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Existence of solutions for fractional integral boundary value problems with $p(t)$-Laplacian operator

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Abstract

This paper aims to investigate the existence of solutions for fractional integral boundary value problems (BVPs for short) with $p(t)$-Laplacian operator. By using the fixed point theorem and the coincidence degree theory, two existence results are obtained, which enrich existing literatures. Some examples are supplied to verify our main results.

Keywords: Fractional differential equation, boundary value problem, $p(t)$-Laplacian operator, fixed point theorem, coincidence degree theory.
2010 MSC: 26A33, 34A08, 34B15, 34B18.

1. Introduction

In the recent years, fractional differential equations have been applied in many research fields (see [4, 20, 23, 24, 30]). For example, Leszczynski and Blaszczyk [23] discussed the following fractional mathematical model which can be used to describe the height of granular material decreasing over time in a silo:

$$^{C}D^{\alpha}_{T^-}D^{\alpha}_{a^+}h^*(t)+\beta h^*(t)=0,\quad t\in[0,T],$$

where $^{C}D^{\alpha}_{T^-}$ is the right Caputo fractional derivative, $D^{\alpha}_{a^+}$ is the left Riemann-Liouville fractional derivative, $\alpha\in(0,1)$. Furthermore, some valuable results which are related to the stability of fractional functional equations (see [10, 17, 28]) and the existence and multiplicity of solutions for fractional boundary value problems (see [1–3, 5, 6, 15, 18]) have been achieved by some scholars. For example, Cabada and Wang [6] considered the existence of positive solutions for the following fractional integral BVP by Guo-Krasnoselskii

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fixed point theorem:

$$\begin{cases} {}^{C}D^{\alpha}x(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \\ x(0) = x''(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)ds, \end{cases}$$

where $2 < \alpha \leq 3$, $0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is a Caputo fractional derivative, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

It is well-known that p-Laplacian operator is a nonlinear operator and occurs in the course of considering glacial sliding (see [27]), torsional creep (see [19]), porous medium (see [22]), etc. For solvability of BVPs for integer differential equations with p-Laplacian operator, we provide readers with some articles (see [11, 13, 14]).

Recently, some scholars have paid more attention to fractional p-Laplacian equations and got some interesting results (see [7, 9, 16, 25]). For example, Chen and Liu [9] investigated the existence of solutions for the anti-periodic BVP of fractional differential equation with p-Laplacian operator by Schaefer's fixed point theorem:

$$\begin{cases} D_{0^+}^{\beta}\varphi_p(D_{0^+}^{\alpha}x(t)) = f(t, x(t)), \quad t \in [0, 1], \\ x(0) = -x(1), \quad D_{0^+}^{\alpha}x(0) = -D_{0^+}^{\alpha}x(1), \end{cases}$$

where $0 < \beta,\ \alpha \leq 1$, $1 < \alpha + \beta \leq 2$, $D_{0^+}^{\alpha}$ is a Caputo fractional derivative, $\varphi_p(\cdot)$ is a p-Laplacian operator, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Mahmudov and Unul [25] studied the existence and uniqueness of solutions for integral BVP of fractional differential equation with p-Laplacian operator by Green's functions and some fixed point theorems:

$$\begin{cases} D_{0^+}^{\beta}\varphi_p(D_{0^+}^{\alpha}x(t)) = f(t, x(t), D_{0^+}^{\gamma}x(t)), \quad t \in [0, 1], \\ x(0) + \mu_1 x(1) = \sigma_1 \int_0^1 g(s, x(s))ds, \\ x'(0) + \mu_2 x'(1) = \sigma_2 \int_0^1 h(s, x(s))ds, \\ D_{0^+}^{\alpha}x(0) = 0, \quad D_{0^+}^{\alpha}x(1) = \nu D_{0^+}^{\alpha}x(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta, \gamma \leq 1$, $0 < \eta < 1$, $\nu, \mu_i, \sigma_i > 0$ $(i = 1, 2)$, $D_{0^+}^{\alpha}$ is a Caputo fractional derivative, $\varphi_p(\cdot)$ is a p-Laplacian operator, f, g, h are continuous.

