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On strong intuitionistic fuzzy metrics

Hakan Efe[a,*], Ebru Yigit[b]

[a]Department of Mathematics, Faculty of Science, Gazi University Teknikokullar, Ankara, 06500, TURKEY.
[b]Graduate School of Natural and Applied Science, Gazi University Teknikokullar, Ankara, 06500, TURKEY.

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Abstract

In this paper we give some properties of a class of intuitionistic fuzzy metrics which is called strong. This new class includes the class of stationary intuitionistic fuzzy metrics. So we examine the relationship between strong intuitionistic fuzzy metric and stationary intuitionistic fuzzy metric.

Keywords: Continuous t-norm, continuous t-conorm, intuitionistic fuzzy metric, strong intuitionistic fuzzy metric, stationary intuitionistic fuzzy metric.
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1. Introduction and Preliminaries

Since the introduction of fuzzy sets by Zadeh [22] in 1965, many authors have introduced the concept of fuzzy metric spaces in different ways [4, 5, 9, 10, 13, 14]. Especially, George and Veeramani [7–9], have introduced a notion of fuzzy metric spaces with the help of continuous t-norms, which constitutes a modification of the one due to Kramosil and Michalek [14]. Then many authors have made contribution to the notion of fuzzy metric spaces [6, 12, 17, 20].

On the other hand Sapena and Morillas [19] have studied notion of strong fuzzy metrics. They have discussed several important properties as strong fuzzy metrics, also some aspects of the completion of this type fuzzy metrics attending to their associated continuous t-norms. Especially they have given the class of stationary fuzzy metrics $(M, *)$, where $*$ is integral, are completable.

Park [16], using the idea of intuitionistic fuzzy sets which was introduced by Atanassov [2], has defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [7]. Besides, he has introduced

*Corresponding author
Email addresses: hakanefe@gazi.edu.tr (Hakan Efe), yigittebru@gmail.com (Ebru Yigit)

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the notion of Cauchy sequences in an intuitionistic fuzzy metric space, has proved the Baire's Theorem and Uniform Limit Theorem for intuitionistic fuzzy metric spaces. Later many authors have studied on intuitionistic fuzzy metric spaces [1, 11] and intuitionistic fuzzy topological spaces [3, 18].

Most of intuitionistic fuzzy metrics in the sense of Park [16], satisfy the following conditions

$$M(x,z,t) \geq M(x,y,t) * M(y,z,t)$$

and

$$N(x,z,t) \leq N(x,y,t) \diamondsuit N(y,z,t).$$

In this paper we study some properties of this class of intuitionistic fuzzy metrics called strong. Also we see that this class (strong) of intuitionistic fuzzy metric includes the class of stationary intuitionistic fuzzy metrics and each strong intuitionistic fuzzy metric is characterized by a family $(\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\})$ of stationary intuitionistic fuzzy metrics.

Definition 1.1 ([21])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

Definition 1.2 ([21])**.** A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamondsuit$ satisfies the following conditions:

(i) $\diamondsuit$ is commutative and associative;
(ii) $\diamondsuit$ is continuous;
(iii) $a \diamondsuit 1 = a$ for all $a \in [0,1]$;
(iv) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

Remark 1.3.

(i) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamondsuit r_2$.
(ii) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamondsuit r_7 \leq r_5$.

Definition 1.4 ([16])**.** A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x,\ y,\ z \in X,\ s, t > 0$,

(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \longrightarrow [0,1]$ is continuous;
(IFM-7) $N(x,y,t) > 0$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t) \diamondsuit N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \longrightarrow [0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of nonnearness between x and y with respect to t, respectively.

Remark 1.5.

(i) Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamondsuit)$ such that t-norm $*$ and t-conorm $\diamondsuit$ are associated (see [15]), that is, $x\diamondsuit y = 1-((1-x)*(1-y))$ for any $x, y \in [0,1]$.

(ii) In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Example 1.6 ([16], Induced Intuitionistic Fuzzy Metric)**.** Let (X, d) be a metric space. Denote $a*b = a.b$ and $a\diamondsuit b = \min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d(x,y,t) = \frac{ht^n}{ht^n + md(x,y)}, N_d(x,y,t) = \frac{d(x,y)}{kt^n + md(x,y)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Remark 1.7. Note the Example 1.6 holds even with the t-norm $a*b = \min\{a,b\}$ and the t-conorm $a\diamondsuit b = \max\{a,b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the Example 1.6 by taking $h = k = m = n = 1$, we get

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}, N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}.$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 1.8 ([16])**.** Let $X = \mathbb{N}$. Define $a*b = \max\{0, a+b-1\}$ and $a\diamondsuit b = a+b-ab$ for all $a, b \in [0,1]$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ as follows:

$$M(x,y,t) = \begin{cases} \frac{x}{y} & \text{if } x \le y, \\ \frac{y}{x} & \text{if } y \le x, \end{cases} \quad , \; N(x,y,t) = \begin{cases} \frac{y-x}{y} & \text{if } x \le y, \\ \frac{x-y}{x} & \text{if } y \le x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space.

Remark 1.9. Note that, in the Example 1.8, t-norm $*$ and t-conorm $\diamondsuit$ are not associated. And there exists no metric d on X satisfying

$$M(x,y,t) = \frac{t}{t + d(x,y)}, \; N(x,y,t) = \frac{d(x,y)}{t + d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in the Example 1.8. Also note the above functions (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined $a*b = \min\{a,b\}$ and $a\diamondsuit b = \max\{a,b\}$.

Definition 1.10 ([16])**.** Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B_{(M,N)}(x,r,t) = \{y \in X : M(x,y,t) > 1-r, \, N(x,y,t) < r\}$$

is called the open ball with center x and radius r with respect to t.

Theorem 1.11 ([16])**.** *Every open ball $B_{(M,N)}(x,r,t)$ is an open set.*

Remark 1.12. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Define
$\tau_{(M,N)} = \left\{A \subset X : \text{for each } x \in A, \text{ there exist } t > 0 \text{ and } r \in (0,1) \ni B_{(M,N)}(x,r,t) \subset A\right\}$.
Then $\tau_{(M,N)}$ is a topology on X.

Remark 1.13.

(i) Since $\left\{B_{(M,N)}\left(x,\frac{1}{n},\frac{1}{n}\right):n=1,2,...\right\}$ is a local base at x, the topology $\tau_{(M,N)}$ is first countable.

(ii) Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n\longrightarrow x$ if and only if $M(x_n,x,t)\longrightarrow 1$ and $N(x_n,x,t)\longrightarrow 0$ as $n\longrightarrow\infty$.

Theorem 1.14 ([16])**.** *Every intuitionistic fuzzy metric space is Hausdorff.*

Definition 1.15 ([16])**.** Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space. Then,

(i) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon>0$ and each $t>0$, there exist $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ and $N(x_n,x_m,t)<\varepsilon$ for all $n,m\geq n_0$.

(ii) $(X,M,N,*,\diamondsuit)$ is called complete if every Cauchy sequence is convergent with respect to $\tau_{(M,N)}$.

2. Main results

Definition 2.1. Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space. The intuitionistic fuzzy metric (M,N) is said to be strong if it satisfies for each $x,y,z\in X$ and each $t>0$

$$M(x,z,t)\geq M(x,y,t)*M(y,z,t)\,, \tag{IFM-5$'$}$$

$$N(x,z,t)\leq N(x,y,t)\diamondsuit N(y,z,t)\,. \tag{IFM-10$'$}$$

Notice that the axioms (IFM-5′) and (IFM-10′) cannot replace axioms (IFM-5) and (IFM-10) in the definition of intuitionistic fuzzy metric, respectively because in this case (M,N) could not be an intuitionistic fuzzy metric on X. Example 2.2 shows this case.

Example 2.2. Consider the usual metric $|\cdot|$ on $\mathbb{R}$. Define the functions $M:\mathbb{R}^2\times\mathbb{R}^+\longrightarrow(0,1]$ and $N:\mathbb{R}^2\times\mathbb{R}^+\longrightarrow(0,1]$ by

$$M(x,y,t)=\frac{\frac{1}{t}}{\frac{1}{t}+|x-y|}\text{ and }N(x,y,t)=\frac{|x-y|}{\frac{1}{t}+|x-y|}.$$

Denote $a*b=ab$ and $a\diamondsuit b=a+b-ab$ for all $a,b\in[0,1]$. It is clear that, (M,N) satisfies (IFM-1), (IFM-2), (IFM-3), (IFM-4), (IFM-6), (IFM-7), (IFM-8), (IFM-9), and (IFM-11) but it does not satisfy (IFM-5) and (IFM-10) with the continuous t-norm and continuous t-conorm defined by $a*b=ab$ and $a\diamondsuit b=a+b-ab$. So indeed, when we choose $x=1$, $y=2$, $z=3$ and $t=s=1$ we get

$$M(x,z,t+s)=\frac{\frac{1}{2}}{\frac{1}{2}+|1-3|}<\frac{\frac{1}{1}}{\frac{1}{1}+|1-2|}.\frac{\frac{1}{1}}{\frac{1}{1}+|2-3|}=M(x,y,t).M(y,z,s)$$

and

$$\begin{aligned}N(x,z,t+s)&=\frac{2}{\frac{1}{2}+2}\\&>\frac{1}{\frac{1}{1}+1}+\frac{1}{\frac{1}{1}+1}-\left[\frac{1}{\frac{1}{1}+1}.\frac{1}{\frac{1}{1}+1}\right]\\&=N(x,y,t)+N(y,z,s)-\left[N(x,y,t).N(y,z,s)\right].\end{aligned}$$

Nonetheless, (M,N) satisfy (IFM-5′) and (IFM-10′) with the same continuous t-norm $*$ and continuous t-conorm $\diamondsuit$. Indeed for all $x,y,z\in X$ and $t>0$,

$$M(x,y,t).M(y,z,t)=\frac{\frac{1}{t^2}}{\frac{1}{t^2}+\frac{1}{t}|y-z|+\frac{1}{t}|x-y|+|x-y|.|y-z|}$$

$$\leq \frac{\frac{1}{t^2}}{\frac{1}{t^2}+\frac{1}{t}\left|y-z\right|+\frac{1}{t}\left|x-y\right|}$$
$$\leq \frac{\frac{1}{t^2}}{\frac{1}{t^2}+\frac{1}{t}\left|x-z\right|}=\frac{\frac{1}{t}}{\frac{1}{t}+\left|x-z\right|}=M(x,z,t)$$

and

$$\begin{aligned}N\left(x,y,t\right)+N(y,z,t)-\left[N\left(x,y,t\right).N(y,z,t)\right] &= \frac{\frac{1}{t^2}+\frac{1}{t}\left|x-y\right|+\frac{1}{t}\left|y-z\right|+\left|x-y\right|.\left|y-z\right|-\frac{1}{t^2}}{\frac{1}{t^2}+\frac{1}{t}\left|y-z\right|+\frac{1}{t}\left|x-y\right|+\left|x-y\right|.\left|y-z\right|}\\ &\geq \frac{\frac{1}{t}\left|y-z\right|+\frac{1}{t}\left|x-y\right|}{\frac{1}{t^2}+\frac{1}{t}\left|y-z\right|+\frac{1}{t}\left|x-y\right|}\\ &\geq \frac{\left|x-z\right|}{\frac{1}{t}+\left|x-z\right|}=N(x,z,t).\end{aligned}$$

So we can say the following remark.

