{\mu r}{M_0} - C_0\right) g(s^{\alpha-1}) \left\{1 + \frac{h(R)}{g(R)}\right\}.$$

By a direct application of Lebesgue dominated convergence Theorem,

$$\|Ty_n - Ty\| \le \sup_{t \in [0,1]} \int_0^1 G(t,s) \rho_n(s) ds \to 0, \text{ as } n \to \infty.$$

Therefore, $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ is continuous.

We now show that T is uniformly bounded. For $y \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, we have

$$\|Ty\| = \sup_{t \in [0,1]} \int_0^1 G(t,s) f^*(s, y(s) - x(s)) ds$$

$$
\begin{aligned}
&\le M_0 \int_0^1 \Phi(s) q(s) g(y(s)-x(s)) \left\{ 1 + \frac{h(y(s)-x(s))}{g(y(s)-x(s))} \right\} ds \\
&\le M_0 \int_0^1 \Phi(s) q(s) K_0 g\left(\frac{\mu r}{M_0} - C_0 \right) g(s^{\alpha-1}) \left\{ 1 + \frac{h(R)}{g(R)} \right\} ds \\
&= M_0 K_0 g\left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_0^1 \Phi(s) q(s) g(s^{\alpha-1}) ds \\
&= a_0 M_0 K_0 g\left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(R)}{g(R)} \right\} < +\infty.
\end{aligned}
$$

Hence, $T(K \cap (\overline{\Omega_2} \backslash \Omega_1))$ is bounded.

We next show that T is equicontinuous. For each $\varepsilon > 0$, $y \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$, $t, t' \in [0,1]$, $t < t'$, since $G(t,s)$ is uniformly continuous on $(t,s) \in [0,1] \times [0,1]$, there exists $\eta > 0$ such that for $t' - t < \eta$ we have

$$
|G(t',s) - G(t,s)| < \frac{\varepsilon}{K_0 b_0 g\left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(R)}{g(R)} \right\}},
$$

and

$$
\begin{aligned}
|(Ty)(t') - (Ty)(t)| &\le \int_0^1 \left| G(t',s) - G(t,s) \right| f^*(s, y(s) - x(s)) ds \\
&\le \int_0^1 \left| G(t',s) - G(t,s) \right| q(s) K_0 g\left(\frac{\mu r}{M_0} - C_0 \right) g(s^{\alpha-1}) \left\{ 1 + \frac{h(R)}{g(R)} \right\} ds \\
&= K_0 g\left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_0^1 \left| G(t',s) - G(t,s) \right| q(s) g(s^{\alpha-1}) ds < \varepsilon.
\end{aligned}
$$

By means of the Arzela-Ascoli Theorem, $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ is compact.

We claim that

$$
\|Ty\| \le \|y\|, \quad y \in K \cap \partial\Omega_1. \tag{3.10}
$$

In fact, for $y \in K \cap \partial\Omega_1$, we get $\|y\| = r$ and

$$
y(t) - x(t) \ge t^{\alpha-1} \left(\frac{\mu r}{M_0} - C_0 \right) > 0, \quad t \in (0,1),
$$

from which we obtain that

$$
\begin{aligned}
(Ty)(t) &= \int_0^1 G(t,s) f^*(s, y(s) - x(s)) ds \\
&\le M_0 \int_0^1 \Phi(s) q(s) [g(y(s) - x(s)) + h(y(s) - x(s))] ds \\
&\le M_0 \int_0^1 \Phi(s) q(s) g(y(s) - x(s)) \left\{ 1 + \frac{h(r)}{g(r)} \right\} ds \\
&\le M_0 K_0 \left\{ 1 + \frac{h(r)}{g(r)} \right\} g\left(\frac{\mu r}{M_0} - C_0 \right) \int_0^1 \Phi(s) q(s) g(s^{\alpha-1}) ds < r.
\end{aligned}
$$

Thus, (3.10) holds, as desired.