Motivated by the work above, our paper aims to investigate the existence of solutions for the following fractional integral BVP with $p(t)$-Laplacian operator under the non-resonance case and resonance case:

$$\begin{cases} D_{0^+}^{\beta}\varphi_{p(t)}(D_{0^+}^{\alpha}x(t)) + f(t, x(t)) = 0, \quad t \in (0, 1), \\ x(0) = 0, \quad D_{0^+}^{\alpha-1}x(1) = \gamma I_{0^+}^{\alpha-1}x(\eta), \quad D_{0^+}^{\alpha}x(0) = 0, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is a Riemann-Liouville fractional derivative, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $\gamma > 0$, $0 < \eta < 1$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. $\varphi_{p(t)}(\cdot)$ is a $p(t)$-Laplacian operator, $p(t) > 1$, $p(t) \in C^1[0, 1]$.

Noting that the $p(t)$-Laplacian operator is the non-standard growth operator which arises from nonlinear electrorheological fluids (see [29]), image restoration (see [8]), elasticity theory (see [32]), etc. There are many valuable results with respect to this type problems (see [12, 31] and references therein). Compared with constant growth operator, it will bring many difficulties. It can turn into the well-known p-Laplacian operator when $p(t) = p$, so our results extend and enrich some existing papers. Moreover, there are almost no papers which considered fractional integral BVPs with $p(t)$-Laplacian operator. For the non-resonance case, by constructing the Green function, we show a new existence result for BVP (1.1) by the Schaefer's fixed point theorem. For the resonance case, by investigating the following equivalence problem (see Lemma 2.8)

$$\begin{cases} D_{0^+}^{\alpha}x(t) = -\varphi_{p(t)}^{-1}(I_{0^+}^{\beta}f(t, x(t)), \quad t \in (0, 1), \\ x(0) = 0, \quad D_{0^+}^{\alpha-1}x(1) = \gamma I_{0^+}^{\alpha-1}x(\eta), \end{cases}$$

we obtain a new existence result for BVP (1.1) by the coincidence degree theory of Mawhin (see [26]).

2. Preliminaries

For basic definitions of Riemann-Liouville fractional integral and fractional derivative, please see [20]. Here, we show some important properties, lemmas and definitions as follows.

Lemma 2.1 ([31]). *For any $(t,x)\in[0,1]\times\mathbb{R}$, $\varphi_{p(t)}(x)=|x|^{p(t)-2}x$, is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and strictly monotone increasing for any fixed t. Moreover, its inverse operator $\varphi^{-1}_{p(t)}(\cdot)$ is defined by*

$$\begin{cases} \varphi^{-1}_{p(t)}(x)=|x|^{\frac{2-p(t)}{p(t)-1}}x, & x\in\mathbb{R}\setminus\{0\},\\ \varphi^{-1}_{p(t)}(0)=0, & x=0,\end{cases}$$

which is continuous and sends bounded sets to bounded sets.

Definition 2.2 ([26]). Let X and Y be real Banach spaces and let $L:\mathrm{dom}L\subset X\to Y$ be a Fredholm operator with index zero. Let $P:X\to X$, $Q:Y\to Y$ be continuous linear projectors such that $\mathrm{Im}P=\mathrm{Ker}L$, $\mathrm{Ker}Q=\mathrm{Im}L$, $X=\mathrm{Ker}L\oplus\mathrm{Ker}P$, $Y=\mathrm{Im}L\oplus\mathrm{Im}Q$. It follows that $L|_{\mathrm{dom}L\cap\mathrm{Ker}P}:\mathrm{dom}L\cap\mathrm{Ker}P\to\mathrm{Im}L$ is invertible. Its inverse is defined by K_P. If Ω is an open bounded subset of X, and $\mathrm{dom}L\cap\overline{\Omega}\neq\varnothing$, the map $N:X\to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN:\overline{\Omega}\to Y$ is bounded and $K_{P,Q}N:=KP(I-Q)N:\overline{\Omega}\to X$ is compact.