Remark 2.3. Notice that it is possible to define a strong intuitionistic fuzzy metric by replacing (IFM-5) by (IFM-5′) and demanding in (IFM-6) that the function $M_{x,y}$ be an increasing continuous function on t and by replacing (IFM-10) by (IFM-10′) and demanding in (IFM-11) that the function $N_{x,y}$ be a decreasing continuous function on t, for each $x,y \in X$. So indeed, in such a case we have

$$M\left(x,z,t+s\right)\geq M(x,y,t+s)*M(y,z,t+s)\geq M(x,y,t)*M(y,z,s)$$

and

$$N\left(x,z,t+s\right)\leq N(x,y,t+s)\Diamond N(y,z,t+s)\leq N(x,y,t)\Diamond N(y,z,s).$$

Definition 2.4. Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space. The intuitionistic fuzzy metric (M,N) is said to be stationary if M and N don't depend on t, in other words the functions $M_{x,y}$ and $N_{x,y}$ are constant for each $x,y \in X$.

If $(X,M,N,*,\Diamond)$ is a stationary intuitionistic fuzzy metric space, we will denote $M(x,y)$, $N(x,y)$ and $B_{(M,N)}(x,r)$ instead of $M(x,y,t)$, $N(x,y,t)$ and $B_{(M,N)}(x,r,t)$, respectively.

Example 2.5. Stationary intuitionistic fuzzy metrics are strong.

Recall that a metric d on X is called non-Archimedean (ultrametric) if

$$d(x,z)\leq \max\left\{d(x,y),d(y,z)\right\},\ \text{for all } x,y,z\in X.$$

Now we give the Definition 2.6.

Definition 2.6. An intuitionistic fuzzy metric $(M,N,*,\Diamond)$ on X is said to be non-Archimedean (ultrametric) if it satisfies

$$M(x,z,t)\geq \min\left\{M(x,y,t),M(y,z,t)\right\}, \tag{2.1}$$
$$N(x,z,t)\leq \max\left\{N(x,y,t),N(y,z,t)\right\} \tag{2.2}$$

for all $x,y,z\in X$, $t>0$.

Example 2.7. Intuitionistic fuzzy ultrametrics are strong.

Proposition 2.8. *Let $f : X \longrightarrow \mathbb{R}^+$ be a one-to-one function and let $\varphi : \mathbb{R}^+ \longrightarrow [0,+\infty)$ be an increasing continuous function. Fixed $\alpha,\beta>0$, denote $a*b=ab$ and $a\Diamond b=a+b-ab$ for all $a,b\in[0,1]$, define M and N by*

$$M(x,y,t)=\left(\frac{\left(\min\left\{f(x),f(y)\right\}\right)^{\alpha}+\varphi(t)}{\left(\max\left\{f(x),f(y)\right\}\right)^{\alpha}+\varphi(t)}\right)^{\beta},$$

$$N(x,y,t) = 1 - \left(\frac{(\min\{f(x),f(y)\})^\alpha + \varphi(t)}{(\max\{f(x),f(y)\})^\alpha + \varphi(t)}\right)^\beta .$$

Then, $(M,N,*,\diamondsuit)$ *is an intuitionistic fuzzy metric on* X.

Proof.

(IFM-1) $M(x,y,t) + N(x,y,t) = 1$ for all $x,y \in X$, $t > 0$.

(IFM-2) It is obvious that $M(x,y,t) > 0$ for all $x,y \in X$, $t > 0$.

(IFM-3)

$$\begin{aligned} M(x,y,t) = \left(\frac{(\min\{f(x),f(y)\})^\alpha + \varphi(t)}{(\max\{f(x),f(y)\})^\alpha + \varphi(t)}\right)^\beta &= 1 \Leftrightarrow \\ \Leftrightarrow \min\{f(x),f(y)\} = \max\{f(x),f(y)\} \Longleftrightarrow f(x) = f(y) \Leftrightarrow x = y. \end{aligned}$$

(IFM-4) It is obvious that $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$, $t > 0$.

(IFM-5) (a) Suppose that $f(x) \le f(z)$. In such a case three cases are possible:

Case1. $f(x) \le f(y) \le f(z)$.

Case2. $f(y) \le f(x) \le f(z)$.

Case3. $f(x) \le f(z) \le f(y)$.

We can write $M(x,z,t+s)$ as a simple way for our operations by

$$M(x,z,t+s) = \left(\frac{f(x)^\alpha + \varphi(t+s)}{f(y)^\alpha + \varphi(t+s)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(t+s)}{f(z)^\alpha + \varphi(t+s)}\right)^\beta .$$

And by using that the function φ is increasing we examine the above three cases.

(1)

$$\begin{aligned} M(x,z,t+s) &= \left(\frac{f(x)^\alpha + \varphi(t+s)}{f(y)^\alpha + \varphi(t+s)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(t+s)}{f(z)^\alpha + \varphi(t+s)}\right)^\beta \\ &\ge \left(\frac{f(x)^\alpha + \varphi(t)}{f(y)^\alpha + \varphi(t)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(s)}{f(z)^\alpha + \varphi(s)}\right)^\beta \\ &= M(x,y,t).M(y,z,s). \end{aligned}$$

(2)

$$\begin{aligned} M(x,z,t+s) &= \left(\frac{f(x)^\alpha + \varphi(t+s)}{f(y)^\alpha + \varphi(t+s)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(t+s)}{f(z)^\alpha + \varphi(t+s)}\right)^\beta \\ &\ge \left(\frac{f(x)^\alpha + \varphi(t)}{f(y)^\alpha + \varphi(t)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(s)}{f(z)^\alpha + \varphi(s)}\right)^\beta \\ &\ge \left(\frac{f(y)^\alpha + \varphi(t)}{f(x)^\alpha + \varphi(t)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(s)}{f(z)^\alpha + \varphi(s)}\right)^\beta \\ &= M(x,y,t).M(y,z,s). \end{aligned}$$

(3)

$$M(x,z,t+s) = \left(\frac{f(x)^\alpha + \varphi(t+s)}{f(y)^\alpha + \varphi(t+s)}\right)^\beta . \left(\frac{f(y)^\alpha + \varphi(t+s)}{f(z)^\alpha + \varphi(t+s)}\right)^\beta$$

$$\begin{aligned}&\geq \left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}\cdot\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\\&\geq \left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}\cdot\left(\frac{f(z)^{\alpha}+\varphi(s)}{f(y)^{\alpha}+\varphi(s)}\right)^{\beta}\\&= M(x,y,t).M(y,z,s).\end{aligned}$$

(b) Similar operations are performed if $f(x) > f(z)$.

(IFM-6) It is obvious that $M(x, y, .) : (0,\infty) \longrightarrow (0, 1]$ is continuous.

(IFM-7) It is obvious that $N(x, y, t) > 0$ for all $x, y \in X$, $t > 0$.

(IFM-8)

$$\begin{aligned}N(x,y,t) = 1-\left(\frac{(\min\{f(x),f(y)\})^{\alpha}+\varphi(t)}{(\max\{f(x),f(y)\})^{\alpha}+\varphi(t)}\right)^{\beta} = 0 \Leftrightarrow\\ \Leftrightarrow \min\{f(x),f(y)\}=\max\{f(x),f(y)\} \Longleftrightarrow f(x)=f(y)\Leftrightarrow x=y.\end{aligned}$$

(IFM-9) It is obvious that $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$, $t > 0$.

(IFM-10) (a) Suppose that $f(x) \leq f(z)$. In this case there are three cases:

Case1. $f(x) \leq f(y) \leq f(z)$.

Case2. $f(y) \leq f(x) \leq f(z)$.

Case3. $f(x) \leq f(z) \leq f(y)$.

Remember the function φ is increasing.

(1)

$$\begin{aligned}N(x,y,t)\Diamond N(y,z,t) &= N(x,y,t)+N(y,z,t)-[N(x,y,t).N(y,z,t)]\\&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}+1-\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\\&\quad-\left(1-\left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}\right)\cdot\left(1-\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\right)\\&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}\cdot\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\\&\geq 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(y)^{\alpha}+\varphi(t+s)}\right)^{\beta}\cdot\left(\frac{f(y)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta}\\&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta} = N(x,z,t+s).\end{aligned}$$

(2)

$$\begin{aligned}N(x,y,t)\Diamond N(y,z,t) &= N(x,y,t)+N(y,z,t)-[N(x,y,t).N(y,z,t)]\\&= 1-\left(\frac{f(y)^{\alpha}+\varphi(t)}{f(x)^{\alpha}+\varphi(t)}\right)^{\beta}+1-\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\\&\quad-\left(1-\left(\frac{f(y)^{\alpha}+\varphi(t)}{f(x)^{\alpha}+\varphi(t)}\right)^{\beta}\right)\cdot\left(1-\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\right)\end{aligned}$$

$$
\begin{aligned}
&= 1-\left(\frac{f(y)^{\alpha}+\varphi(t)}{f(x)^{\alpha}+\varphi(t)}\right)^{\beta}\cdot\left(\frac{f(y)^{\alpha}+\varphi(s)}{f(z)^{\alpha}+\varphi(s)}\right)^{\beta}\\
&\geq 1-\left(\frac{f(y)^{\alpha}+\varphi(t+s)}{f(x)^{\alpha}+\varphi(t+s)}\right)^{\beta}\cdot\left(\frac{f(y)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta}\\
&\geq 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(x)^{\alpha}+\varphi(t+s)}\right)^{\beta}\cdot\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta}\\
&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta} = N(x,z,t+s).
\end{aligned}
$$

(3)

$$
\begin{aligned}
N(x,y,t)\Diamond N(y,z,t) &= N(x,y,t)+N(y,z,t)-[N(x,y,t)+N(y,z,t)]\\
&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}+1-\left(\frac{f(z)^{\alpha}+\varphi(s)}{f(y)^{\alpha}+\varphi(s)}\right)^{\beta}\\
&\quad-\left(1-\left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}\right)\cdot\left(1-\left(\frac{f(z)^{\alpha}+\varphi(s)}{f(y)^{\alpha}+\varphi(s)}\right)^{\beta}\right)\\
&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t)}{f(y)^{\alpha}+\varphi(t)}\right)^{\beta}\cdot\left(\frac{f(z)^{\alpha}+\varphi(s)}{f(y)^{\alpha}+\varphi(s)}\right)^{\beta}\\
&\geq 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(y)^{\alpha}+\varphi(t+s)}\right)^{\beta}\cdot\left(\frac{f(z)^{\alpha}+\varphi(t+s)}{f(y)^{\alpha}+\varphi(t+s)}\right)^{\beta}\\
&\geq 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta}\cdot\left(\frac{f(z)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta}\\
&= 1-\left(\frac{f(x)^{\alpha}+\varphi(t+s)}{f(z)^{\alpha}+\varphi(t+s)}\right)^{\beta} = N(x,z,t+s).
\end{aligned}
$$

(IFM-11) It is obvious that $N(x,y,.):(0,\infty)\longrightarrow(0,1]$ is continuous.

