Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4003–4015
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On the split equality common fixed point problem for asymptotically nonexpansive semigroups in Banach spaces

Zhaoli Ma[a], Lin Wang[b,*]

[a]School of Information Engineering, The College of Arts and Sciences Yunnan Normal University, Kunming, Yunnan, 650222, China.
[b]College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China.

Communicated by Y. J. Cho

Abstract

In this article, we propose an iteration methods for finding a split equality common fixed point of asymptotically nonexpansive semigroups in Banach spaces. The weak and strong convergence theorems of the iteration scheme proposed are obtained. As application, we shall utilize our results to study the split equality variational inequality problems to support the main results. The results presented in the article are new and improve and extend some recent corresponding results.

Keywords: Split equality problem, convergence, asymptotically nonexpansive semigroup, Banach spaces.
2010 MSC: 47H09, 47H05, 47J20.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point $q \in H_1$ with the property:

$$q \in C \quad and \quad Aq \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Assuming that SFP (1.1) is consistent (that is, (1.1) has a solution), it is not hard to see that $x \in C$ is a solution of (1.1) if and only if it solves the following fixed point equation

$$x = P_C(I - \gamma A^*(I - P_Q)A)x, \qquad x \in C,$$

*Corresponding author:Lin Wang
Email addresses: kmszmzl@126.com (Zhaoli Ma), WL64mail@aliyun.com and Wanglin64@outlook.com (Lin Wang)

Received 2016-05-17

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A.

The split feasibility problem in finite dimensional Hilbert spaces was introduced by Censor and Elfving [5] in 1994 for modeling inverse problems which arise from phase retrievals and in medical imagine reconstruction [3]. Recently, it has been found that split feasibility problems can be used in various disciplines, such as imagine restoration, computer tomography, and radiation therapy treatment planning [4, 6, 7]. As well as the convex feasibility formalism is at the core of the modeling of many inverse problems and has been used to model significant real-world problems.

If C and Q are the sets of fixed points of two nonlinear mappings, respectively, and C and Q are nonempty closed convex subsets, then q is said to be a split common fixed point for the two nonlinear mappings. That is, the split common fixed point problem ($SCFP$) for mappings S and T is to find a point $q \in H_1$ with the property:

$$q \in C := F(S) \quad and \quad Aq \in Q := F(T), \tag{1.2}$$

where $F(S)$ and $F(T)$ denote the sets of fixed points of S and T, respectively. We use Γ to denote the set of solution of $SCFP$ (1.2), that is, $\Gamma = \{q \in F(S) : Aq \in F(T)\}$.

Recently, Moudafi [12] proposed a new split feasibility problem, which is also called split equality fixed point problem. Let H_1, H_2, and H_3 be real Hilbert spaces, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear mappings with nonempty fixed point sets $C := FixU$ and $Q := FixT$, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. The split equality fixed point problem for U and T is

$$Finding\ x \in C\ and\ y \in Q\ such\ that\ Ax = By, \tag{1.3}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance in decomposition methods for $PDE's$, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables, for further details, the interested reader is referred to [1]. In IMRT, this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity, for further details, the interested reader is referred to [4, 6].

We use Γ to denote the set of solutions of the new split feasibility problem (1.3), that is,

$$\Gamma = \{(x, y) : Ax = By, x \in C, y \in Q\}. \tag{1.4}$$

Let E be a real normed linear space and C be a nonempty closed convex subset of E. The mapping $T : C \to C$ is said to be nonexpansive if for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|.$$

The mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in C$ and each $n \geq 1$

$$\|T^n x - T^n y\| \leq k_n \|x - y\|.$$

Being an important generalization of the class of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [10] in 1972, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping, then T has a fixed point.

Definition 1.1 ([19])**.** A one-parameter family $\mathcal{F} := \{T(t) : t \geq 0\}$ of E into itself is called a strongly continuous semigroup of Lipschitzian mappings on E if it satisfies the following conditions:

(i) $T(0)x = x$, for all $x \in E$;

(ii) $T(s+t) = T(s)T(t)$, for all $s, t \geq 0$;
(iii) for each $x \in E$, the mapping $t \mapsto T(t)x$ is continuous;
(iv) for each $t > 0$, there exists a bounded measurable function $L(t) : [0, \infty) \to [0, \infty)$ such that

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \text{ } for \text{ } all \text{ } x, y \in E.$$

A strongly continuous semigroup of Lipschitzian mappings $\mathcal{F}$ is called strongly continuous semigroup of nonexpansive mappings if $L(t) = 1$ for all $t > 0$ and strongly continuous semigroup of asymptotically nonexpansive if $\limsup_{t\to\infty} L(t) \leq 1$. Note that for asymptotically nonexpansive semigroup $\mathcal{F}$, we can always assume that $\{L(t)\}_{t>0}$ is such that $L(t) \geq 1$ for each $t > 0$, $L(t)$ is nonincreasing in t, and $\lim_{t\to\infty} L(t) = 1$; otherwise we replace $L(t)$, for each $t > 0$, with $L(t) := \max\{sup_{s\geq t} L(s), 1\}$. We denote by $F(\mathcal{F})$ the set of all common fixed points of $\mathcal{F}$, that is,

$$F(\mathcal{F}) := \{x \in E : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t\geq 0} F(T(t)).$$

If $\mathcal{F}$ satisfies (i)-(iii) and

$$\limsup_{t\to\infty x\in D} \|T(t)x - T(s)T(t)x\| = 0, \text{ } for \text{ } all \text{ } s > 0 \text{ } and \text{ } bounded \text{ } D \subseteq C,$$

then $\mathcal{F}$ is called uniformly asymptotically regular on C.