Lemma 2.3 ([26]). *Let $L:\mathrm{dom}L\subset X\to Y$ be a Fredholm operator of index zero and $N:X\to Y$ be $L-$compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(\mathrm{dom}L\setminus \mathrm{Ker}L)]\cap\partial\Omega\times(0,1)$;*
(ii) *$Nx\notin \mathrm{Im}L$ for every $x\in \mathrm{Ker}L\cap\partial\Omega$;*
(iii) *$\deg(QN|_{\mathrm{Ker}L},\mathrm{Ker}L\cap\Omega,0)\neq 0$, where $Q:Y\to Y$ is a projection such that $\mathrm{Im}L=\mathrm{Ker}Q$. Then the equation $Lx=Nx$ has at least one solution in $\mathrm{dom}L\cap\overline{\Omega}$.*

Lemma 2.4 ([20]). *Some properties for the Riemann-Liouville fractional integral and fractional derivative are as follows:*

(i) *If $\alpha\geq 0$, $\lambda>-1$, $\lambda\neq\alpha-i$, $i=1,2,...,[\alpha]+1$, we have*

$$D^{\alpha}_{0^+}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}.$$

Moreover, $D^{\alpha}_{0^+}t^{\alpha-i}=0$, $i=1,2,...,[\alpha]+1$.

(ii) *If $\alpha>0$, $\lambda>-1$, we have*

$$I^{\alpha}_{0^+}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)}t^{\lambda+\alpha}.$$

Lemma 2.5. *If $y(t)\in C[0,1]$ and $0<\gamma\eta^{2\alpha-2}<\Gamma(2\alpha-1)$, the unique solution of*

$$\begin{cases} D^{\beta}_{0^+}\varphi_{p(t)}(D^{\alpha}_{0^+}x(t))+y(t)=0, & t\in(0,1),\\ x(0)=0, \quad D^{\alpha-1}_{0^+}x(1)=\gamma I^{\alpha-1}_{0^+}x(\eta), \quad D^{\alpha}_{0^+}x(0)=0,\end{cases}$$

can be expressed as the following integral equation

$$x(t)=\int_0^1 G(t,s)\varphi^{-1}_{p(s)}(I^{\beta}_{0^+}y(s))ds, \tag{2.1}$$

where

$$G(t,s)=\begin{cases} \frac{t^{\alpha-1}\Gamma(2\alpha-1)-\gamma t^{\alpha-1}(\eta-s)^{2\alpha-2}-(t-s)^{\alpha-1}(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}, & 0\leq s\leq t\leq 1, \quad s\leq\eta,\\ \frac{t^{\alpha-1}\Gamma(2\alpha-1)-\gamma t^{\alpha-1}(\eta-s)^{2\alpha-2}}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}, & 0\leq t\leq s\leq\eta<1,\\ \frac{t^{\alpha-1}\Gamma(2\alpha-1)-(t-s)^{\alpha-1}(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}, & 0<\eta\leq s\leq t\leq 1,\\ \frac{t^{\alpha-1}\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}, & 0\leq t\leq s\leq 1, \quad s\geq\eta.\end{cases}$$

Proof. Based on the definitions of Riemann-Liouville fractional integral, we have

$$\varphi_{p(t)}(D_{0+}^{\alpha}x(t)) = -I_{0+}^{\beta}y(t) + ct^{\beta-1},\ c \in \mathbb{R}.$$

Combining with $D_{0+}^{\alpha}x(0) = 0$, for fixed $t = 0$, we have $c = 0$ and

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds + c_1t^{\alpha-1} + c_2t^{\alpha-2},$$

where $c_i \in \mathbb{R}, i = 1, 2$. By $x(0) = 0$, we obtain $c_2 = 0$. In view of Lemma 2.4, it follows that

$$I_{0+}^{\alpha-1}x(t) = -\frac{1}{\Gamma(2\alpha-1)}\int_0^t (t-s)^{2\alpha-2}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds + c_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)}t^{2\alpha-2},$$

and

$$D_{0+}^{\alpha-1}x(t) = -\int_0^t \varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds + c_1\Gamma(\alpha).$$

Based on the boundary value condition $D_{0+}^{\alpha-1}x(1) = \gamma I_{0+}^{\alpha-1}x(\eta)$, it follows

$$c_1 = \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}\left[\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds - \frac{\gamma}{\Gamma(2\alpha-1)}\int_0^{\eta}(\eta-s)^{2\alpha-2}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds\right].$$

Thus, we have

$$\begin{aligned}x(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds + \frac{t^{\alpha-1}\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}\left[\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds\right.\\ &\left.-\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^{\eta}(\eta-s)^{2\alpha-2}\varphi_{p(s)}^{-1}(I_{0+}^{\beta}y(s))ds\right].\end{aligned}$$