□

Example 2.9. Let $X=\mathbb{R}^{+}$. Define $a*b=ab$ and $a\Diamond b=a+b-ab$ for all $a,b\in[0,1]$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$ as follows:

$$
M(x,y,t)=\frac{\min\{x,y\}+\varphi(t)}{\max\{x,y\}+\varphi(t)},\ N(x,y,t)=\frac{\max\{x,y\}-\min\{x,y\}}{\max\{x,y\}+\varphi(t)}
$$

for all $x,y\in X$, $t>0$. Then the intuitionistic fuzzy metric $(M,N,*,\Diamond)$ is strong. Where $\varphi:(0,\infty)\longrightarrow(0,\infty)$ is an increasing and continuous function.

In the Proposition 2.8, if we choose $\alpha=\beta=1$ we have the functions M and N in this example. So $(M,N,*,\Diamond)$ is an intuitionistic fuzzy metric on X. Now we show the conditions (IFM-5′) and (IFM-10′) to see $(M,N,*,\Diamond)$ is strong. Take $x,y,z\in\mathbb{R}^{+}$ and $t>0$.

(IFM-5′) (a) Suppose that $x\leq z$. In such a case three cases are possible:

Case1. Let $x\leq y\leq z$.

$$
M(x,z,t)=\frac{x+\varphi(t)}{z+\varphi(t)}=\frac{x+\varphi(t)}{y+\varphi(t)}.\frac{y+\varphi(t)}{z+\varphi(t)}=M(x,y,t).M(y,z,t).
$$

Case2. Let $y\leq x\leq z$.

$$
M(x,z,t)=\frac{x+\varphi(t)}{z+\varphi(t)}\geq\frac{y+\varphi(t)}{x+\varphi(t)}.\frac{x+\varphi(t)}{z+\varphi(t)}\geq\frac{y+\varphi(t)}{x+\varphi(t)}.\frac{y+\varphi(t)}{z+\varphi(t)}=M(x,y,t).M(y,z,t).
$$

Case3. Let $x \leq z \leq y$.

$$M(x,z,t) = \frac{x+\varphi(t)}{z+\varphi(t)} = \frac{x+\varphi(t)}{y+\varphi(t)}.\frac{y+\varphi(t)}{z+\varphi(t)} \geq \frac{x+\varphi(t)}{y+\varphi(t)}.\frac{z+\varphi(t)}{y+\varphi(t)} = M(x,y,t).M(y,z,t).$$

(b) Similar operations are performed if $z < x$.

(IFM-10′) Firstly we show equivalent of $N(x,y,t) + N(y,z,t) - [N(x,y,t) + N(y,z,t)]$, then examine the cases.

$$\begin{aligned}
&N(x,y,t)\Diamond N(y,z,t) \\
&\quad = N(x,y,t) + N(y,z,t) - [N(x,y,t).N(y,z,t)] = \\
&\quad = \frac{\max\{x,y\} - \min\{x,y\}}{\max\{x,y\}+\varphi(t)} + \frac{\max\{y,z\} - \min\{y,z\}}{\max\{y,z\}+\varphi(t)} \\
&\qquad - \left[\frac{\max\{x,y\} - \min\{x,y\}}{\max\{x,y\}+\varphi(t)}.\frac{\max\{y,z\} - \min\{y,z\}}{\max\{y,z\}+\varphi(t)}\right] \\
&\quad = \frac{\varphi(t)\max\{x,y\} - \varphi(t)\min\{x,y\} + \max\{x,y\}\max\{y,z\}}{[\max\{x,y\}+\varphi(t)]\,[\max\{y,z\}+\varphi(t)]} + \\
&\qquad + \frac{\varphi(t)\max\{y,z\} - \varphi(t)\min\{y,z\} - \min\{x,y\}\min\{y,z\}}{[\max\{x,y\}+\varphi(t)]\,[\max\{y,z\}+\varphi(t)]} \\
&\quad = \frac{\max\{x,y\}\,[\max\{y,z\}+\varphi(t)] + \varphi(t)\,[\max\{y,z\}+\varphi(t)]}{[\max\{x,y\}+\varphi(t)]\,[\max\{y,z\}+\varphi(t)]} + \\
&\qquad + \frac{-\min\{x,y\}\,[\min\{y,z\}+\varphi(t)] - \varphi(t)\,[\min\{y,z\}+\varphi(t)]}{[\max\{x,y\}+\varphi(t)]\,[\max\{y,z\}+\varphi(t)]} \\
&\quad = \frac{[\max\{x,y\}+\varphi(t)]\,[\max\{y,z\}+\varphi(t)] - [\min\{x,y\}+\varphi(t)]\,[\min\{y,z\}+\varphi(t)]}{[\max\{x,y\}+\varphi(t)]\,[\max\{y,z\}+\varphi(t)]}.
\end{aligned}$$

(a) Suppose that $x \leq z$. In such a case three cases are possible:
Case1. Let $x \leq y \leq z$.

$$\begin{aligned}
N(x,y,t) &+ N(y,z,t) - [N(x,y,t) + N(y,z,t)] \\
&= \frac{[y+\varphi(t)]\,[z+\varphi(t)] - [x+\varphi(t)]\,[y+\varphi(t)]}{[y+\varphi(t)]\,[z+\varphi(t)]} \\
&= 1 - \frac{[x+\varphi(t)]}{[z+\varphi(t)]} = \frac{z-x}{z+\varphi(t)} = N(x,z,t).
\end{aligned}$$

Case2. Let $y \leq x \leq z$.

$$\begin{aligned}
N(x,y,t) &+ N(y,z,t) - [N(x,y,t) + N(y,z,t)] \\
&= \frac{[x+\varphi(t)]\,[z+\varphi(t)] - [y+\varphi(t)]\,[y+\varphi(t)]}{[x+\varphi(t)]\,[z+\varphi(t)]} \\
&= 1 - \frac{[y+\varphi(t)]}{[x+\varphi(t)]}.\frac{[y+\varphi(t)]}{[z+\varphi(t)]} \\
&\geq 1 - \frac{[x+\varphi(t)]}{[x+\varphi(t)]}.\frac{[x+\varphi(t)]}{[z+\varphi(t)]} \\
&= \frac{z-x}{z+\varphi(t)} = N(x,z,t).
\end{aligned}$$

Case3. Let $x \leq z \leq y$.

$$N(x,y,t) + N(y,z,t) - [N(x,y,t) + N(y,z,t)]$$

$$
\begin{aligned}
&= \frac{[y+\varphi(t)]\,[y+\varphi(t)] - [x+\varphi(t)]\,[z+\varphi(t)]}{[y+\varphi(t)]\,[y+\varphi(t)]} \\
&= 1 - \frac{[x+\varphi(t)]}{[y+\varphi(t)]}.\frac{[z+\varphi(t)]}{[y+\varphi(t)]} \\
&\geq 1 - \frac{[x+\varphi(t)]}{[z+\varphi(t)]}.\frac{[z+\varphi(t)]}{[z+\varphi(t)]} \\
&= \frac{z-x}{z+\varphi(t)} = N(x,z,t).
\end{aligned}
$$

(b) Similar operations are performed if $z < x$.

Proposition 2.10. *Let (K,P) be a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = ab$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$. And let the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing and continuous, $t > 0$ and M, N be fuzzy sets on $X^2 \times \mathbb{R}^+$ defined by*

$$
M(x,y,t) = \frac{\varphi(t)}{\varphi(t) + 1 - K(x,y)} \text{ and } N(x,y,t) = \frac{P(x,y)}{\varphi(t) + P(x,y)}.
$$

Then $(M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric on X.

Proof.

(IFM-1) We show that

$$
M(x,y,t) + N(x,y,t) = \frac{\varphi(t)}{\varphi(t) + 1 - K(x,y)} + \frac{P(x,y)}{\varphi(t) + P(x,y)} \leq 1
$$

is equivalent to

$$
\frac{\varphi(t)^2 + \varphi(t)P(x,y) + \varphi(t)P(x,y) + P(x,y) - P(x,y).K(x,y)}{[\varphi(t) + 1 - K(x,y)]\,[\varphi(t) + P(x,y)]} - 1 \leq 0.
$$

The last inequality means that

$$
\frac{\varphi(t)P(x,y) - \varphi(t) + \varphi(t)K(x,y)}{[\varphi(t) + 1 - K(x,y)]\,[\varphi(t) + P(x,y)]} \leq 0
$$

and this is equivalent to

$$
\varphi(t)\,[K(x,y) + P(x,y) - 1] \leq 0.
$$

Since $\varphi(t) \in \mathbb{R}^+$ for all $t > 0$, it is sufficient to see that $[K(x,y) + P(x,y) - 1] \leq 0$. Since (K,P) is a stationary intuitionistic fuzzy metric on X, $K(x,y) + P(x,y) \leq 1$. Then $K(x,y) + P(x,y) - 1 \leq 0$.

(IFM-2) It is clear that $M(x,y,t) > 0$ for all $x, y \in X$ and $t > 0$.