Example 1.2 ([14], Example of asymptotically nonexpansive semi-group)**.** Let E be an uniformly convex Banach space which admits a weakly continuous duality mapping. Let $L(E)$ be the space of all bounded linear operators on E. For $\Psi \in L(E)$, define $\mathcal{F} := \{T(t) : t \in R^+\}$ of bounded linear operators by using the following exponential expression:

$$T(t) = e^{-t\Psi} := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \Psi^k.$$

Then, clearly, the family $\mathcal{F} := \{T(t) : t \in R^+\}$ satisfies the semigroup properties. Moreover, this family forms a one parameter semigroup of self-mappings of E because $e^{t\Psi} = [e^{-t\Psi}]^{-1} : E \to E$ exists for each $t \in R^+$.

Concerning the weak and strong convergence of iterative sequences to approximate a solution of split feasibility problem and split equality problem have been studied by some authors in the setting of Hilbert space (see, for example, [3, 6, 7, 9, 11, 13, 20] and the references therein). But according to the literature, we can find that there is no relevant literature about the convergence of the split feasibility problem and the split equality common fixed point problem for the operator semigroups in Banach spaces. Very recently, In 2015, Takahashi and Yao [15] obtained some strong and weak convergence theorems by using hybrid methods for the split feasibility problem and split common null point problem in the setting of one Hilbert space and one Banach space. Then, Tang et al. [18] proved a weak convergence theorem and a strong convergence theorem for split common fixed point problem involving a quasi-strict pseudo contractive mapping and an asymptotical nonexpansive mapping in the setting of two Banach spaces.

In this paper, motivated by the works above, we propose the following iterative algorithm to approximate a solution of the split equality fixed point problems of two asymptotically nonexpansive semigroups in the setting of two Banach spaces. For any given $x_0 \in E_1$ and $y_0 \in E_2$, the sequence $\{(x_n, y_n)\}$ is defined as follows:

$$\begin{cases} z_n \in J_3(Ax_n - By_n) \\ u_n = S(t_n)(x_n - \gamma J_1^{-1} A^* z_n) \\ v_n = T(t_n)(y_n + \gamma J_2^{-1} B^* z_n) \\ y_{n+1} = \beta_n v_n + (1 - \beta_n)(y_n + \gamma J_2^{-1} B^* z_n) \\ x_{n+1} = \beta_n u_n + (1 - \beta_n)(x_n - \gamma J_1^{-1} A^* z_n). \end{cases}$$

Under some suitable conditions strong and weak convergence theorems are established. As application, we shall utilize our results to study the split equality variational inequality problem. The results presented in this paper are new and improve and extend some recent corresponding results.

2. Preliminaries

We now recall some definitions and elementary facts which will be used in the proofs of our main results.

Let E be a real Banach space with the dual E^*. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\|,\ \|y\| \leq 1,\ \|x - y\| \geq \epsilon\}$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. A Hilbert space is 2-uniformly convex, while L^p is $\max\{p, 2\}-$uniformly convex for every $p > 1$.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in U, \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. A typical example of uniformly smooth Banach space is L^p, where $p > 1$. More precisely, L^p is $min\{p, 2\}$-uniformly smooth for every $p > 1$. Let q be a fixed real number with $q > 1$, then a Banach space E is said to be $q-$uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$. It is well known that every $q-$uniformly smooth Banach space is uniformly smooth.

Lemma 2.1 ([21]). *Given a number $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for all $x, y \in E$, $t \in [0, 1]$, with $\|x\| \leq r$ and $\|y\| \leq r$.

Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. Then T is said to be demiclosed at zero if for any $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, $x = Tx$.

A mapping $T : C \to C$ is said to be semi−compact, if for any sequence $\{x_n\}$ in C such that $\|x_n -Tx_n\| \to 0, (n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{x_j}\}$ converges strongly to $x^* \in C$.

A Banach space E is said to satisfy Opial's property if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$, for any $y \in E$ with $y \neq x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Lemma 2.2 ([21]). *Let E be a 2-uniformly smooth Banach space with the best smoothness constants $K > 0$. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \forall x, y \in E.$$

Lemma 2.3 ([8]). *Let E be a real uniformly convex Banach space, C be a nonempty closed subset of E, and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\} \subset C$ converges weakly to a point $p \in C$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $p = Tp$.*

Lemma 2.4 ([17])**.** *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real number sequences and satisfy*

$$a_{n+1} \leq (1+b_n)a_n, \quad \forall n \geq 1,$$

where $a_n \geq 0$, $b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then

(1) $\lim_{n\to\infty} a_n$ *exists;*
(2) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

Throughout this section, we assume that:

(1) Let E_1 and E_2 be real uniformly convex and 2-uniformly smooth Banach spaces satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, E_3 be a real Banach space.
(2) Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be two bounded linear operators with adjoints A^* and B^*, respectively.
(3) Let $\{S(t) : t \geq 0\} : E_1 \to E_1$ be uniformly asymptotically regular asymptotically nonexpansive semigroup with a bounded measurable function $L^{(1)}(t) : [0,\infty) \to [1,\infty)$ satisfying $\lim_{t\to\infty} L^{(1)}(t) = 1$ and $C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$, $\{T(t) : t \geq 0\} : E_2 \to E_2$ be uniformly asymptotically regular family of asymptotically nonexpansive semigroup with a bounded measurable function $L^{(2)}(t) : [0,\infty) \to [1,\infty)$ satisfying $lim_{t\to\infty} L^{(2)}(t) = 1$ and $Q := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$, respectively.