Therefore, (2.1) holds. □

Lemma 2.6. *If* $0 < \gamma\eta^{2\alpha-2} < \Gamma(2\alpha-1)$, $G(t,s)$ *satisfies the condition*

$$0 < G(t,s) < \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}, \quad \forall s,t \in (0,1).$$

Proof. By definition of $G(t,s)$, it is clear that $G(t,s) < \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}$ for all $s,t \in (0,1)$. On the other hand, if $s \le t,\ s \le \eta$, then set

$$g(t,s) = t^{\alpha-1}\Gamma(2\alpha-1) - \gamma t^{\alpha-1}(\eta-s)^{2\alpha-2} - (t-s)^{\alpha-1}(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2}).$$

We can obtain

$$\begin{aligned}g(t,s) &\ge t^{\alpha-1}\Gamma(2\alpha-1) - t^{\alpha-1}\gamma\eta^{2\alpha-2} - (t-s)^{\alpha-1}(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})\\ &= (\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})[t^{\alpha-1}-(t-s)^{\alpha-1}]\\ &\ge (\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})[t^{\alpha-1}-(t-ts)^{\alpha-1}]\\ &= t^{\alpha-1}(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})[1-(1-s)^{\alpha-1}] > 0.\end{aligned}$$

For other situations, it is clear that $G(t,s) > 0$, so we omit the proof. □

Remark 2.7. Noting that if $\gamma\eta^{2\alpha-2} = \Gamma(2\alpha-1)$, BVP (1.1) is the resonance case. However, the Mawhin's continuation theorem is not suitable for the nonlinear operator case. Thus, we need the following lemma to turn the nonlinear operator case into the linear operator case.

Lemma 2.8. *BVP* (1.1) *is equivalent to the following problem*

$$\begin{cases} D_{0+}^{\alpha}x(t) = -\varphi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t)), \quad t\in(0,1), \\ x(0)=0, \quad D_{0+}^{\alpha-1}x(1)=\gamma I_{0+}^{\alpha-1}x(\eta). \end{cases} \tag{2.2}$$

Proof. On one hand, based on the boundary condition $D_{0+}^{\alpha}x(0)=0$, we conclude (2.2) from (1.1). On the other hand, if $D_{0+}^{\alpha}x(t) = -\varphi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t))$, substituting $t=0$ into the above equality, we obtain $D_{0+}^{\alpha}x(0)=0$. Moreover, applying the operator $\varphi_{p(t)}^{-1}$ and D_{0+}^{β} to the both side of above equality, we have $D_{0+}^{\beta}\varphi_{p(t)}(D_{0+}^{\alpha}x(t))+f(t,x(t))=0$. Thus, the claim is proved. □

3. Main result

3.1. The non-resonance case

Let $X=C[0,1]$ with norm $\|x\|_{\infty}=\max_{t\in[0,1]}|x(t)|$. In the sequel, we assume that $0<\gamma\eta^{2\alpha-2}<\Gamma(2\alpha-1)$. Furthermore, in order to state Theorem 3.1, let $P_L:=\min_{t\in[0,1]}p(t),\ P_M:=\max_{t\in[0,1]}p(t)$.

Theorem 3.1. *Assume that the following condition holds.*

(H_1) *There exist constants $a,b>0$ such that*

$$|f(t,x)|\le a+b|x|^{\theta-1}, \quad 1<\theta\le P_L.$$

Then BVP (1.1) *has at least one solution, provided that*

$$\frac{2^{\frac{1}{P_L-1}}\Gamma(2\alpha-1)\max\{b^{\frac{1}{P_L-1}},b^{\frac{1}{P_M-1}}\}}{(\Gamma(\beta+1))^{\frac{1}{P_M-1}}\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}<1. \tag{3.1}$$

Proof. The operator $T: C[0,1]\to C[0,1]$ is defined by

$$Tx(t)=\int_0^1 G(t,s)\varphi_{p(s)}^{-1}(I_{0+}^{\beta}f(s,x(s)))ds.$$

By the continuity of f, it is easy to find that T is continuous. Let Ω be any bounded open subset of $C[0,1]$. Since $\varphi_{p(t)}^{-1}(\cdot)$ and f are continuous, there exists a constant $M>0$ such that $|\varphi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t)))|\le M$ on $[0,1]\times\overline{\Omega}$. Thus, we can obtain

$$\|Tx\|=\max_{t\in[0,1]}|Tx|\le\int_0^1\frac{M\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}ds=\frac{M\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1)-\gamma\eta^{2\alpha-2})}.$$