Next, we show that

$$
\|Ty\| \ge \|y\|, \quad y \in K \cap \partial\Omega_2. \tag{3.11}
$$

To see this let $y \in K \cap \partial\Omega_2$, then $\|y\| = R$ and

$$
y(t) - x(t) \ge t^{\alpha-1} \left(\frac{\mu R}{M_0} - C_0 \right) = t^{\alpha-1} R \left(\frac{\mu}{M_0} - \frac{C_0}{R} \right) \ge \epsilon R t^{\alpha-1}, \quad t \in [0,1].
$$

By using (3.8), we see that

$$
\begin{aligned}
(Ty)(\sigma) &= \int_0^1 G(\sigma,s)f^*(s,y(s)-x(s))ds \\
&\geq \int_\theta^{1-\theta} G(\sigma,s)q_1(s)g_1(y(s)-x(s))\left\{1+\frac{h_1(y(s)-x(s))}{g_1(y(s)-x(s))}\right\}ds \\
&\geq \int_\theta^{1-\theta} G(\sigma,s)q_1(s)g_1(R)\left\{1+\frac{h_1(\epsilon R\theta^{\alpha-1})}{g_1(\epsilon R\theta^{\alpha-1})}\right\}ds \\
&= g_1(R)\left\{1+\frac{h_1(\epsilon R\theta^{\alpha-1})}{g_1(\epsilon R\theta^{\alpha-1})}\right\}\int_\theta^{1-\theta} G(\sigma,s)q_1(s)ds \\
&\geq g_1(R)\left\{1+\frac{h_1(\epsilon R\theta^{\alpha-1})}{g_1(\epsilon R\theta^{\alpha-1})}\right\}\frac{Rg_1(\epsilon R\theta^{\alpha-1})}{g_1(R)g_1(\epsilon R\theta^{\alpha-1})+g_1(R)h_1(\epsilon R\theta^{\alpha-1})} = R,
\end{aligned}
$$

which implies that

$$(Ty)(\sigma) \geq R = \|y\|.$$

Thus $\|Ty\| \geq \|y\|$, and (3.11) is held. By Theorem 2.8, we conclude that T has a fixed point $y_1 \in K\cap(\overline{\Omega_2}\backslash\Omega_1)$, then $r \leq \|y_1\| \leq R$ and $y_1(t) \geq \frac{\mu t^{\alpha-1}}{M_0}r$ for $t\in[0,1]$. Thus $y_1(t)$ is a solution of BVP (3.9), and

$$y_1(t)-x(t) \geq t^{\alpha-1}\left(\frac{\mu r}{M_0}-C_0\right) > 0, \quad t\in(0,1).$$

Therefore, $u(t) = y_1(t)-x(t)$ is a positive solution of BVP (1.1). □

Theorem 3.2. *Suppose that conditions* (3.1)-(3.6) *hold. Furthermore, assume that*

$$
\begin{cases}
\textit{For each constant } L>0, \textit{ there exists a function } \varphi_L\in C[0,1],\ \varphi_L(t)>0 \textit{ for } t\in(0,1) \\
\textit{such that } f^*(t,u)\geq\varphi_L(t) \textit{ for } (t,u)\in[0,1]\times(0,L], \textit{ and} \\
\varphi_r(t)>e(t),\ t\in(0,1), \textit{ where } r \textit{ is as in } (3.5).
\end{cases}
\tag{3.12}
$$

Then BVP (1.1) *has at least one positive solution* u *with* $0<\|u\|<r$.

Proof. The idea is that we first show that BVP (3.9) has a positive solution y satisfying $y(t) > x(t)$ for $t\in(0,1)$. If this is true, it is easy to see that $u(t) = y(t)-x(t)$ will be a positive solution of BVP (1.1). Similar to the proof of Theorem 3.1, we can prove that

$$y(t) = \int_0^1 G(t,s)f^*(s,y(s)-x(s))ds \tag{3.13}$$

has a positive solution. Since (3.5) holds, there exists $n_0\in\{1,2,\cdots\}$ such that $\frac{1}{n_0} < r-\frac{M_0}{\mu}C_0$ and

$$M_0K_0a_0g\left(\frac{\mu r}{M_0}-C_0\right)\left\{1+\frac{h(r)}{g(r)}\right\}+\frac{1}{n_0} < r.$$