Theorem 3.1. *Let E_1, E_2, E_3, A, B, C, Q, $\{S(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ be the same as above. For any given $(x_0, y_0) \in E_1 \times E_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{cases} z_n \in J_3(Ax_n - By_n) \\ u_n = S(t_n)(x_n - \gamma J_1^{-1}A^* z_n) \\ v_n = T(t_n)(y_n + \gamma J_2^{-1}B^* z_n) \\ y_{n+1} = \beta_n v_n + (1-\beta_n)(y_n + \gamma J_2^{-1}B^* z_n) \\ x_{n+1} = \beta_n u_n + (1-\beta_n)(x_n - \gamma J_1^{-1}A^* z_n), \end{cases} \tag{3.1}$$

where $\{t_n\}$ is a sequence of real numbers, $\{\beta_n\}$ is a sequence in $(0,1)$ and γ is a positive number satisfying

(1) $t_n > 0$ *and* $\lim_{n\to\infty} t_n = \infty$;
(2) $L(t) = \max\{L^{(1)}(t), L^{(2)}(t)\}$ *and* $\sum_{n=1}^{\infty}(L^2(t_n) - 1) < \infty$;
(3) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\frac{1}{\|A\|^2+\|B\|^2} < \gamma < \frac{2}{\|A\|^2+\|B\|^2}$.

If $\Gamma = \{(x^*, y^*) \in E_1 \times E_2 : Ax^* = By^*, x^* \in C, y^* \in Q\} \neq \emptyset$, then

(I) *the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x^*, y^*) \in \Gamma$ of* (1.4).
(II) *In addition, if there exists at least one $S(t) \in \{S(t) : t \geq 0\}$ and one $T(t) \in \{T(t) : t \geq 0\}$ are semi-compact, respectively, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Gamma$ of* (1.4).

Proof. Now we prove the conclusion (I).

It is well known that the normalized duality mapping J of a smooth, reflexive and strictly convex Banach space is single-valued, one to one, and surjective. Since E_1 and E_2 are real uniformly convex and 2-uniformly smooth Banach spaces, and E_3 is a real Banach space, therefore, the iteration scheme of (3.1) is well defined.

We shall divide the proof into four steps.

Step 1. We first show that $\lim_{n\to\infty}\Gamma_{n+1}(x,y)$ exists.

Setting $e_n = x_n - \gamma J_1^{-1}A^*z_n$ and $w_n = y_n + \gamma J_2^{-1}B^*z_n$. Let $(x,y)\in\Gamma$, it follows from Lemma 2.1 that

$$\begin{aligned}\|x_{n+1}-x\|^2 &= \|(1-\beta_n)e_n+\beta_n u_n - x\|^2\\ &= \|(1-\beta_n)(e_n-x)+\beta_n(u_n-x)\|^2\\ &\le (1-\beta_n)\|e_n-x\|^2+\beta_n\|u_n-x\|^2-\beta_n(1-\beta_n)g_1(\|e_n-u_n\|)\\ &= (1-\beta_n)\|e_n-x\|^2+\beta_n\|S(t_n)e_n-x\|^2-\beta_n(1-\beta_n)g_1(\|e_n-u_n\|)\\ &\le (1-\beta_n)\|e_n-x\|^2+\beta_n L^2(t_n)\|e_n-x\|^2-\beta_n(1-\beta_n)g_1(\|e_n-u_n\|)\\ &= (1+\beta_n(L^2(t_n)-1))\|e_n-x\|^2-\beta_n(1-\beta_n)g_1(\|e_n-u_n\|).\end{aligned}\tag{3.2}$$

Further, from Lemma 2.2, we have

$$\begin{aligned}\|e_n-x\|^2 &= \|x_n-\gamma J_1^{-1}A^*z_n-x\|^2\\ &= \|(x_n-x)-\gamma J_1^{-1}A^*z_n\|^2\\ &\le \|\gamma J_1^{-1}A^*J_3z_n\|^2+2\gamma\langle x-x_n, J_1J_1^{-1}A^*z_n\rangle+2k^2\|x-x_n\|^2\\ &\le \gamma^2\|A\|^2\|z_n\|^2+2\gamma\langle Ax-Ax_n,z_n)\rangle+2k^2\|x_n-x\|^2.\end{aligned}\tag{3.3}$$

By (3.2) and (3.3), we have

$$\begin{aligned}\|x_{n+1}-x\|^2 \le{}& (1+\beta_n(L^2(t_n)-1))[\gamma^2\|A\|^2\|z_n\|^2+2k^2\|x_n-x\|^2\\ &+2\gamma\langle Ax-Ax_n,z_n\rangle]-\beta_n(1-\beta_n)g_1(\|z_n-u_n\|)\\ ={}& (1+\beta_n(L^2(t_n)-1))2k^2\|x_n-x\|^2+(1+\beta_n(L^2(t_n)-1))[\gamma^2\|A\|^2\|z_n\|^2\\ &+2\gamma\langle Ax-Ax_n,z_n)\rangle]-\beta_n(1-\beta_n)g_1(\|e_n-u_n\|).\end{aligned}\tag{3.4}$$

By using the similar argument as given above, we have

$$\begin{aligned}\|y_{n+1}-y\|^2 \le{}& (1+\beta_n(L^2(t_n)-1))2k^2\|y_n-y\|^2+(1+\beta_n(L^2(t_n)-1))[\gamma^2\|B\|^2\|z_n\|^2\\ &+2\gamma\langle By_n-By,z_n\rangle]-\beta_n(1-\beta_n)g_2(\|w_n-v_n\|).\end{aligned}\tag{3.5}$$