Thus, $T\overline{\Omega}$ is uniformly bounded. On the other hand, for all $t_1,\ t_2\in[0,1]$, assume that $t_1\le t_2$, for any $x\in\overline{\Omega}$, we have

$$\begin{aligned} |Tx(t_2)-Tx(t_1)| &= \left|\int_0^1 G(t_2,s)\varphi_{p(s)}^{-1}(I_{0+}^{\beta}f(s,x(s)))ds-\int_0^1 G(t_1,s)\varphi_{p(s)}^{-1}(I_{0+}^{\beta}f(s,x(s)))ds\right| \\ &\le \frac{M}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\ &\quad+\frac{(\Gamma(2\alpha)+\gamma\eta^{2\alpha-1})M}{\Gamma(\alpha)[\Gamma(2\alpha)-(2\alpha-1)\gamma\eta^{2\alpha-2}]}(t_2^{\alpha-1}-t_1^{\alpha-1}) \\ &= \frac{M}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha})+\frac{(\Gamma(2\alpha)+\gamma\eta^{2\alpha-1})M}{\Gamma(\alpha)[\Gamma(2\alpha)-(2\alpha-1)\gamma\eta^{2\alpha-2}]}(t_2^{\alpha-1}-t_1^{\alpha-1}). \end{aligned}$$

Thus, one has

$$|Tx(t_2) - Tx(t_1)| \to 0 \ \text{ uniformly as } \ t_1 \to t_2.$$

Therefore, T is equicontinuous on $\overline{\Omega}$. By the Arzelá-Ascoli theorem, we can obtain T is completely continuous. Define

$$V = \{x \in X | x = \lambda Tx, \ \lambda \in (0,1)\}.$$

According to Schaefer's fixed point theorem, we just need to prove that V is bounded. For $x \in V$, we have

$$\begin{aligned} |I_{0^+}^{\beta} f(t,x(t))| &\le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |f(s,x(s))| ds \\ &\le \frac{1}{\Gamma(\beta)} \int_0^1 (t-s)^{\beta-1} (a + b|x(s)|^{\theta-1}) ds \\ &\le \frac{1}{\Gamma(\beta+1)} (a + b\|x\|_\infty^{\theta-1}). \end{aligned}$$

By the basic inequality $(x+y)^p \le 2^p(x^p + y^p)$ for $x, y, p > 0$ (see [21]), we have

$$\begin{aligned} |x(t)| = \lambda |Tx(t)| &\le \int_0^1 G(t,s) \varphi_{p(s)}^{-1} (I_{0^+}^{\beta} |f(s,x(s))|) ds \\ &\le \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)(\Gamma(2\alpha-1) - \gamma\eta^{2\alpha-2})} \int_0^1 [\frac{2^{\frac{1}{p(s)-1}}}{(\Gamma(\beta+1))^{\frac{1}{p(s)-1}}} (a^{\frac{1}{p(s)-1}} + b^{\frac{1}{p(s)-1}} \|x\|_\infty^{\frac{\theta-1}{p(s)-1}})] ds. \end{aligned}$$

Since $\frac{\theta-1}{p(t)-1} \in (0,1]$, by the basic inequality $x^\kappa \le x+1$ for $x > 0, \kappa \in (0,1]$, we have

$$\|x\|_\infty \le \frac{2^{\frac{1}{P_L-1}} \Gamma(2\alpha-1)}{(\Gamma(\beta+1))^{\frac{1}{P_M-1}} \Gamma(\alpha)(\Gamma(2\alpha-1) - \gamma\eta^{2\alpha-2})} \int_0^1 [a^{\frac{1}{p(s)-1}} + b^{\frac{1}{p(s)-1}} (\|x\|_\infty + 1)] ds.$$

By (3.1), there exists a constant $M_1 > 0$ such that $\|x\|_\infty \le M_1$. Thus, the operator T has a fixed point, which implies BVP (1.1) has at least one solution. □