(IFM-3) By using the (K,P) which is a stationary intuitionistic fuzzy metric on X, we get

$$
M(x,y,t) = \frac{\varphi(t)}{\varphi(t) + 1 - K(x,y)} = 1 \Leftrightarrow K(x,y) = 1 \Leftrightarrow x = y.
$$

(IFM-4) Since (K,P) is a stationary intuitionistic fuzzy metric on X, $K(x,y) = K(y,x)$ for all $x, y \in X$. Then

$$
M(x,y,t) = \frac{\varphi(t)}{\varphi(t) + 1 - K(x,y)} = \frac{\varphi(t)}{\varphi(t) + 1 - K(y,x)} = M(y,x,t).
$$

(IFM-5) We show that $M(x,z,t+s) \geq M(x,y,t).M(y,z,s)$ for all $x,y,z \in X$, $t>0$. By using the (K,P) is a stationary intuitionistic fuzzy metric on X,

$$[1-K(x,y)]\,.\,[1-K(y,z)] = 1-K(x,y)-K(y,z)+K(x,y)K(y,z) \geq 0.$$

So we write

$$\varphi(t)\,[(1-K(x,y))\,.\,(1-K(y,z))] + 1 - K(x,y) - K(y,z) + K(x,y).K(y,z) \geq 0$$

$$\begin{aligned}
&\Rightarrow \varphi(t) - \varphi(t)K(x,y) - \varphi(t)K(y,z) + 1 + K(x,y)K(y,z) - K(x,y) - K(y,z)\\
&\quad \geq -\varphi(t)K(x,y)K(y,z)\\
&\Rightarrow [\varphi(t)+1-K(y,z)]\,[\varphi(t)+1-K(x,y)] \geq \varphi(t)\,[\varphi(t)+1-K(x,y)K(y,z)]\\
&\quad \geq [\varphi(t)+1-K(x,z)]\\
&\Rightarrow \frac{1}{[\varphi(t)+1-K(x,z)]} \geq \frac{\varphi(t)}{[\varphi(t)+1-K(y,z)]\,.\,[\varphi(t)+1-K(x,y)]}\\
&\Rightarrow \frac{\varphi(t)}{[\varphi(t)+1-K(x,z)]} \geq \frac{\varphi(t)}{[\varphi(t)+1-K(x,y)]}.\frac{\varphi(t)}{[\varphi(t)+1-K(y,z)]},
\end{aligned}$$

this means that $M(x,z,t) \geq M(x,y,t).M(y,z,t)$, also we can write $M(x,z,t+s) \geq M(x,y,t+s).M(y,z,t+s)$. Since the function M is increasing and continuous, we write $M(x,z,t+s) \geq M(x,y,t).M(y,z,s)$.

(IFM-6) It is clear that $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

(IFM-7) It is clear that $N(x,y,t) > 0$ for all $x,y \in X$ and $t>0$.

(IFM-8)

$$N(x,y,t) = \frac{P(x,y)}{\varphi(t)+P(x,y)} = 0 \Leftrightarrow P(x,y) = 0 \Leftrightarrow x = y.$$

(IFM-9) Since (K,P) is a stationary intuitionistic fuzzy metric on X, $P(x,y) = P(y,x)$ for all $x,y \in X$. Then

$$N(x,y,t) = \frac{P(x,y)}{\varphi(t)+P(x,y)} = \frac{P(y,x)}{\varphi(t)+P(y,x)} = N(y,x,t).$$

(IFM-10) We show that $N(x,z,t+s) \leq N(x,y,t) + N(y,z,s) - N(x,y,t).N(y,z,s)$ for all $x,y,z \in X$, $t,s>0$. By using the (K,P) which is a stationary intuitionistic fuzzy metric on X and P satisfies the condition $P(x,z) \leq P(x,y) + P(y,z) - P(x,y).P(y,z)$,

$$\begin{aligned}
N(x,y,t)\Diamond N(y,z,t) &= N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t) =\\
&= \frac{P(x,y)}{\varphi(t)+P(x,y)} + \frac{P(y,z)}{\varphi(t)+P(y,z)} - \left[\frac{P(x,y)}{\varphi(t)+P(x,y)}.\frac{P(y,z)}{\varphi(t)+P(y,z)}\right]\\
&= \frac{\varphi(t)^2 + \varphi(t)P(x,y) + \varphi(t)P(y,z) + P(x,y)P(y,z) - \varphi(t)^2}{\varphi(t)^2 + \varphi(t)P(x,y) + \varphi(t)P(y,z) + P(x,y)P(y,z)}\\
&= 1 - \frac{\varphi(t)^2}{\varphi(t)^2 + \varphi(t)P(x,y) + \varphi(t)P(y,z) + P(x,y)P(y,z)}\\
&\geq 1 - \frac{\varphi(t)^2}{\varphi(t)^2 + \varphi(t)P(x,y) + \varphi(t)P(y,z)}\\
&\geq 1 - \frac{\varphi(t)}{\varphi(t) + P(x,y) + P(y,z) - P(x,y)P(y,z)}
\end{aligned}$$

$$\geq 1-\frac{\varphi(t)}{\varphi(t)+P(x,z)}=\frac{P(x,z)}{\varphi(t)+P(x,z)}=N(x,z,t).$$

Also, by using N which is decreasing, we can write,

$$N(x,y,t+s)\leq N(x,y,t)$$

and

$$N(y,z,t+s)\leq N(y,z,s).$$

Then

$$\begin{aligned}1-N(x,y,t+s)&\geq 1-N(x,y,t)\\ 1-N(y,z,t+s)&\geq 1-N(y,z,s).\end{aligned}$$

If we multiply the last two inequalities side by side, we attain

$$\begin{aligned}&N(x,y,t+s)+N(y,z,t+s)-N(x,y,t+s)N(y,z,t+s)\\ &\leq N(x,y,t)+N(y,z,s)-N(x,y,t)N(y,z,s)\end{aligned}$$

then,

$$N(x,z,t+s)\leq N(x,y,t)+N(y,z,s)-N(x,y,t)N(y,z,s).$$

(IFM-11) It is clear that $N(x,y,.):(0,\infty)\to(0,1]$ is continuous.

□

In the Example 2.11 and Example 2.12 intuitionistic fuzzy metric (M,N) defined by means of a stationary intuitionistic fuzzy metric (K,P).

Example 2.11. Let (K,P) be a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a*b=ab$ and $a\diamondsuit b=a+b-ab$ for all $a,b\in[0,1]$, $t>0$ and M, N be fuzzy sets on $X^2\times\mathbb{R}^+$ defined by

$$M(x,y,t)=\frac{t}{t+1-K(x,y)}\text{ and }N(x,y,t)=\frac{P(x,y)}{t+P(x,y)}.$$

Then $(M,N,*,\diamondsuit)$ is a strong intuitionistic fuzzy metric on X.

It is clear that if we choose $\varphi(t)=t$ in the Proposition 2.10, $(M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric on X. So in this example we will only show the conditions (IFM-5′) and (IFM-10′) to see $(M,N,*,\diamondsuit)$ is strong.

(IFM-5′) We show that $M(x,z,t)\geq M(x,y,t).M(y,z,t)$ for all $x,y,z\in X$, $t>0$, that is,

$$\frac{t}{t+1-K(x,z)}\geq\frac{t}{t+1-K(x,y)}.\frac{t}{t+1-K(y,z)}$$

is equivalent to

$$t\left[1+K(x,z)-K(y,z)-K(x,y)\right]+1+K(x,y).K(y,z)-K(x,y)-K(y,z)\geq 0.$$

If we show that this inequality is true, we will say that the condition (IFM-5′) is satisfied. Since $t>0$ and

$$1+K(x,y).K(y,z)-K(x,y)-K(y,z)=\left[1-K(x,y)\right].\left[1-K(y,z)\right]\geq 0,$$

we need to show that $[1+K(x,z)-K(y,z)-K(x,y)]>0$. Since (K,P) is a stationary fuzzy metric on X, K satisfies $K(x,z)\geq K(x,y).K(y,z)$ (the condition IFM-5′). So we can write

$$\begin{aligned}1+K(x,z)-K(y,z)-K(x,y)&\geq 1+K(x,y).K(y,z)-K(x,y)-K(y,z)\\ &=\left[1-K(x,y)\right]\left[1-K(y,z)\right]>0.\end{aligned}$$

(IFM-10′) We show that $N(x,z,t) \leq N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t)$ for all $x,y,z \in X$, $t > 0$, that is,

$$\frac{P(x,z)}{t+P(x,z)} \leq \frac{P(x,y)}{t+P(x,y)} + \frac{P(y,z)}{t+P(y,z)} - \left[\frac{P(x,y)}{t+P(x,y)}.\frac{P(y,z)}{t+P(y,z)}\right]$$

is equivalent to

$$t^2\left[P(x,y) + P(y,z) - P(x,z)\right] + tP(x,y)P(y,z) \geq 0.$$

Then we need to show that $[P(x,y) + P(y,z) - P(x,z)] \geq 0$. Since (K,P) is a stationary intuitionistic fuzzy metric on X, P satisfies $P(x,z) \leq P(x,y) + P(y,z) - P(x,y).P(y,z)$ (the condition (IFM-10′). So we can write

$$-P(x,z) \geq -P(x,y) - P(y,z) + P(x,y).P(y,z)$$

and then

$$P(x,y) + P(y,z) - P(x,z) \geq P(x,y).P(y,z) \geq 0.$$

Example 2.12. Let (K,P) be a stationary intuitionistic fuzzy metric on X with the continuous t-norm and continuous t-conorm defined by $a * b = ab$ and $a\diamondsuit b = a + b - ab$ for all $a,b \in [0,1]$. And let $t > 0$ and M, N be fuzzy sets on $X^2 \times \mathbb{R}^+$ defined by

$$M(x,y,t) = \frac{t+K(x,y)}{t+1},\ N(x,y,t) = \frac{P(x,y)}{t+1}.$$

Then $(M,N,*,\diamondsuit)$ is a strong intuitionistic fuzzy metric on X.

Firstly we show that $(M,N,*,\diamondsuit)$ is an intuitionistic fuzzy metric on X. We only proof the conditions (IFM-5) and (IFM-10) since the others are obvious.

(IFM-5) It is clear that for all $a,b \in [0,1]$, $t,s \in \mathbb{R}^+$

$$\frac{t+a}{t+1}.\frac{t+b}{t+1} \leq \frac{(t+s)+a}{(t+s)+1}.\frac{(t+s)+b}{(t+s)+1} \leq \frac{(t+s)+a.b}{(t+s)+1}. \tag{2.3}$$

Since $K(x,y).K(y,z) \in [0,1]$ and also $K(x,z) \geq K(x,y).K(y,z)$, by using (2.3)

$$\begin{aligned} M(x,y,t) * M(y,z,s) = M(x,y,t).M(y,z,s) &= \frac{t+K(x,y)}{t+1}.\frac{s+K(y,z)}{s+1} \\ &\leq \frac{(t+s)+K(x,y)}{(t+s)+1}.\frac{(t+s)+K(y,z)}{(t+s)+1} \\ &\leq \frac{(t+s)+K(x,y).K(y,z)}{(t+s)+1} \leq \frac{(t+s)+K(x,z)}{(t+s)+1} = M(x,z,t+s). \end{aligned}$$

(IFM-10) Since (K,P) is a stationary intuitionistic fuzzy metric on X, P satisfies $P(x,z) \leq P(x,y) + P(y,z) - P(x,y).P(y,z)$. Then we can write

$$\frac{P(x,z)}{(t+s)+1} \leq \frac{P(x,y)}{(t+s)+1} + \frac{P(y,z)}{(t+s)+1} - \frac{P(x,y)P(y,z)}{(t+s)+1}. \tag{2.4}$$

Also, it is clear that for all $t,s \in \mathbb{R}^+$

$$\frac{P(x,y)}{(t+s)+1} \leq \frac{P(x,y)}{t+1} \text{ and } \frac{P(y,z)}{(t+s)+1} \leq \frac{P(y,z)}{t+1}.$$