By adding (3.4) and (3.5), and by taking into account the fact that $Ax = By$ and $z_n \in J_3(Ax_n - By_n)$, we have

$$\begin{aligned}\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2 \le{}& (1+\beta_n(L^2(t_n)-1))2k^2(\|x-x_n\|^2+\|y-y_n\|^2)\\ &+(1+\beta_n(L^2(t_n)-1))\gamma^2(\|A\|^2+\|B\|^2)\|z_n\|^2\\ &-2(1+\beta_n(L^2(t_n)-1))\gamma\langle Ax_n-By_n,z_n\rangle\\ &-\beta_n(1-\beta_n)[g_1(\|e_n-u_n\|)+g_2(\|w_n-v_n\|)]\\ \le{}& (1+\beta_n(L^2(t_n)-1))2k^2(\|x-x_n\|^2+\|y-y_n\|^2)\\ &+(1+\beta_n(L^2(t_n)-1))\gamma[\gamma(\|A\|^2+\|B\|^2)\|Ax_n-By_n\|^2\\ &-2\|Ax_n-By_n\|^2]-\beta_n(1-\beta_n)[g_1(\|e_n-u_n\|)+g_2(\|w_n-v_n\|)]\\ \le{}& (1+\beta_n(L^2(t_n)-1))2k^2(\|x-x_n\|^2+\|y-y_n\|^2)\\ &-(1+\beta_n(L^2(t_n)-1))\gamma[2-\gamma(\|A\|^2+\|B\|^2)]\|Ax_n-By_n\|^2\\ &-\beta_n(1-\beta_n)[g_1(\|e_n-u_n\|)+g_2(\|w_n-v_n\|)].\end{aligned}$$

Setting $\Gamma_n(x,y) := \|x_n-x\|^2+\|y_n-y\|^2$, we have

$$\begin{aligned}\Gamma_{n+1}(x,y) \le{}& (1+\beta_n(L^2(t_n)-1))2k^2\Gamma_n(x,y)\\ &-(1+\beta_n(L^2(t_n)-1))\gamma[2-\gamma(\|A\|^2+\|B\|^2)]\|Ax_n-By_n\|^2\\ &-\beta_n(1-\beta_n)[g_1(\|e_n-u_n\|)+g_2(\|w_n-v_n\|)].\end{aligned}\tag{3.6}$$

Since $\lim_{t_n\to\infty} L(t_n) = 1$, $\sum_{n=1}^{\infty}(L^2(t_n) - 1) < \infty$, $0 < k < \frac{1}{\sqrt{2}}$, $\frac{1}{\|A\|^2+\|B\|^2} < \gamma < \frac{2}{\|A\|^2+\|B\|^2}$, so, $0 < 2 - \gamma(\|A\|^2 + \|B\|^2) < 1$. It follows from (3.6) and Lemma 2.4 that the $\lim_{n\to\infty} \Gamma_{n+1}(x, y)$ exists.

Step 2. We prove that $\lim_{n\to\infty} \|Ax_n - By_n\| = 0$, $\lim_{n\to\infty} \|x_n - u_n\| = 0$, and $\lim_{n\to\infty} \|y_n - v_n\| = 0$.

It follows from (3.6) that

$$\begin{aligned}
&(1 + \beta_n(L^2(t_n) - 1))\gamma[2 - \gamma(\|A\|^2 + \|B\|^2]\|Ax_n - By_n\|^2 \\
&\quad + \beta_n(1 - \beta_n)[g(\|z_n - u_n\|) + g(\|w_n - u_n\|)] \\
&\qquad \le (1 + \beta_n(L^2(t_n) - 1))2k^2\Gamma_n(x, y) - \Gamma_{n+1}(x, y) \\
&\qquad \le \Gamma_n(x, y) + \beta_n(L^2(t_n) - 1))2k^2\Gamma_n(x, y) - \Gamma_{n+1}(x, y).
\end{aligned}$$

Therefore, we obtain that

$$\lim_{n\to\infty} g_1(\|e_n - u_n\|) = 0, \quad \lim_{n\to\infty} g_2(\|w_n - v_n\|) = 0,$$

and

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{3.7}$$

By virtue of Lemma 2.1 and the properties of g_1 and g_2, we may get

$$\lim_{n\to\infty} \|e_n - u_n\| = 0, \tag{3.8}$$

and

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.9}$$

Since

$$\|x_n - e_n\| = \|J_1(x_n - e_n)\| = \|\gamma A^* J_3(Ax_n - By_n)\| \le \gamma\|A\|\|Ax_n - By_n\|,$$

and

$$\|y_n - w_n\| = \|J_2(y_n - w_n)\| = \|\gamma B^* J_3(Ax_n - By_n)\| \le \gamma\|B\|\|Ax_n - By_n\|.$$

From (3.7), we may get

$$\lim_{n\to\infty} \|x_n - e_n\| = 0, \tag{3.10}$$

and

$$\lim_{n\to\infty} \|y_n - w_n\| = 0.$$

It follows from (3.8) and (3.10) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.11}$$

By (3.9) and (3.11), we have

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{3.12}$$

Step 3. We prove that $\lim_{n\to\infty} \|x_n - S(t)x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$, for all $t \in [0, +\infty)$.

It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n u_n + (1 - \beta_n)(x_n - \gamma J_1^{-1} A^* z_n) - x_n\| \\
&= \|\beta_n u_n - \beta_n x_n - (1 - \beta_n)\gamma J_1^{-1} A^* z_n)\| \\
&\le \beta_n\|u_n - x_n\| + (1 - \beta_n)\|\gamma J_1^{-1} A^* J_3(Ax_n - By_n))\|.
\end{aligned} \tag{3.13}$$

From (3.7), (3.11), and (3.13), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Similarly, we can obtain

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.15}$$

In addition, from (3.1) we can get

$$\begin{aligned}\|u_n - S(t_n)x_n\|^2 &= \|S(t_n)(x_n - \gamma J_1^{-1}A^* z_n) - S(t_n)x_n\|^2\\ &\le L^2(t_n)\|\gamma J_1^{-1}A^* z_n)\|^2\\ &\le L^2(t_n)\|\gamma J_1^{-1}A^* J_3(Ax_n - By_n)\|^2,\end{aligned}$$

and

$$\begin{aligned}\|v_n - T(t_n)y_n\|^2 &= \|T(t_n)(y_n + \gamma J_2^{-1}B^* z_n) - T(t_n)y_n\|^2\\ &\le L^2(t_n)\|\gamma J_2^{-1}B^* J_3(Ax_n - By_n)\|^2.\end{aligned}$$