Example 3.2. Consider the following BVP:

$$\begin{cases} D_{0^+}^{\frac{3}{4}} \varphi_{t^2+2}(D_{0^+}^{\frac{3}{2}} x(t)) + f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = 0, \quad D_{0^+}^{\frac{1}{2}} x(1) = \frac{1}{2} I_{0^+}^{\frac{1}{2}} x(\frac{1}{2}), \quad D_{0^+}^{\frac{3}{2}} x(0) = 0, \end{cases}$$

where $\alpha = \frac{3}{2}$, $\beta = \frac{3}{4}$, $p(t) = t^2 + 2$, $\theta = 2$, $\eta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $f(t,x(t)) = \frac{9}{64}x^2$, $a = 0$, $b = \frac{9}{64}$. Clearly, (H_1) holds. Moreover, in view of (3.1), we have

$$\frac{1}{\Gamma(\frac{3}{2})(\Gamma(\frac{7}{4}))^{\frac{1}{2}}} < 1.$$

Thus, it has at least one solution.

3.2. The resonance case

In this part, let $X = Y = C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Noting that if $\gamma\eta^{2\alpha-2} = \Gamma(2\alpha-1)$, BVP (1.1) turns into resonance case.

By Lemma 2.8, the original problem can be turned into the following problem:

$$\begin{cases} D_{0^+}^{\alpha} x(t) = -\varphi_{p(t)}^{-1} (I_{0^+}^{\beta} f(t,x(t)), \quad t \in (0,1), \\ x(0) = 0, \quad D_{0^+}^{\alpha-1} x(1) = \gamma I_{0^+}^{\alpha-1} x(\eta). \end{cases}$$

Define the operator $L : \mathrm{dom}L \subset X \to Y$ by

$$Lx = D_{0^+}^{\alpha} x,$$

where

$$\mathrm{dom}L = \{x \in X | D_{0^+}^{\alpha} x(t) \in Y,\ x(0) = 0,\ D_{0^+}^{\alpha-1} x(1) = \gamma I_{0^+}^{\alpha-1} x(\eta)\}.$$

Let $N : X \to Y$ be the Nemytski operator

$$Nx(t) = -\varphi_{p(t)}^{-1}(I_{0^+}^{\beta} f(t, x(t)), \quad \forall t \in [0, 1].$$

Then BVP (1.1) is equivalent to the operator equation

$$Lx = Nx, \quad x \in \mathrm{dom}L.$$

It is clear that

$$\begin{aligned}
&\mathrm{Ker}L = \{x \in X | x(t) = ct^{\alpha-1},\ \forall t \in [0, 1],\ c \in \mathbb{R}\},\\
&\mathrm{Im}L = \{y \in Y | \int_0^1 y(s)ds - \frac{\gamma}{\Gamma(2\alpha-1)} \int_0^{\eta} (\eta - s)^{2\alpha-2} y(s)ds = 0\}.
\end{aligned}$$

Define the linear continuous projection operators $P : X \to X$ and $Q : Y \to Y$:

$$\begin{aligned}
&Px(t) = \frac{D_{0^+}^{\alpha-1} x(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad \forall t \in [0, 1],\\
&Qy(t) = \Delta[\int_0^1 y(s)ds - \frac{\gamma}{\Gamma(2\alpha-1)} \int_0^{\eta} (\eta - s)^{2\alpha-2} y(s)ds], \quad \forall t \in [0, 1],
\end{aligned}$$

where

$$\Delta := \frac{\Gamma(2\alpha)}{\Gamma(2\alpha) - \gamma\eta^{2\alpha-1}} = \frac{\Gamma(2\alpha)}{\Gamma(2\alpha) - \eta\Gamma(2\alpha-1)} = \frac{2\alpha-1}{2\alpha-1-\eta} > 0.$$

It is easy to find that $P^2 = P$, $Q^2 = Q$,

$$X = \mathrm{Ker}L \oplus \mathrm{Ker}P, \quad \text{and} \quad Y = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Noting that

$$\dim \mathrm{Ker}L = \dim \mathrm{Im}Q = \mathrm{codim}\ \mathrm{Im}L = 1.$$

Thus, L is a Fredholm operator of index zero. Let $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$, which can be given by $K_P y = I_{0^+}^{\alpha} y$. K_P is the inverse of $L|_{\mathrm{dom}L \cap \mathrm{Ker}P}$. Since f and $\varphi_{p(t)}^{-1}(\cdot)$ are continuous, it is easy to obtain that N is L-compact on $\overline{\Omega}$ whose proof is similar to some parts of Lemma 3.3 in [15], so we omit it here.