At the same time, $(t+s)+1 \leq (t+1).(s+1)$ is satisfied for all $t,s \in \mathbb{R}^+$, then we write

$$\frac{1}{(t+s)+1} \geq \frac{1}{(t+1).(s+1)}$$

and

$$-\frac{P(x,y).P(y,z)}{(t+s)+1} \leq -\frac{P(x,y).P(y,z)}{(t+1).(s+1)}$$

and

$$\begin{aligned}\frac{P(x,y)}{(t+s)+1}+\frac{P(y,z)}{(t+s)+1}-\frac{P(x,y)P(y,z)}{(t+s)+1} &\leq \frac{P(x,y)}{(t+s)+1}+\frac{P(y,z)}{(t+s)+1}-\frac{P(x,y).P(y,z)}{(t+1).(s+1)}\\ &\leq \frac{P(x,y)}{t+1}+\frac{P(y,z)}{s+1}-\frac{P(x,y)}{(t+1)}.\frac{P(y,z)}{s+1}\\ &= N(x,y,t)+N(y,z,s)-N(x,y,t).N(y,z,s).\end{aligned} \tag{2.5}$$

From (2.4) and (2.5)

$$\frac{P(x,z)}{(t+s)+1} \leq \frac{P(x,y)}{t+1}+\frac{P(y,z)}{s+1}-\frac{P(x,y)}{(t+1)}.\frac{P(y,z)}{s+1}.$$

The last inequality means that $N(x,z,t+s) \leq N(x,y,t)+N(y,z,s)-N(x,y,t).N(y,z,s)$. Now we show the conditions (IFM-5′) and (IFM-10′) to see $(M,N,*,\diamondsuit)$ is strong.

(IFM-5′) We will show that $M(x,z,t) \geq M(x,y,t).M(y,z,t)$ for all $x,y,z \in X$, $t>0$ that is,

$$\frac{t+K(x,z)}{t+1} \geq \frac{t+K(x,y)}{t+1}.\frac{t+K(y,z)}{t+1}$$

is equivalent to

$$(t+1).[t+K(x,z)] \geq [t+K(x,y)].[t+K(y,z)],$$

also the above equation is equivalent to

$$t[K(x,z)+1]+K(x,z)-K(x,y).K(y,z) \geq t[K(x,y)+K(y,z)].$$

Since $K(x,z) \geq K(x,y).K(y,z)$, it is sufficient to see that $t[K(x,z)+1] \geq t[K(x,y)+K(y,z)]$. So if we show that $K(x,z)+1-K(x,y)-K(y,z) \geq 0$, the proof is completed. By using the (K,P) which is a stationary intuitionistic fuzzy metric on X (K satisfies $K(x,z) \geq K(x,y).K(y,z)$),

$$\begin{aligned}K(x,z)+1-K(x,y)-K(y,z) &\geq K(x,y).K(y,z)+1-K(x,y)-K(y,z)\\ &= K(x,y)[K(y,z)-1]+[1-K(y,z)]\\ &= [K(y,z)-1].[K(x,y)-1] \geq 0.\end{aligned}$$

(IFM-10′) We will show that $N(x,z,t) \leq N(x,y,t)+N(y,z,t)-N(x,y,t).N(y,z,t)$ for all $x,y,z \in X$, $t>0$. Since (K,P) is a stationary intuitionistic fuzzy metric on X, P satisfies $P(x,z) \leq P(x,y)+P(y,z)-P(x,y)P(y,z)$ for all $x,y,z \in X$, then we write

$$\frac{P(x,z)}{t+1} \leq \frac{P(x,y)}{t+1}+\frac{P(y,z)}{t+1}-\frac{P(x,y)P(y,z)}{t+1}. \tag{2.6}$$

Also, $t+1 \leq (t+1).(t+1)$ is satisfied for all $t,s \in \mathbb{R}^+$, then we write

$$\frac{1}{t+1} \geq \frac{1}{(t+1).(t+1)}$$

and

$$-\frac{P(x,y).P(y,z)}{t+1} \leq -\frac{P(x,y).P(y,z)}{(t+1).(t+1)}$$

and

$$\frac{P(x,y)}{t+1}+\frac{P(y,z)}{t+1}-\frac{P(x,y)P(y,z)}{t+1} \leq \frac{P(x,y)}{t+1}+\frac{P(y,z)}{t+1}-\frac{P(x,y).P(y,z)}{(t+1).(t+1)}. \tag{2.7}$$

From (2.6) and (2.7)

$$\frac{P(x,z)}{t+1} \leq \frac{P(x,y)}{t+1} + \frac{P(y,z)}{(t+1} - \frac{P(x,y)}{(t+1)}.\frac{P(y,z)}{(t+1)}.$$

The last inequality means that $N(x,z,t) \leq N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t)$.

In the Example 2.13, Example 2.14 and Example 2.15 d is a metric on X. And intuitionistic fuzzy metric (M,N) defined by the means of a metric d.

Example 2.13. Let $\varphi : \mathbb{R}^+ \longrightarrow (0,1]$ be an increasing and continuous function. Define $a * b = ab$ and $a\diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$ and let M, N be fuzzy sets on $X^2 \times \mathbb{R}^+$ defined by

$$M(x,y,t) = \frac{\varphi(t)}{\varphi(t) + d(x,y)},\ N(x,y,t) = \frac{d(x,y)}{\varphi(t) + d(x,y)}.$$

Then $(M,N,*,\diamondsuit)$ is strong on X.

We show that $(M,N,*,\diamondsuit)$ is an intuitionistic fuzzy metric on X by using $(M,N,*,\diamondsuit)$ is strong. It is obvious that the conditions (IFM-1), (IFM-2), (IFM-3), (IFM-4), (IFM-6), (IFM-7), (IFM-8), (IFM-9) and (IFM-11) are satisfied, now we will see only (IFM-5) and (IFM-10) by using (IFM-5′) and (IFM-10′), since M is increasing N is decreasing and M, N are continuous functions with respect to $t \in \mathbb{R}^+$.

(IFM-5′) Since $\varphi : \mathbb{R}^+ \longrightarrow (0,1]$ is an increasing and continuous function, $M(x,y,t)$ is increasing. We show $M(x,z,t) \geq M(x,y,t).M(y,z,t)$ for all $x,y,z \in X$, $t > 0$ that is,

$$\frac{\varphi(t)}{\varphi(t) + d(x,z)} \geq \frac{\varphi(t)}{\varphi(t) + d(x,y)}.\frac{\varphi(t)}{\varphi(t) + d(y,z)} \tag{2.8}$$

is equivalent to

$$[\varphi(t) + d(x,y)]\,.\,[\varphi(t) + d(y,z)] \geq \varphi(t)\,[\varphi(t) + d(x,z)]\,, \tag{2.9}$$

then, to demonstrate the validity of (2.8), it is sufficient to show the validity of (2.9).

$$\begin{aligned} &\varphi(t)^2 + \varphi(t)d(y,z) + \varphi(t)d(x,y) + d(x,y).d(y,z)\\ &\quad = \varphi(t)^2 + \varphi(t)\,[d(x,y) + d(y,z)] + d(x,y).d(y,z)\\ &\quad \geq \varphi(t)^2 + \varphi(t)\,[d(x,z)] + d(x,y).d(y,z)\\ &\quad \geq \varphi(t)^2 + \varphi(t)\,[d(x,z)] = \varphi(t)\,[\varphi(t) + d(x,z)]\,. \end{aligned}$$

Consequently, $M(x,z,t) \geq M(x,y,t).M(y,z,t)$ and so we can write $M(x,z,t+s) \geq M(x,y,t+s).M(y,z,t+s)$. Also, since M_{xy} is increasing and continuous, we write $M(x,z,t+s) \geq M(x,y,t+s).M(y,z,t+s) \geq M(x,y,t).M(y,z,s)$. So we demonstrated the condition (IFM-5) by using (IFM-5′).

(IFM-10′) Since $\varphi : \mathbb{R}^+ \longrightarrow (0,1]$ is an increasing and continuous function, $N(x,y,t)$ is decreasing and continuous. We show $N(x,z,t) \leq N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t)$ for all $x,y,z \in X$, $t > 0$.

$$\begin{aligned} &N(x,y,t) + N(y,z,t) - N(x,y,t)N(y,z,t)\\ &\quad = \frac{d(x,y)}{\varphi(t) + d(x,y)} + \frac{d(y,z)}{\varphi(t) + d(y,z)} - \frac{d(x,y)}{\varphi(t) + d(x,y)}.\frac{d(y,z)}{\varphi(t) + d(y,z)}\\ &\quad = \frac{\varphi(t)^2 + \varphi(t)d(x,y) + \varphi(t)d(y,z) + d(x,y).d(y,z) - \varphi(t)^2}{\varphi(t)^2 + \varphi(t)d(x,y) + \varphi(t)d(y,z) + d(x,y).d(y,z)}\\ &\quad = 1 - \frac{\varphi(t)^2}{\varphi(t)^2 + \varphi(t)\,[d(x,y) + d(y,z)] + d(x,y).d(y,z)}\\ &\quad \geq 1 - \frac{\varphi(t)^2}{\varphi(t)^2 + \varphi(t)[d(x,y) + d(y,z)]} \end{aligned}$$

$$\geq 1 - \frac{\varphi(t)}{\varphi(t) + d(x,z)} = \frac{d(x,z)}{\varphi(t) + d(x,z)} = N(x,z,t).$$

Consequently, $N(x,z,t) \leq N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t)$ and $N(x,z,t+s) \leq N(x,y,t+s) + N(y,z,t+s) - N(x,y,t+s).N(y,z,t+s)$. Also, since N_{xy} is decreasing and continuous we write $N(x,y,t+s) \leq N(x,y,t)$ and $N(y,z,t+s) \leq N(y,z,s)$, then $1 - N(x,y,t+s) \geq 1 - N(x,y,t)$ and $1 - N(y,z,t+s) \geq 1 - N(y,z,s)$. If we multiply the last two inequalities side by side, we attain

$$N(x,y,t+s) + N(y,z,t+s) - N(x,y,t+s).N(y,z,t+s) \leq N(x,y,t) + N(y,z,s) - N(x,y,t).N(y,z,s)$$

then,

$$N(x,z,t+s) \leq N(x,y,t) + N(y,z,s) - N(x,y,t).N(y,z,s).$$

So we demonstrated the conditions (IFM-10) by using (IFM-10$'$). Thus, $(M,N,*,\diamondsuit)$ is strong and $(M,N,*,\diamondsuit)$ is an intuitionistic fuzzy metric on X.

In particular, the standard intuitionistic fuzzy metric $(M_d, N_d, *, \diamondsuit)$ is strong with the continuous t-norm and continuous t-conorm defined by $a * b = a.b$, $a \diamondsuit b = a + b - a.b$ for all $a, b \in [0,1]$.