By (3.7), we obtain

$$\lim_{n\to\infty} \|u_n - S(t_n)x_n\| = 0, \tag{3.16}$$

and

$$\lim_{n\to\infty} \|v_n - T(t_n)y_n\| = 0. \tag{3.17}$$

It follows from (3.11), (3.12), (3.16), and (3.17) that

$$\lim_{n\to\infty} \|x_n - S(t_n)x_n\| = 0, \tag{3.18}$$

and

$$\lim_{n\to\infty} \|y_n - T(t_n)y_n\| = 0.$$

Since $\|x_n - x\|^2 \le \Gamma_n(x, y)$, $\|y_n - y\|^2 \le \Gamma_n(x, y)$, and $\lim_{n\to\infty} \Gamma_n$ exists, we know that $\{x_n\}$ and $\{y_n\}$ are bounded. Therefore, there exist bounded subsets $C_1 \subseteq E_1$ and $Q_1 \subseteq E_2$ such that $\{x_n\} \subseteq C_1$ and $\{y_n\} \subseteq Q_1$, respectively. Since $\{S(t) : t \ge 0\}$ and $\{T(t) : t \ge 0\}$ are uniformly asymptotically regular, and $\lim_{n\to\infty} t_n = \infty$, then for all $t \ge 0$,

$$\lim_{n\to\infty} \|S(t)S(t_n)x_n - S(t_n)x_n\| \le \limsup_{n\to\infty x\in C_1} \|S(t)S(t_n)x - S(t_n)x\| = 0,$$

and

$$\lim_{n\to\infty} \|T(t)T(t_n)y_n - T(t)y_n\| \le \limsup_{n\to\infty y\in Q_1} \|T(t)T(t_n)Ay - T(t)Ay\| = 0.$$

Since $\{S(t)x\}$ is continuous on t for all $x \in E_1$, and

$$\|x_n - S(t)x_n\| \le \|x_n - S(t_n)x_n\| + \|S(t_n)x_n - S(t)S(t_n)x_n\| + \|S(t)S(t_n)x_n - S(t)x_n\|. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\lim_{n\to\infty} \|x_n - S(t)x_n\| = 0.$$

Similarly,

$$\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0.$$

Step 4. We prove that (x^*, y^*) is the unique weak cluster point of $\{(x_n, y_n)\}$.

Since E_1 and E_2 are uniformly convex, they are reflexive. On the other hand, since $\{(x_n, y_n)\} \subseteq C_1 \times Q_1$, so we may assume that (x^*, y^*) is a weak cluster point of $\{(x_n, y_n)\}$. Since each asymptotically nonexpansive mapping on real uniformly convex Banach spaces is demiclosed at zero, we know from Lemma 2.3 that $x^* \in C = \cap_{t\ge 0} F(S(t))$, $y^* \in Q = \cap_{t\ge 0} F(T(t))$.

Since A and B are bounded linear operators, we know that $Ax^* - By^*$ is a weak cluster point of $\{Ax_n - By_n\}$. From the weakly lower semi-continuous property of the norm and (3.7), we get

$$\|Ax^* - By^*\| \leq \liminf_{n\to\infty} \|Ax_n - By_n\| = 0.$$

So, $Ax^* = By^*$. This implies $(x^*, y^*) \in \Gamma$.

We now prove that (x^*, y^*) is the unique weak cluster point of $\{(x_n, y_n)\}$.

Assume that there exists another subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $\{(x_{n_k}, y_{n_k})\}$ converges weakly to a point (p, q) with $(p, q) \neq (x^*, y^*)$. Similar with the argument above, we know that $(p, q) \in \Gamma$, too. Since E_1 and E_2 satisfy Opial's property, we have

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - p\| &< \liminf_{i\to\infty} \|x_{n_i} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\| \\ &= \liminf_{k\to\infty} \|x_{n_k} - x^*\| < \liminf_{k\to\infty} \|x_{n_k} - p\| \\ &= \lim_{n\to\infty} \|x_n - p\| = \liminf_{i\to\infty} \|x_{n_i} - p\|,\end{aligned}$$

and

$$\begin{aligned}\liminf_{i\to\infty} \|y_{n_i} - q\| &< \liminf_{i\to\infty} \|y_{n_i} - y^*\| = \lim_{n\to\infty} \|y_n - y^*\| \\ &= \liminf_{k\to\infty} \|y_{n_k} - y^*\| < \liminf_{k\to\infty} \|y_{n_k} - q\| \\ &= \lim_{n\to\infty} \|y_n - q\| = \liminf_{i\to\infty} \|y_{n_i} - q\|,\end{aligned}$$

which is a contradiction. This implies that $(p, q) = (x^*, y^*)$. The proof of conclusion (I) is completed.

Next, we prove the conclusion (II).