Theorem 3.3. *Assume that* (H_1) *and the following condition hold.*

(H_2) *there exists a constant $B > 0$ such that if $|x(t)| > B$ for any $t \in [\eta, 1]$, either*

$$sgn\{x(t)\}QN(x(t)) < 0,$$

or

$$sgn\{x(t)\}QN(x(t)) > 0.$$

Then BVP (1.1) *has at least one solution, provided that*

$$\frac{b2^{\theta}(1 + \eta^{\alpha-1})^{\theta-1}}{\Gamma(\beta+1)(\Gamma(\alpha+1)\eta^{\alpha-1})^{\theta-1}} < 1. \tag{3.2}$$

Proof. Let

$$\Omega_1 = \{x \in \mathrm{dom}L \setminus \mathrm{Ker}L \mid Lx = \lambda Nx,\ \lambda \in (0,1)\}.$$

Since $x \in \Omega_1$, we have $Nx \in \mathrm{Im}L = \mathrm{Ker}Q$. Thus, $QNx = 0$. By (H_2), there exists a constant $\xi \in [\eta, 1]$ such that $|x(\xi)| \le B$. By $x(0) = 0$, we have

$$x(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) + c_1 t^{\alpha-1}.$$

Hence, we can obtain

$$|c_1| \le \frac{1}{\xi^{\alpha-1}}[|x(\xi)| + \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}|D_{0+}^{\alpha}x(s)|ds] \le \frac{1}{\eta^{\alpha-1}}(B + \frac{1}{\Gamma(\alpha+1)}\|D_{0+}^{\alpha}x\|_{\infty}),$$

and

$$\|x\|_{\infty} \le \frac{1+\eta^{\alpha-1}}{\Gamma(\alpha+1)\eta^{\alpha-1}}\,\|D_{0+}^{\alpha}x\|_{\infty} + \frac{B}{\eta^{\alpha-1}}. \tag{3.3}$$

Based on $Lu = \lambda Nu$, we can get

$$D_{0+}^{\alpha}x(t) = -\lambda\varphi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t))).$$

Applying the operator $\varphi_{p(t)}^{-1}$ to the both side of above equality, one has

$$\varphi_{p(t)}(D_{0+}^{\alpha}x(t)) = -\lambda^{p(t)-1}(I_{0+}^{\beta}f(t,x(t))).$$

From (H_1) and $\lambda \in (0,1)$, we have

$$|D_{0+}^{\alpha}x(t)|^{p(t)-1} \le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\,|f(t,x(t))|ds \le \frac{1}{\Gamma(\beta+1)}(a + b\,\|x\|_{\infty}^{\theta-1}).$$

In view of (3.3), we have

$$|D_{0+}^{\alpha}x(t)|^{p(t)-1} \le \frac{1}{\Gamma(\beta+1)}[a + b(\frac{1+\eta^{\alpha-1}}{\Gamma(\alpha+1)\eta^{\alpha-1}}\,\|D_{0+}^{\alpha}x\|_{\infty} + \frac{B}{\eta^{\alpha-1}})^{\theta-1}].$$

By the basic inequality $(x+y)^p \le 2^p(x^p + y^p),\ x, y, p > 0$, we have

$$|D_{0+}^{\alpha}x(t)|^{p(t)-1} \le \Lambda_1 + \Lambda_2\,\|D_{0+}^{\alpha}x\|_{\infty}^{\theta-1},$$

where

$$\Lambda_1 = \frac{b(2B)^{\theta-1} + a\eta^{(\alpha-1)(\theta-1)}}{\Gamma(\beta+1)\eta^{(\alpha-1)(\theta-1)}}, \quad \Lambda_2 = \frac{b2^{\theta-1}(1+\eta^{\alpha-1})^{\theta-1}}{\Gamma(\beta+1)(\Gamma(\alpha+1)\eta^{\alpha-1})^{\theta-1}}.$$