Example 2.14. Let $a * b = ab$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$, $t > 0$ and let M, N be fuzzy sets on $X^2 \times \mathbb{R}^+$ defined by

$$M(x,y,t) = e^{\frac{-d(x,y)}{t}} \text{ and } N(x,y,t) = \frac{e^{\frac{d(x,y)}{t}} - 1}{e^{\frac{d(x,y)}{t}}}.$$

$(M,N,*,\diamondsuit)$ is strong.

We show the conditions (IFM-5$'$) and (IFM-10$'$) are provided for M and N. Take $x, y, z \in X$, $t > 0$.

(IFM-5$'$) As d is a metric on X, it satisfies $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality). So, we can write

$$-\frac{d(x,z)}{t} \geq -\frac{d(x,y)}{t} - \frac{d(y,z)}{t}.$$

Then,

$$e^{-\frac{d(x,z)}{t}} \geq e^{-\frac{d(x,y)}{t} - \frac{d(y,z)}{t}}$$

and it is the same with

$$e^{-\frac{d(x,z)}{t}} \geq e^{-\frac{d(x,y)}{t}}.e^{-\frac{d(y,z)}{t}},$$

this means that

$$M(x,z,t) \geq M(x,y,t).M(y,z,t).$$

(IFM-10$'$)

$$\begin{aligned}
&N(x,y,t) + N(y,z,t) - [N(x,y,t) + N(y,z,t)] \\
&= \frac{e^{\frac{d(x,y)}{t}} - 1}{e^{\frac{d(x,y)}{t}}} + \frac{e^{\frac{d(y,z)}{t}} - 1}{e^{\frac{d(y,z)}{t}}} - \left[\frac{e^{\frac{d(x,y)}{t}} - 1}{e^{\frac{d(x,y)}{t}}} . \frac{e^{\frac{d(y,z)}{t}} - 1}{e^{\frac{d(y,z)}{t}}}\right] \\
&= \frac{e^{\frac{d(x,y)}{t} + \frac{d(y,z)}{t}} - 1}{e^{\frac{d(x,y)}{t} + \frac{d(y,z)}{t}}} = 1 - e^{-\left[\frac{d(x,y)+d(y,z)}{t}\right]} \\
&\geq 1 - e^{-\frac{d(x,z)}{t}} = \frac{e^{\frac{d(x,z)}{t}} - 1}{e^{\frac{d(x,z)}{t}}} = N(x,z,t).
\end{aligned}$$

Example 2.15.

(i) The standard intuitionistic fuzzy metric (M_d, N_d) is an intuitionistic fuzzy ultrametric on X (thus, it is strong with the continuous t-norm and continuous t-conorm defined by $a * b = \min\{a, b\}$ and $a \diamondsuit b = \max\{a, b\}$ for all $a, b \in [0,1]$) if and only if d is an ultrametric on X.

(ii) If d is a metric which is not an ultrametric on X, then (M_d, N_d) is an intuitionistic fuzzy metric which is not strong with the continuous t-norm and continuous t-conorm defined by $a * b = \min\{a, b\}$ and $a \diamondsuit b = \max\{a, b\}$ for all $a, b \in [0, 1]$.

(i) Let d be ultrametric. We show that (M_d, N_d) is an intuitionistic fuzzy ultrametric. As d is ultrametric, it satisfies $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

$$\begin{aligned}
M_d(x, z, t) &= \frac{t}{t + d(x, z)} \\
&\geq \frac{t}{t + \max\{d(x, y), d(y, z)\}} \\
&= \min\left\{\frac{t}{t + d(x, y)}, \frac{t}{t + d(y, z)}\right\} \\
&= \min\{M_d(x, y, t), M_d(y, z, t)\}
\end{aligned}$$

and

$$\begin{aligned}
d(x, z) &\leq \max\{d(x, y), d(y, z)\} \\
&\Rightarrow t + d(x, z) \leq \max\{t + d(x, y), t + d(y, z)\} \\
&\Rightarrow -\frac{t}{t + d(x, z)} \leq \max\left\{-\frac{t}{t + d(x, y)}, -\frac{t}{t + d(y, z)}\right\} \\
&\Rightarrow 1 - \frac{t}{t + d(x, z)} \leq \max\left\{1 - \frac{t}{t + d(x, y)}, 1 - \frac{t}{t + d(y, z)}\right\} \\
&\Rightarrow \frac{d(x, z)}{t + d(x, z)} \leq \max\left\{\frac{d(x, y)}{t + d(x, y)}, \frac{d(y, z)}{t + d(y, z)}\right\},
\end{aligned}$$

this means $N_d(x, z, t) \leq \max\{N_d(x, y, t), N_d(y, z, t)\}$. So, (M_d, N_d) is intuitionistic fuzzy ultrametric on X.

Conversely, let (M_d, N_d) is an intuitionistic fuzzy ultrametric on X. We show that d is ultrametric on X. As (M_d, N_d) is intuitionistic fuzzy ultrametric, it satisfies $M_d(x, z, t) \geq \min\{M_d(x, y, t), M_d(y, z, t)\}$ and $N_d(x, z, t) \leq \max\{N_d(x, y, t), N_d(y, z, t)\}$ for all $x, y, z \in X, t > 0$. Then,

$$\begin{aligned}
d(x, z) &= t\left[\frac{1}{M_d(x, z, t)} - 1\right] \\
&\leq t\left[\frac{1}{\min\{M_d(x, y, t), M_d(y, z, t)\}} - 1\right] \\
&= t\left[\frac{1}{\min\left\{\frac{t}{t+d(x,y)}, \frac{t}{t+d(y,z)}\right\}} - 1\right] = t\left[\max\left\{\frac{t + d(x, y)}{t}, \frac{t + d(y, z)}{t}\right\} - 1\right] \\
&= t\left[\max\left\{\frac{d(x, y)}{t}, \frac{d(y, z)}{t}\right\}\right] = \max\{d(x, y), d(y, z)\}
\end{aligned}$$

and

$$\begin{aligned}
d(x, z) &= t\left[\frac{1}{1 - N_d(x, z, t)} - 1\right] \\
&\leq t\left[\frac{1}{1 - \max\{N_d(x, y, t), N_d(y, z, t)\}} - 1\right] \\
&= t\left[\frac{1}{\min\left\{1 - \frac{d(x,y)}{t+d(x,y)}, 1 - \frac{d(y,z)}{t+d(y,z)}\right\}} - 1\right]
\end{aligned}$$

$$= t\left[\max\left\{\frac{t+d(x,y)}{t},\frac{t+d(y,z)}{t}\right\}-1\right]$$
$$= t\left[\max\left\{\frac{d(x,y)}{t},\frac{d(y,z)}{t}\right\}\right]=\max\left\{d(x,y),d(y,z)\right\}.$$

(ii) As d is not ultrametric, $d(x,z)>\max\{d(x,y),d(y,z)\}\ \exists\ x,y,z\in X$. It is obvious that (M_d,N_d) is non-strong intuitionistic fuzzy metric on X with the continuous t-norm $*$ and the continuous t-conorm $\diamondsuit$ defined by $a*b=\min\{a,b\}$ and $a\diamondsuit b=\max\{a,b\}$ for all $a,b\in[0,1]$.

Let $(M,N,*,\diamondsuit)$ be a non-stationary intuitionistic fuzzy metric. We define the family of functions $\{(M_t,N_t):t\in\mathbb{R}^+\}$ where $M_t:X^2\longrightarrow(0,1]$ and $N_t:X^2\longrightarrow(0,1]$ are given by $M_t(x,y)=M(x,y,t)$ and $N_t(x,y)=N(x,y,t)$, respectively. With this notation we have the Proposition 2.16.

Proposition 2.16. *Let $(M,N,*,\diamondsuit)$ be a non-stationary intuitionistic fuzzy metric on X. Then:*

(i) *$(M,N,*,\diamondsuit)$ is strong if and only if $(M_t,N_t,*,\diamondsuit)$ is a stationary intuitionistic fuzzy metric on X for each $t\in\mathbb{R}^+$.*

(ii) *If $(M,N,*,\diamondsuit)$ is strong then $\tau_{(M,N)}=\vee\left\{\tau_{(M_t,N_t)}:t\in\mathbb{R}^+\right\}$.*

If (M,N) is a strong intuitionistic fuzzy metric we will say that $\{(M_t,N_t):t\in\mathbb{R}^+\}$ is the family of stationary intuitionistic fuzzy metrics deduced from (M,N).

Proof. (i) Let $(M,N,*,\diamondsuit)$ be a non-stationary and strong intuitionistic fuzzy metric on X and $t\in\mathbb{R}^+$. We show that $(M_t,N_t,*,\diamondsuit)$ is stationary intuitionistic fuzzy metric on X, for each $t\in\mathbb{R}^+$. It is obvious that, when we choose a fixed $t\in\mathbb{R}^+$, we see the functions M_t and N_t are stationary since they are independent from t. So for all different values of $t\in\mathbb{R}^+$, M_t and N_t are stationary. Now we show that the $(M_t,N_t,*,\diamondsuit)$ is an intuitionistic fuzzy metric on X for all $t\in\mathbb{R}^+$.

By using the $(M,N,*,\diamondsuit)$ which is a strong intuitionistic fuzzy metric on X, for all $x,y,z\in X$, $t,s>0$;

(IFM-1) $M_t(x,y)+N_t(x,y)=M(x,y,t)+N(x,y,t)\leq 1$,

(IFM-2) $M_t(x,y)=M(x,y,t)>0$,

(IFM-3) $M_t(x,y)=M(x,y,t)=1\Leftrightarrow x=y$,

(IFM-4) $M_t(x,y)=M(x,y,t)=M(y,x,t)=M_t(y,x)$,

(IFM-5) $M_t(x,z)=M(x,z,t)\geq M(x,y,t)*M(y,z,t)=M_t(x,y)*M_t(y,z)$,

(IFM-6) $M_t(x,y)=M(x,y,t):(0,\infty)\to(0,1]$ is continuous,

(IFM-7) $N_t(x,y)=N(x,y,t)>0$,

(IFM-8) $N_t(x,y)=N(x,y,t)=0\Leftrightarrow x=y$,

(IFM-9) $N_t(x,y)=N(x,y,t)=N(y,x,t)=N_t(y,x)$,

(IFM-10) $N_t(x,z)=N(x,z,t)\leq N(x,y,t)\diamondsuit N(y,z,t)=N_t(x,y)\diamondsuit N_t(y,z)$,

(IFM-11) $N_t(x,y)=N(x,y,t):(0,\infty)\to(0,1]$ is continuous.

Then, $(M_t,N_t,*,\diamondsuit)$ is stationary intuitionistic fuzzy metric on X.