Since there exist one $S(t) \in \{S(t) : t \geq 0\}$ and one $T(t) \in \{T(t) : t \geq 0\}$ are semi-compact, $\{(x_n, y_n)\}$ is bounded and $\lim_{n\to\infty} ||x_n - S(t)x_n|| = 0$, $\lim_{n\to\infty} ||y_n - T(t)y_n|| = 0$ for all $t \geq 0$, then there exists subsequences $\{(x_{n_j}, y_{n_j})\}$ of $\{(x_n, y_n)\}$ such that $\{(x_{n_j}, y_{n_j})\}$ converges strongly to (u^*, v^*). Due to $\{(x_n, y_n)\}$ converges weakly to (x^*, y^*), we know that $(u^*, v^*) = (x^*, y^*)$. It follows from Lemma 2.3 that $(x^*, y^*) \in C \times Q$. Further, due to the norm $\|\cdot\|$ is weakly lower semi-continuous and $Ax_{n_j} - By_{n_j} \to Ax^* - By^*$, we get

$$\|Ax^* - By^*\|^2 \leq \liminf_{n\to\infty} \|Ax_{n_j} - By_{n_j}\|^2 = 0.$$

So, $Ax^* = By^*$. This implies $(x^*, y^*) \in \Gamma$.

On the other hand, since $\Gamma_n(x, y) = \|x_n - x\|^2 + \|y_n - y\|^2$ for any $(x, y) \in \Gamma$, we know that $\lim_{j\to\infty} \Gamma_{n_j}(x^*, y^*) = 0$. From Conclusion (I), we know that $\lim_{n\to\infty} \Gamma_n(x^*, y^*)$ exists, therefore, $\lim_{n\to\infty} \Gamma_n(x^*, y^*) = 0$. From the facts that $0 \leq \|x_n - x^*\| \leq \Gamma_n$ and $0 \leq \|y_n - y^*\| \leq \Gamma_n$, we can obtain that $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and $\lim_{n\to\infty} \|y_n - y^*\| = 0$. This completes the proof of the Conclusion (II).

□

Corollary 3.2. *Let E_1, E_2, E_3, A, and B be the same as above. Let $\{S(t) : t \geq 0\} : E_1 \to E_1$ and $\{T(t) : t \geq 0\} : E_2 \to E_2$ be two nonexpansive semigroups satisfying $C := \cap_{t\geq 0} F(S(t)) \neq \emptyset$ and $Q := \cap_{t\geq 0} F(T(t)) \neq \emptyset$, respectively. For any given $(x_0, y_0) \in E_1 \times E_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{cases} z_n \in J_3(Ax_n - By_n) \\ u_n = S(t_n)(x_n - \gamma J_1^{-1} A^* z_n) \\ v_n = T(t_n)(y_n + \gamma J_2^{-1} B^* z_n) \\ y_{n+1} = \beta_n v_n + (1 - \beta_n)(y_n + \gamma J_2^{-1} B^* z_n) \\ x_{n+1} = \beta_n u_n + (1 - \beta_n)(x_n - \gamma J_1^{-1} A^* z_n), \end{cases}$$

where $\{t_n\}$ is a sequence of real numbers, $\{\beta_n\}$ is a sequence in $(0, 1)$ and γ is a positive real number satisfying

(1) $t_n > 0$ *and* $\lim_{n\to\infty} t_n = \infty$;
(2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\frac{1}{\|A\|^2+\|B\|^2} < \gamma < \frac{2}{\|A\|^2+\|B\|^2}$.

If $\Gamma = \{(x^*, y^*) \in E_1 \times E_2 : Ax^* = By^*, x^* \in C, y^* \in Q\} \neq \emptyset$, *then*

(I) *the sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution* $(x^*, y^*) \in \Gamma$ *of* (1.4).
(II) *In addition, if there exists at least one* $S(t) \in \{S(t) : t \geq 0\}$ *and one* $T(t) \in \{T(t) : t \geq 0\}$ *are semi-compact, respectively, then the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution* $(x^*, y^*) \in \Gamma$ *of* (1.4).

In Theorem 3.1, taking $B = I$, $E_2 = E_3$, $J_2 = J_3$, similar with the proofs in Theorem 3.1, we can obtain the following result for split common fixed point problem (1.2).

Corollary 3.3. *Let* E_1, A, $\{S(t) : t \geq 0\}$ *and* $\{T(t) : t \geq 0\}$ *be the same as Theorem* 3.1, E_2 *be a real Banach space. For any given* $(x_0, y_0) \in E_1 \times E_2$, *the sequence* $\{(x_n, y_n)\}$ *is generated by*

$$\begin{cases} z_n \in J_2(Ax_n - y_n) \\ u_n = S(t_n)(x_n - \gamma J_1^{-1} A^* z_n) \\ v_n = T(t_n)(y_n + \gamma(Ax_n - y_n)) \\ y_{n+1} = \beta_n v_n + (1-\beta_n)(y_n + \gamma(Ax_n - y_n)) \\ x_{n+1} = \beta_n u_n + (1-\beta_n)(x_n - \gamma J_1^{-1} A^* z_n), \end{cases}$$

where $\{t_n\}$ *is a sequence of real numbers,* $\{\beta_n\}$ *is a sequence in* $(0, 1)$ *and* γ *is a positive real number satisfying*

(1) $t_n > 0$ *and* $\lim_{n\to\infty} t_n = \infty$;
(2) $L(t) = \max\{L^{(1)}(t), L^{(2)}(t)\}$ *and* $\sum_{n=1}^{\infty}(L^2(t_n) - 1) < \infty$;
(3) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\frac{1}{\|A\|^2+1} < \gamma < \frac{2}{\|A\|^2+1}$.

If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, *then*

(I) *the sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution* $(x^*, y^*) \in \Gamma$ *of* (1.2).
(II) *In addition, if there exists at least one* $S(t) \in \{S(t) : t \geq 0\}$ *and one* $T(t) \in \{T(t) : t \geq 0\}$ *are semi-compact, respectively, then the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution* $(x^*, y^*) \in \Gamma$ *of* (1.2).