Let $N_0 = \{n_0, n_0+1,\cdots\}$. Consider the family of equations

$$(T_ny)(t) = \int_0^1 G(t,s)f_n^*\left(s,y(s)-x(s)\right)ds+\frac{1}{n}, \quad t\in(0,1), \tag{3.14}$$

where $n\in N_0$, $f_n^*(t,v) = f^*(t,\max\{\frac{1}{n},v\})$. Similar to the proof of Theorem 3.1, we can easy to verify that T_n is well defined and maps K into K, and T_n is completely continuous. By Leray-Schauder alternative principle, we need to consider

$$y(t) = \lambda\int_0^1 G(t,s)f_n^*\left(s,y(s)-x(s)\right)ds+\frac{1}{n}, \quad t\in(0,1), \tag{3.15}$$

where $n \in N_0,\ \lambda \in (0,1]$. We claim that any fixed point y of (3.15) for any $\lambda \in (0,1]$ must satisfy $\|y\| \neq r$. Otherwise, assume that y is a fixed point of (3.15) for some $\lambda \in (0,1]$ such that $\|y\| = r$. Note that

$$y(t) - \frac{1}{n} = \lambda \int_0^1 G(t,s) f_n^*(s, y(s) - x(s)) ds \leq \lambda M_0 \int_0^1 \Phi(s) f_n^*(s, y(s) - x(s)) ds,$$

we conclude that

$$\left\| y - \frac{1}{n} \right\| \leq \lambda M_0 \int_0^1 \Phi(s) f_n^* \left(s, y(s) - x(s)\right) ds.$$

On the other hand, we have

$$y(t) - \frac{1}{n} = \lambda \int_0^1 G(t,s) f_n^* \left(s, y(s) - x(s)\right) ds \geq \lambda \mu t^{\alpha-1} \int_0^1 \Phi(s) f_n^* \left(s, y(s) - x(s)\right) ds \geq \frac{\mu t^{\alpha-1}}{M_0} \left\| y - \frac{1}{n} \right\|.$$

By the choice of n_0, $\frac{1}{n} \leq \frac{1}{n_0} < r - \frac{M_0}{\mu} C_0$, we have

$$y(t) \geq \frac{\mu t^{\alpha-1}}{M_0} \left| y(t) - \frac{1}{n} \right| + \frac{1}{n} \geq \frac{\mu t^{\alpha-1}}{M_0} \left(\|y\| - \frac{1}{n} \right) + \frac{1}{n} \geq \frac{\mu t^{\alpha-1}}{M_0} r + \left(1 - \frac{\mu t^{\alpha-1}}{M_0} \right) \frac{1}{n}, \quad t \in [0,1].$$

Then, for each $t \in [0,1]$, we estimate

$$y(t) - x(t) \geq \frac{\mu t^{\alpha-1}}{M_0} r + \left(1 - \frac{\mu t^{\alpha-1}}{M_0} \right) \frac{1}{n} - t^{\alpha-1} C_0 = \frac{\mu t^{\alpha-1}}{M_0} \left(r - \frac{M_0}{\mu} C_0 - \frac{1}{n} \right) + \frac{1}{n} > \frac{1}{n},$$

and

$$\begin{aligned} y(t) - x(t) \geq & \frac{\mu t^{\alpha-1}}{M_0} r + \left(1 - \frac{\mu t^{\alpha-1}}{M_0} \right) \frac{1}{n} - t^{\alpha-1} C_0 \\ = & \frac{\mu t^{\alpha-1}}{M_0} \left(r - \frac{M_0}{\mu} C_0 \right) + \left(1 - \frac{\mu t^{\alpha-1}}{M_0} \right) \frac{1}{n} \\ > & \frac{\mu t^{\alpha-1}}{M_0} \left(r - \frac{M_0}{\mu} C_0 \right) \\ = & t^{\alpha-1} \left(\frac{\mu r}{M_0} - C_0 \right). \end{aligned}$$

Thus, for each $t \in [0,1]$, it holds that

$$\begin{aligned} y(t) = & \lambda \int_0^1 G(t,s) f_n^* \left(s, y(s) - x(s)\right) ds + \frac{1}{n} \\ = & \lambda \int_0^1 G(t,s) f^* \left(s, y(s) - x(s)\right) ds + \frac{1}{n} \\ \leq & M_0 \int_0^1 \Phi(s) q(s) g \left(y(s) - x(s)\right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} ds + \frac{1}{n} \\ \leq & M_0 K_0 \int_0^1 \Phi(s) q(s) g(s^{\alpha-1}) g \left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} ds + \frac{1}{n} \\ \leq & M_0 K_0 g \left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} a_0 + \frac{1}{n}. \end{aligned}$$

Therefore,

$$r = \|y\| \leq M_0 K_0 a_0 g \left(\frac{\mu r}{M_0} - C_0 \right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} + \frac{1}{n}.$$

This is a contradiction to the choice of n_0 and the claim is proved. Hence, by Theorem 2.7,

$$y(t) = \int_0^1 G(t,s) f_n^*(s, y(s) - x(s)) ds + \frac{1}{n}$$

has a fixed point, denoted by y_n, in $B_r = \{y \in E : \|y\| < r\}$.