Conversely, let $(M_t,N_t,*,\diamondsuit)$ be a stationary intuitionistic fuzzy metric on X for all $t\in\mathbb{R}^+$. Now we show $(M,N,*,\diamondsuit)$ is strong. Because $(M_t,N_t,*,\diamondsuit)$ is stationary for all $t\in\mathbb{R}^+$, $(M_t,N_t,*,\diamondsuit)$ is strong. Then,

$$M(x,z,t)=M_t(x,z)\geq M_t(x,y)*M_t(y,z)=M(x,y,t)*M(y,z,t)$$

and

$$N(x,z,t)=N_t(x,z)\le N_t(x,y)\diamondsuit N_t(y,z)=N(x,y,t)\diamondsuit N(y,z,t)$$

so, $(M,N,*,\diamondsuit)$ is strong.

(ii) Let $(M,N,*,\diamondsuit)$ be non-stationary and strong intuitionistic fuzzy metric on X. We show that $\tau_{(M,N)}=\vee\left\{\tau_{(M_t,N_t)}:t\in\mathbb{R}^+\right\}$. To demonstrate the validity of this equation, it is sufficient to show that $B_{(M,N)}(x,r,t)=B_{(M_t,N_t)}(x,r)$ for all $t\in\mathbb{R}^+$, $x\in X$, $r\in(0,1)$.

$$\begin{aligned}B_{(M,N)}(x,r,t)&=\{y\in X:M(x,y,t)>1-r\ ,\ N(x,y,t)<r\ \}\\&=\{y\in X:M_t(x,y)=M(x,y,t)>1-r\ ,\ N_t(x,y)=N(x,y,t)<r\ \}\\&=\{y\in X:M_t(x,y)>1-r\ ,\ N_t(x,y)<r\ \}\\&=B_{(M_t,N_t)}(x,r).\end{aligned}$$

So, the open ball $B_{(M,N)}(x,r,t)$ coincides with the open ball $B_{(M_t,N_t)}(x,r)$ for all $x\in X,r\in(0,1),t\in\mathbb{R}^+$ then $\tau_{(M,N)}=\vee\left\{\tau_{(M_t,N_t)}:t\in\mathbb{R}^+\right\}$. □

Example 2.17, Example 2.18, and Example 2.19 illustrate the Proposition 2.16.

Example 2.17. Let d be a metric on X. Define $a*b=\min\{a,b\}$ and $a\diamondsuit b=\max\{a,b\}$ for all $a,b\in[0,1]$. Then $(M_{d_t},N_{d_t},*,\diamondsuit)$ is a stationary intuitionistic fuzzy metric on X for each $t>0$ if and only if $(M_d,N_d,*,\diamondsuit)$ is strong if and only if d is an ultrametric on X.

For the proof of "$(M_d,N_d,*,\diamondsuit)$ is strong if and only if d is an ultrametric on X."see Example 2.15 (i).

Let $(M_{d_t},N_{d_t},*,\diamondsuit)$ is a stationary intuitionistic fuzzy metric on X for each $t>0$. We will show that $(M_d,N_d,*,\diamondsuit)$ is strong. Since $(M_{d_t},N_{d_t},*,\diamondsuit)$ is stationary intuitionistic fuzzy metric, it is strong for each $t>0$. Then,

$$M_d(x,z,t)=M_{d_t}(x,z)\ge\min\{M_{d_t}(x,y),M_{d_t}(y,z)\}=\min\{M_d(x,y,t),M_d(y,z,t)\}$$

and

$$N_d(x,z,t)=N_{d_t}(x,z)\le\max\{N_{d_t}(x,y),N_{d_t}(y,z)\}=\max\{N_d(x,y,t),N_d(y,z,t)\}.$$

Consequently (M_d,N_d) is strong with the minimum t-norm and maximum t-conorm.

The converse of the proof is clear.

Example 2.18. Consider the strong intuitionistic fuzzy metric (M,N) in the Example 2.9 and choose $\varphi(t)=t$. Then,

$$M_t(x,y)=\frac{\min\{x,y\}+t}{\max\{x,y\}+t}\text{ and }N_t(x,y)=\frac{\max\{x,y\}-\min\{x,y\}}{\max\{x,y\}+t}$$

is a stationary intuitionistic fuzzy metric for each $t>0$ and it is easy to verify that $\tau_{(M_t,N_t)}=\tau_{(M,N)}$ for each $t>0$.

In the Example 2.9 we have shown that non-stationary intuitionistic fuzzy metric $(M,N,*,\diamondsuit)$ is strong with the continuous t-norm and continuous t-conorm defined by $a*b=ab$ and $a\diamondsuit b=a+b-ab$ for all $a,b\in[0,1]$. Now we will see $(M_t,N_t,*,\diamondsuit)$ is a stationary intuitionistic fuzzy metric on $X=\mathbb{R}^+$. Note that we choose the values of t from interval $(0,\infty)$. For example, let $t=\frac{1}{2}$. Then, since

$$M_{\frac{1}{2}}(x,y)=\frac{\min\{x,y\}+\frac{1}{2}}{\max\{x,y\}+\frac{1}{2}}\text{ and }N_{\frac{1}{2}}(x,y)=\frac{\max\{x,y\}-\min\{x,y\}}{\max\{x,y\}+\frac{1}{2}}$$

for $t=\frac{1}{2}$ and all $x,y\in\mathbb{R}^+$, $\left(M_{\frac{1}{2}},N_{\frac{1}{2}},*,\diamondsuit\right)$ is a stationary intuitionistic fuzzy metric on X. So, it is clear that for all values of t from interval $(0,\infty)$, $(M_t,N_t,*,\diamondsuit)$ is stationary intuitionistic fuzzy metric on X. Also it is clear that $\tau_{(M,N)}=\tau_{(M_t,N_t)}$ for all $t\in\mathbb{R}^+$.

Example 2.19. Consider the functions M and N on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ given by

$$M(x,y,t) = \begin{cases} 1 & \text{if } x = y \\ \frac{\min\{x,y\}}{\max\{x,y\}}.\varphi(t) & \text{if } x \neq y \end{cases}, \; N(x,y,t) = \begin{cases} 0 & \text{if } x = y \\ \frac{\max\{x,y\} - \varphi(t)\min\{x,y\}}{\max\{x,y\}} & \text{if } x \neq y \end{cases},$$

where

$$\varphi(t) = \begin{cases} t & 0 < t \leq 1 \\ 1 & t > 1 \end{cases}.$$

Then (M,N) is a strong intuitionistic fuzzy metric on $\mathbb{R}^+$ with the continuous t-norm and continuous t-conorm defined by $a * b = ab$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$.

It is clear that for all $t \in \mathbb{R}^+$ $(M_t, N_t, *, \diamondsuit)$ is stationary intuitionistic fuzzy metric on $\mathbb{R}^+$.Then we can easily say from the Proposition 2.16 that $(M, N, *, \diamondsuit)$ is strong. But still we will show this below.

(For this example's proof we will consider the functions K and P on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ defined by

$$K(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}}, \; P(x,y) = \frac{\max\{x,y\} - \min\{x,y\}}{\max\{x,y\}}.$$

It is easy to verify that (K,P) is an intuitionistic fuzzy metric on $\mathbb{R}^+$ with the continuous t-norm and continuous t-conorm defined by $a * b = ab$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$. So by using the condition

$$K(x,z) \geq K(x,y).K(y,z).$$

that is,

$$\frac{\min\{x,z\}}{\max\{x,z\}} \geq \frac{\min\{x,y\}}{\max\{x,y\}}.\frac{\min\{y,z\}}{\max\{y,z\}} \tag{2.10}$$

we will complete the proof). Since it is easy to verify the conditions of intuitionistic fuzzy metric space, we will only show the conditions (IFM-5′) and (IFM-10′) to see (M,N) is strong.

(IFM-5′) In the events of $x = y$, $y = z$, $x = z$ and $x = y = z$, the proof is obvious. Suppose that $x \neq y \neq z \neq x$. Remember that $\varphi(t) \in (0,1]$, $x, y, z \in \mathbb{R}^+$ and by using (2.10)

$$\begin{aligned} M(x,z,t) = \frac{\min\{x,z\}}{\max\{x,z\}}.\varphi(t) &\geq \frac{\min\{x,y\}}{\max\{x,y\}}.\frac{\min\{y,z\}}{\max\{y,z\}}.\varphi(t) \\ &\geq \frac{\min\{x,y\}}{\max\{x,y\}}.\varphi(t).\frac{\min\{y,z\}}{\max\{y,z\}}.\varphi(t) = M(x,y,t).M(y,z,t). \end{aligned}$$

(IFM-10′)

$$\begin{aligned} &N(x,y,t) + N(y,z,t) - N(x,y,t).N(y,z,t) \\ &= \frac{\max\{x,y\}.\max\{y,z\} - \varphi(t)^2 \min\{x,y\}.\min\{y,z\}}{\max\{x,y\}.\max\{y,z\}} \\ &= 1 - \frac{\min\{x,y\}}{\max\{x,y\}}.\frac{\min\{y,z\}}{\max\{y,z\}}.\varphi(t)^2 \\ &\geq 1 - \frac{\min\{x,z\}}{\max\{x,z\}}.\varphi(t) \\ &= \frac{\max\{x,z\} - \varphi(t)\min\{x,z\}}{\max\{x,z\}} = N(x,z,t). \end{aligned}$$

In this example we saw that if $(M, N, *, \diamondsuit)$ is a non-stationary intuitionistic fuzzy metric on $\mathbb{R}^+$ and $(M_t, N_t, *, \diamondsuit)$ is a stationary intuitionistic fuzzy metric on $\mathbb{R}^+$ for each $t \in \mathbb{R}^+$, $(M, N, *, \diamondsuit)$ is strong. At the same time $\tau_{(M,N)}$ is the discrete topology on $\mathbb{R}^+$.

For $t \geq 1$ we get that $M_t(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}}$, $N_t(x,y) = \frac{\max\{x,y\}-\min\{x,y\}}{\max\{x,y\}}$ and $\tau_{(M_t,N_t)}$ is the usual topology of $\mathbb{R}$ relative to $\mathbb{R}^+$.

For $t < 1$ we get that $M_t(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}}.t$, $N_t(x,y) = \frac{\max\{x,y\}-t.\min\{x,y\}}{\max\{x,y\}}$ and thus, $\tau_{(M_t,N_t)}$ is the discrete topology.

Now it occurs the natural question of when a family $(M_t, N_t, *, \diamondsuit)$ of stationary intuitionistic fuzzy metrics on X for $t \in \mathbb{R}^+$, defines a intuitionistic fuzzy metric $(M, N, *, \diamondsuit)$ on X by means of the formula $M_t(x,y) = M(x,y,t)$ and $N_t(x,y) = N(x,y,t)$ for each $x, y \in X$, $t \in \mathbb{R}^+$. The Proposition 2.20 answers this question.