Corollary 3.4. *Let* E_1, E_2, E_3, A, *and* B *be the same as Theorem* 3.1, $S : E_1 \to E_1$ *and* $T : E_2 \to E_2$ *be two asymptotically nonexpansive mapping with the sequence* $\{k_n\} \subseteq [1, \infty)$ *and* $\{l_n\} \subseteq [1, \infty)$ *satisfying* $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$, *and* $\sum_{n=1}^{\infty}(l_n - 1) < +\infty$, *respectively. For any given* $(x_0, y_0) \in E_1 \times E_2$, *the sequence* $\{(x_n, y_n)\}$ *is generated by*

$$\begin{cases} z_n \in J_3(Ax_n - By_n) \\ u_n = S^n(x_n - \gamma J_1^{-1} A^* z_n) \\ v_n = T^n(y_n + \gamma J_2^{-1} B^* z_n) \\ y_{n+1} = \beta_n v_n + (1-\beta_n)(y_n + \gamma J_2^{-1} B^* z_n) \\ x_{n+1} = \beta_n u_n + (1-\beta_n)(x_n - \gamma J_1^{-1} A^* z_n), \end{cases}$$

where $\{\beta_n\}$ *is a sequence in* $(0, 1)$ *and* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,$\gamma$ *is a positive number satisfying* $\frac{1}{\|A\|^2+\|B\|^2} < \gamma < \frac{2}{\|A\|^2+\|B\|^2}$. *If* $\Gamma = \{(x^*, y^*) \in E_1 \times E_2 : Ax^* = By^*, x^* \in C, y^* \in Q\} \neq \emptyset$, *where* $C := F(S) \neq \emptyset$ *and* $Q := F(T) \neq \emptyset$, *then*

(I) *the sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution* $(x^*, y^*) \in \Gamma$ *of* (1.4).
(II) *In addition, if* S *and* T *are semi-compact, then the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution* $(x^*, y^*) \in \Gamma$ *of* (1.4).

4. Application to the split equality variational inequality problem in Banach spaces

Throughout this section, we assume that C and Q are nonempty and closed convex subsets of E_1 and E_2, respectively.

Let $M : C \to E_1$ be a mapping. Variational inequality problem (VIP) in Banach space is formulated as the problem of finding a point x^* with property $x^* \in C$ such that for some $j(z - x^*) \in J(x - x^*)$,

$$\langle Mx^*, j(z - x^*)\rangle \geq 0, \forall z \in C.$$

We will denote the solution set of VIP by VI(M, C).

A mapping $M : C \to E_1$ is said to be α-strongly accretive if for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Mx - My, j(x - y)\rangle \geq \alpha\|x - y\|^2,\ for\ \alpha > 0.$$

A mapping $M : C \to E_1$ is said to be β-inverse strongly accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Mx - My, j(x - y)\rangle \geq \beta\|Mx - My\|^2,\ for\ \beta > 0.$$

The equilibrium problem (for short, EP) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C.$$

The set of solutions of EP is denoted by $EP(F)$. Given a mapping $T : C \to C$, let $F(x, y) =< Tx, j(y - x) >$ for all $x, y \in C$. Then $x^* \in EP(F)$ if and only if $x^* \in C$ is a solution of the variational inequality $< Tx, j(y - x) >\geq 0$ for all $y \in C$, that is, x^* is a solution of the variational inequality.

Setting $F(x, y) = \langle Mx, j(y - x)\rangle$, it is easy to show that F satisfies the following conditions (A1)–(A4) as M is a β-inverse strongly accretive mapping.

(A1) $F(x, x) = 0,\ \forall x \in C$;

(A2) $F(x, y) + F(y, x) \leq 0,\ \forall x, y \in C$;

(A3) For all $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) For each $x \in C$, the function $y \longmapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 4.1 ([2]). *Let C be a closed convex subset of a reflexive, strictly convex and smooth Banach space E. Suppose F is a bifunction from $C \times C$ to R satisfying* (A1)–(A4)*, $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, j(z - x)\rangle \geq 0, \quad \forall y \in C.$$

Lemma 4.2 ([16]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Suppose F is a bifunction from $C \times C$ to R satisfying* (A1)–(A4)*. For $r > 0$ and $x \in E$, define a mapping $T_r : X \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r} < y - z, j(z - x) >\geq 0,\ \forall y \in C\}.$$

Then,

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, that is, $\forall x, y \in E$,*

$$\langle T_r(x) - T_r(y), j(T_r(x)) - j(T_r(y))\rangle \leq< T_r(x) - T_r(y), j(x) - j(y) >;$$

(3) *$F(T_r) = EP(F)$;*
(4) *$EP(F)$ is closed and convex and T_r is a relatively nonexpansive mappings.*

Let $B_1 : C \to E_1$ and $B_2 : Q \to E_2$ be two β-inverse-strongly accretive mappings, where C and Q are nonempty and closed convex subsets of E_1 and E_2, respectively. The "so-called" split equality variational inequality problem is shown that it is equivalent to find $x^* \in C, y^* \in Q$ such that

$$< B_1(x^*), j_1(x - x^*) >\geq 0,\ for\ all\ x \in C,$$

and

$$< B_2(y^*), j_2(y - y^*) >\geq 0, for\ all\ y \in Q,$$

and such that

$$Ax^* = By^*. \tag{4.1}$$

We will denote the solution set of split equality variational inequality problem by Ω, that is,

$$\Omega = \{x^*, y^*) \in VI(B_1, C) \times VI(B_2, Q) : Ax^* = By^*\}.$$

Setting $F(x, y) =< B_1x, j_1(y - x) >$, and $G(x, y) =< B_2x, j_2(y - x) >$, it is easy to show that F and G satisfy the conditions (A1)–(A4) as B_i $(i = 1, 2)$ is a β_i-inverse-strongly accretive mapping. For $r > 0$, $x \in E_1$ and $u \in E_2$, define mappings $T_r : E_1 \to C$ and $S_r : E_2 \to Q$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r} < y - z, j_1(z - x) >\geq 0,\ \forall y \in C\},$$

and

$$S_r(u) = \{z \in Q : G(z, v) + \frac{1}{r} < v - z, j_2(z - u) >\geq 0,\ \forall v \in Q\}.$$

It follows from Lemma 4.2 that $F(T_r) = VI(B_1, C) \neq \emptyset$, $F(S_r) = VI(B_2, Q) \neq \emptyset$, $T_r(x)$ and $S_r(u)$ are single-valued and firmly nonexpansive mappings, respectively. Therefore the split equality variational inequality problem with respect to B_1 and B_2 is equivalent to the following split equality fixed point problem:

$$to\ find\ x^* \in F(T_r),\ y^* \in F(S_r)\ such\ that\ Ax^* = By^*.$$

Then it follows from Theorem 3.1 that the following result holds.