Next, we claim that $y_n(t)-x(t)$ have a uniform positive lower bound, that is, there exists a constant $\delta>0$, independent of $n\in N_0$, such that

$$\min_{t\in[0,1]}\{y_n(t)-x(t)\}\geq \delta t^{\alpha-1} \tag{3.16}$$

for all $n\in N_0$. Since (3.12) holds, there exists a continuous function $\varphi_r(t)>0$ such that $f^*(t,u)\geq\varphi_r(t)>e(t)$ for all $(t,u)\in(0,1]\times(0,r]$. Since $y_n(t)-x(t)<r$, it holds that

$$\begin{aligned}y_n(t)-x(t)&=\int_0^1 G(t,s)f_n^*(s,y(s)-x(s))ds+\frac{1}{n}-\int_0^1 G(t,s)e(s)ds\\&\geq\int_0^1 G(t,s)(\varphi_r(s)-e(s))ds\geq\mu t^{\alpha-1}\int_0^1\Phi(s)(\varphi_r(s)-e(s))ds:=\delta t^{\alpha-1}.\end{aligned}$$

In order to pass the fixed point y_n of the truncation equation (3.14) to that of the equation (3.13), we need the following fact

$$\{y_n\}\text{ is equicontinuous on }[0,1]\text{ for all }n\in N_0. \tag{3.17}$$

In fact, for each $\varepsilon>0$, $y_n\in B_r$, $t,t'\in[0,1]$, $t<t'$, since $G(t,s)$ is uniformly continuous on $(t,s)\in[0,1]\times[0,1]$, there exists $\tau>0$ such that for $t'-t<\tau$, we have

$$|G(t',s)-G(t,s)|<\frac{\varepsilon}{K_0b_0g(\delta)\left\{1+\frac{h(r)}{g(r)}\right\}},$$

then

$$\begin{aligned}|y_n(t')-y_n(t)|&\leq\int_0^1\left|G(t',s)-G(t,s)\right|f^*(s,y(s)-x(s))ds\\&\leq\int_0^1\left|G(t',s)-G(t,s)\right|q(s)K_0g(\delta)g(s^{\alpha-1})\left\{1+\frac{h(r)}{g(r)}\right\}ds\\&=K_0g(\delta)\left\{1+\frac{h(r)}{g(r)}\right\}\int_0^1\left|G(t',s)-G(t,s)\right|q(s)g(s^{\alpha-1})ds<\varepsilon.\end{aligned}$$

The facts $\|y_n\|<r$ and (3.17) imply that $\{y_n\}_{n\in N_0}$ is a bounded and equicontinuous family on [0,1]. Now the Arzela-Ascoli Theorem implies that $\{y_n\}_{n\in N_0}$ has a subsequence $\{y_{n_k}\}_{n_k\in N_0}$ converging uniformly on [0,1] to a function $y\in E$. From the facts $\|y_n\|<r$ and (3.16), y satisfies $\delta t^{\alpha-1}\leq y(t)-x(t)<r$ for $t\in[0,1]$. Moreover, y_{n_k} satisfies the integral equation

$$y_{n_k}(t)=\int_0^1 G(t,s)f_{n_k}^*(s,y_{n_k}(s)-x(s))ds+\frac{1}{n_k},\quad t\in(0,1).$$

Passing to the limit as $k\to\infty$, we obtain

$$y(t)=\int_0^1 G(t,s)f^*(s,y(s)-x(s))ds,\quad t\in(0,1).$$

Hence, y is a positive solution of (3.9) and satisfies $0<\|y\|<r$. Therefore, $u(t)=y(t)-x(t)$ is a positive solution of BVP (1.1). □

Theorem 3.3. *Assume that conditions* (3.1)-(3.8) *and* (3.12) *hold. Then BVP* (1.1) *has at least two positive solutions.*

Proof. From the proof of Theorem 3.1, we have that (3.9) has a positive solution $y_1(t)>x(t)$ for $t\in(0,1)$ with $r\leq\|y_1\|\leq R$, and by Theorem 3.2, we have that (3.9) has another positive solution $y_2(t)>x(t)$ for $t\in(0,1)$ with $\|y_2\|<r$. Thus BVP (1.1) has at least two positive solutions $u_1(t)=y_1(t)-x(t)$ and $u_2(t)=y_2(t)-x(t)$. □

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