Proposition 2.20. *Let $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ be a family of stationary intuitionistic fuzzy metrics on X.*

(i) *Consider the functions M and N on $X^2 \times \mathbb{R}^+$ defined by $M(x,y,t) = M_t(x,y)$ and $N(x,y,t) = N_t(x,y)$ then (M,N) is an intuitionistic fuzzy metric when considering the t-norm $*$, t-conorm $\diamondsuit$, if and only if $\{M_t : t \in \mathbb{R}^+\}$ is an increasing family (that is, $M_t \leq M_{t'}$ if $t < t'$), $\{N_t : t \in \mathbb{R}^+\}$ is a decreasing family (that is, $N_{t'} \leq N_t$ if $t < t'$) and the functions $M_{xy}, N_{xy} : \mathbb{R}^+ \to (0,1]$ defined by $M_{xy}(t) = M_t(x,y)$ and $N_{xy}(t) = N_t(x,y)$ are continuous functions, for each $x, y \in X$.*

(ii) *If conditions (i) are satisfied then $(M, N, *, \diamondsuit)$ is strong and $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ is the family of stationary intuitionistic fuzzy metrics deduced from (M,N). By (ii) we can notice that a strong intuitionistic fuzzy metric is characterized by its family $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ of stationary intuitionistic fuzzy metrics.*

Proof. (i) If (M,N) is an intuitionistic fuzzy metric on X, the conclusion is obvious. Indeed, it is obvious that, since $(M_t, N_t, *, \diamondsuit)$ is an intuitionistic fuzzy metric on X for all $t \in \mathbb{R}^+$, $(M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric on X, too. So the functions $M_{xy}(t) = M_t(x,y)$ and $N_{xy}(t) = N_t(x,y)$ are continuous. Also, since $(M_t, N_t, *, \diamondsuit)$ is stationary for each $t \in \mathbb{R}^+$, $(M_t, N_t, *, \diamondsuit)$ is strong. Then we can write

$$M(x,z,t) = M_t(x,z) \geq M_t(x,y) * M_t(y,z) = M(x,y,t) * M(y,z,t)$$

and

$$N(x,z,t) = N_t(x,z) \leq N_t(x,y) \diamondsuit N_t(y,z) = N(x,y,t) \diamondsuit N(y,z,t).$$

It means that $(M, N, *, \diamondsuit)$ is strong and for all $t > 0$, M is increasing, N is decreasing, then, $\{M_t : t \in \mathbb{R}^+\}$ is an increasing family (that is, $M_t \leq M_{t'}$ if $t < t'$), $\{N_t : t \in \mathbb{R}^+\}$ is a decreasing family (that is, $N_{t'} \leq N_t$ if $t < t'$).

Conversely, let $\{M_t : t \in \mathbb{R}^+\}$ be an increasing family (that is, $M_t \leq M_{t'}$ if $t < t'$), $\{N_t : t \in \mathbb{R}^+\}$ be a decreasing family (that is, $N_{t'} \leq N_t$ if $t < t'$) and the functions $M_{xy}, N_{xy} : \mathbb{R}^+ \to (0,1]$ defined by $M_{xy}(t) = M_t(x,y)$ and $N_{xy}(t) = N_t(x,y)$ are continuous functions, for each $x, y \in X$. Now we show that $(M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric on X. It is clear that the axioms (IFM-1), (IFM-2), (IFM-3), (IFM-4), (IFM-6), (IFM-7), (IFM-8), (IFM-9) and (IFM-11) are satisfied. We only show the triangle inequality for M and N. Because $(M_t, N_t, *, \diamondsuit)$ is stationary for each $t \in \mathbb{R}^+$, $(M_t, N_t, *, \diamondsuit)$ is strong. Then,

$$M(x,z,t+s) = M_{t+s}(x,z) \geq M_{t+s}(x,y) * M_{t+s}(y,z) \geq M_t(x,y) * M_s(y,z) = M(x,y,t) * M(y,z,s)$$

and

$$N(x,z,t+s) = N_{t+s}(x,z) \leq N_{t+s}(x,y) \diamondsuit N_{t+s}(y,z) \leq N_t(x,y) \diamondsuit N_s(y,z) = N(x,y,t) \diamondsuit N(y,z,s).$$

(ii) $\forall x, y, z \in X$, $t > 0$

$$M(x,z,t) = M_t(x,z) \geq M_t(x,y) * M_t(y,z) = M(x,y,t) * M(y,z,t)$$

and

$$N(x,z,t) = N_t(x,z) \leq N_t(x,y) \diamondsuit N_t(y,z) = N(x,y,t) \diamondsuit N(y,z,t).$$

Then $(M, N, *, \diamondsuit)$ is strong and $(M, N, *, \diamondsuit)$ is characterized by its family $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ of stationary intuitionistic fuzzy metrics. □

The Example 2.21 and Example 2.22 illustrate the Proposition 2.20.

Example 2.21. Consider on $\mathbb{R}$ the family of stationary intuitionistic fuzzy metrics $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ given by

$$M_t(x,y) = \begin{cases} \frac{1}{1+|x-y|} & \text{if } t \le 1 \\ \frac{2}{2+|x-y|} & \text{if } t > 1 \end{cases} \quad \text{and } N_t(x,y) = \begin{cases} \frac{|x-y|}{1+|x-y|} & \text{if } t \le 1 \\ \frac{|x-y|}{2+|x-y|} & \text{if } t > 1 \end{cases}.$$

Define the functions M, N on $\mathbb{R}^2 \times \mathbb{R}^+$ by $M(x,y,t) = M_t(x,y)$ and $N(x,y,t) = N_t(x,y)$. Denote $a * b = ab$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$. Then $(M, N, *, \diamondsuit)$ is not an intuitionistic fuzzy metric on $\mathbb{R}$. So indeed, M_{xy} is an increasing function, N_{xy} is a decreasing function on $\mathbb{R}$. But $(M, N, *, \diamondsuit)$ is not an intuitionistic fuzzy metric on $\mathbb{R}$ since M_{xy} and N_{xy} are not continuous at $t = 1$ if $x \neq y$.

Example 2.22. Consider on $\mathbb{R}$ the family of stationary intuitionistic fuzzy metrics $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ given by

$$M_t(x,y) = \begin{cases} \frac{1}{1+|x-y|} & \text{if } t \le 1 \\ \frac{\frac{1}{t}}{\frac{1}{t}+|x-y|} & \text{if } t > 1 \end{cases} \quad \text{and } N_t(x,y) = \begin{cases} \frac{|x-y|}{1+|x-y|} & \text{if } t \le 1 \\ \frac{|x-y|}{\frac{1}{t}+|x-y|} & \text{if } t > 1 \end{cases}.$$

Define the functions M, N on $\mathbb{R}^2 \times \mathbb{R}^+$ by $M(x,y,t) = M_t(x,y)$ and $N(x,y,t) = N_t(x,y)$. Denote $a * b = ab$ and $a \diamondsuit b = a + b - ab$ for all $a, b \in [0,1]$. Then $(M, N, *, \diamondsuit)$ is not an intuitionistic fuzzy metric on $\mathbb{R}$. So indeed, M_{xy} and N_{xy} are continuous functions on $\mathbb{R}$. But $(M, N, *, \diamondsuit)$ is not an intuitionistic fuzzy metric on $\mathbb{R}$ since M_{xy} is not an increasing function and N_{xy} is not a decreasing function on $\mathbb{R}$.

An easy consequence of the previous definitions is the Proposition 2.23.

Proposition 2.23. *Let $\{(M_t, N_t, *, \diamondsuit) : t \in \mathbb{R}^+\}$ be the family of stationary intuitionistic fuzzy metrics deduced from the strong intuitionistic fuzzy metric $(M, N, *, \diamondsuit)$ on X. Then the sequence $\{x_n\}$ in X is $MN-$Cauchy if and only if $\{x_n\}$ is M_tN_t-Cauchy for each $t > 0$.*

Corollary 2.24. *Let $(X, M, N, *, \diamondsuit)$ be a strong intuitionistic fuzzy metric space. $(X, M, N, *, \diamondsuit)$ is complete if and only if $(X, M_t, N_t, *, \diamondsuit)$ is complete for each $t > 0$.*

Proof. The proof is clear since $\tau_{(M,N)} = \vee \left\{\tau_{(M_t,N_t)} : t > 0\right\}$. □

References

[1] C. Alaca, D. Turkoglu, C. Yildiz, *Fixed points in intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals, **29** (2006), 1073–1078. 1

[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Sys., **20** (1986), 87–96. 1

[3] D. Çoker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets Sys., **88** (1997), 81–89. 1

[4] Z. Deng, *Fuzzy pseudo-metric spaces*, J. Math. Anal. Appl., **86** (1982), 74–95. 1

[5] M. A. Erceg, *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., **69** (1979), 205–230. 1

[6] J. G. García, S. Romaguera, *Examples of non-strong fuzzy metrics*, Fuzzy Sets Sys., **162** (2011), 91–93. 1

[7] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Sys., **64** (1994), 395–399. 1

[8] A. George, P. Veeramani, *Some theorems in fuzzy metric spaces*, J. Fuzzy Math., **3** (1995), 933–940.

[9] A. George, P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets Sys., **90** (1997), 365–368. 1

[10] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets Sys., **27** (1988), 385–389. 1

[11] V. Gregori, S. Romaguera, P. Veeramani, *A note on intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals, **28** (2006), 902–905. 1

[12] V. Gregori, S. Morillas, A. López-Crevillén, *On continuity and uniform continuity in fuzzy metric spaces*, In Proceedings of the Workshop in Applied Topology WIAT'09, (2009), 85–91. 1

[13] O. Kaleva, S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets Sys., **12** (1984), 215–229. 1

[14] I. Kramosil, J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11** (1975), 336–344. 1

[15] R. Lowen, *Fuzzy Set Theory*, Kluwer Academic Publishers, Dordrecht, (1996). 1.5

[16] J. H. Park, *Intuitionistic fuzzy metric spaces,* Chaos Solitons Fractals, **22** (2004), 1039–1046. 1, 1.4, 1.6, 1.8, 1.10, 1.11, 1.14, 1.15

[17] S. Romaguera, *On completion of fuzzy metric spaces*, Fuzzy Sets Sys., **130** (2002), 399–404. 1

[18] R. Saadati, J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos Solitons Fractals, **27** (2006), 331–344. 1

[19] A. Sapena, S. Morillas, *On strong fuzzy metrics*, In Proceedings of the Workshop in Applied Topology WIAT'09, (2009), 135–141. 1

[20] A. Savchenko, *A remark on stationary fuzzy metric spaces*, Carpathian Math. Publ., **3** (2011), 124–129. 1

[21] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math, **10** (1960), 313–334. 1.1, 1.2

[22] L. A. Zadeh, *Fuzzy sets*, Inform. Control, **8** (1965), 338–353. 1