Theorem 4.3. *Let E_1 and E_2 be real uniformly convex and 2-uniformly smooth Banach spaces satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, E_3 be a real Banach space, $C \subseteq E_1$ and $Q \subseteq E_2$ be nonempty closed convex subsets of E_1 and E_2, respectively. Let B_i $(i = 1, 2)$ is a η_i-inverse strongly accretive mappings, and $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be two bounded linear operators with adjoints A^* and B^*, respectively, T_r and S_r be the resolvent operator of the equilibrium function F and G, respectively. For any given $(x_0, y_0) \in E_1 \times E_2$, the sequence $\{(x_n, y_n)\}$ is generated by*

$$\begin{cases} z_n \in J_3(Ax_n - By_n) \\ u_n = T_r(x_n - \gamma J_1^{-1}A^*z_n) \\ v_n = S_r(y_n + \gamma J_2^{-1}B^*z_n) \\ y_{n+1} = \beta_n v_n + (1 - \beta_n)(y_n + \gamma J_2^{-1}B^*z_n) \\ x_{n+1} = \beta_n u_n + (1 - \beta_n)(x_n - \gamma J_1^{-1}A^*z_n), \end{cases}$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, γ is a positive real number satisfying $\frac{1}{\|A\|^2+\|B\|^2} < \gamma < \frac{2}{\|A\|^2+\|B\|^2}$. If $\Omega = \{x^*, y^*) \in VI(B_1, C) \times VI(B_2, Q) : Ax^* = By^*\} \neq \emptyset$, then

(I) *the sequence $\{(x_n, y_n)\}$ converges weakly to a solution $(x^*, y^*) \in \Omega$ of (4.1);*
(II) *In addition, if S_r and T_r are semi-compact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$ of (4.1).*

Acknowledgment

The authors wish to thank the editor and the referee for valuable suggestions. This work was supported by the National Natural Science Foundation of China (Grant No. 11361070).

References

[1] H. Attouch, J. Bolte, P. Redont, A. Soubeyran, *Alternating proximal algorithms for weakly coupled convex minimization problems, Applications to dynamical games and PDEs.*, J. Convex Anal., **15** (2008), 485–506. 1

[2] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems.* Math. Stud., **63** (1994), 123–145. 4.1

[3] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, **18** (2002), 441–453. 1, 1

[4] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., **51** (2006), 2353–2365. 1, 1

[5] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, **8** (1994), 221–239. 1

[6] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, *The multiple-sets split feasibility problem and its applications*, Inverse Problems, **21** (2005), 2071–2084. 1, 1, 1

[7] Y. Censor, A. Motova, A. Segal, *Perturbed projections and subgradient projections for the multiple-sets split feasibility problem*, J. Math. Anal. Appl., **327** (2007), 1244–1256. 1, 1

[8] S.-S. Chang, Y. J. Cho, H. Zhou, *Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings*, J. Korean Math. Soc., **38** (2001), 1245–1260. 2.3

[9] P. Cholamjiak, Y. Shehu, *Fixed point problem involving an asymptotically nonexpansive semigroup and a total asymptotically strict pseudocontraction*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 1

[10] K. Goebel, W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35** (1972), 171–174. 1

[11] Z. L. Ma, L. Wang, *An algorithm with strong convergence for the split common fixed point problem of total asymptotically strict pseudocontraction mappings*, J. Inequal. Appl., **2015** (2015), 13 pages. 1

[12] A. Moudafi, *A relaxed alternating CQ-algorithms for convex feasibility problems*, Nonlinear Anal., **79** (2013), 117–121. 1

[13] A. Moudafi, E. Al-Shemas, *Simultaneous iterative methods for split equality problems*, Trans. Math. Program. Appl., **1** (2013), 1–11. 1

[14] P. Sunthrayuth, P. Kumam, *Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2012** (2012), 18 pages. 1.2

[15] W. Takahashi, J.-C. Yao, *Strong convergence theorems by hybrid methods for the split common null point problem in Banach spaces*, Fixed point theory Appl., **2015** (2015), 13 pages. 1

[16] W. Takahashi, K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Nonlinear Anal., **70** (2009), 45–57. 4.2

[17] K.-K. Tan, H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., **178** (1993), 301–308. 2.4

[18] J. F. Tang, S.-S. Chang, L. Wang, X. R. Wang, *On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces*, J. Inequal. Appl., **2015** (2015), 11 pages. 1

[19] R. Wangkeeree, P. Preechasilp, *The General Iterative Methods for Asymptotically Nonexpansive Semigroups in Banach Spaces*, Abstr. Appl. Anal., **2012** (2012), 20 pages. 1.1

[20] Y. J. Wu, R. D. Chen, L. Y. Shi, *Split equality problem and multiple-sets split equality problem for quasi-nonexpansive multi-valued mappings*, J. Inequal. Appl., **2014** (2014), 8 pages. 1

[21] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127–1138. 2.1, 2.2