Hence, we can obtain

$$\|D_{0+}^{\alpha}x\|_{\infty} \le 2^{\frac{1}{p(t)-1}}(\Lambda_1^{\frac{1}{p(t)-1}} + \Lambda_2^{\frac{1}{p(t)-1}}\,\|D_{0+}^{\alpha}x\|_{\infty}^{\frac{\theta-1}{p(t)-1}}).$$

Since $\frac{\theta-1}{p(t)-1} \in (0,1]$, by the basic inequality $x^{\kappa} \le x+1$ for $x > 0, \kappa \in (0,1]$, we have

$$\|D_{0+}^{\alpha}x\|_{\infty} \le 2^{\frac{1}{p(t)-1}}\Lambda_1^{\frac{1}{p(t)-1}} + 2^{\frac{1}{p(t)-1}}\Lambda_2^{\frac{1}{p(t)-1}}(\|D_{0+}^{\alpha}x\|_{\infty} + 1).$$

In view of (3.2), we can obtain that there exists a constant $M_2 > 0$ such that

$$\|D_{0+}^{\alpha}x\|_{\infty} \le M_2, \quad \|x\|_{\infty} \le \frac{1+\eta^{\alpha-1}}{\Gamma(\alpha+1)\eta^{\alpha-1}}M_2 + \frac{B}{\eta^{\alpha-1}} := M_3.$$

So, Ω_1 is bounded. Let

$$\Omega_2 = \{x | x \in \mathrm{Ker}L,\ Nx \in \mathrm{Im}L\}.$$

For $x \in \Omega_2$, we have $x(t) = ct^{\alpha-1},\ c \in \mathbb{R}$ and $Nx \in \mathrm{Im}L$. Thus, we can obtain $QN(ct^{\alpha-1}) = 0$, which together with (H$_2$) implies $|c| \le \frac{B}{\eta^{\alpha-1}}$. Hence, Ω_2 is bounded. Let

$$\Omega_3 = \{x \in \mathrm{Ker}L | -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $J : \mathrm{Ker}L \to \mathrm{Im}Q$ is defined by $J(ct^{\alpha-1}) = c,\ c \in \mathbb{R}\ , t \in [0,1]$. Thus, we have

$$\lambda c + (1-\lambda)\Delta\left[\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta} f(s, cs^{\alpha-1}))ds - \frac{\gamma}{\Gamma(2\alpha-1)}\int_0^{\eta}(\eta-s)^{2\alpha-2}\varphi_{p(s)}^{-1}(I_{0+}^{\beta} f(s, cs^{\alpha-1}))ds\right] = 0. \quad (3.4)$$

If $\lambda = 0$, by the first part of (H$_2$), we have $|c| \le \frac{B}{\eta^{\alpha-1}}$. If $\lambda \in (0,1]$, we can also obtain $|c| \le \frac{B}{\eta^{\alpha-1}}$. Otherwise, if $|c| > \frac{B}{\eta^{\alpha-1}}$, in view of the first part of (H$_2$), one has

$$\begin{aligned}\lambda \mathrm{sgn}(ct^{\alpha-1})c + (1-\lambda)\Delta \mathrm{sgn}(ct^{\alpha-1})&\left[\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta} f(s, cs^{\alpha-1}))ds\right.\\ &\left.-\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^{\eta}(\eta-s)^{2\alpha-2}\varphi_{p(s)}^{-1}(I_{0+}^{\beta} f(s, cs^{\alpha-1}))ds\right] > 0\end{aligned}$$

for any $t \in [\eta, 1]$. Thus, by choosing $t = 1$, we obtain

$$\begin{aligned}\lambda \mathrm{sgn}(c)c + (1-\lambda)\Delta \mathrm{sgn}(c)&\left[\int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^{\beta} f(s, cs^{\alpha-1}))ds\right.\\ &\left.-\frac{\gamma}{\Gamma(2\alpha-1)}\int_0^{\eta}(\eta-s)^{2\alpha-2}\varphi_{p(s)}^{-1}(I_{0+}^{\beta} f(s, cs^{\alpha-1}))ds\right] > 0,\end{aligned}$$

which contradicts to (3.4). Thus, Ω_3 is bounded. Let

$$\Omega_3' = \{x \in \mathrm{Ker}L | \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}.$$

Similarly, we can prove that Ω_3' is bounded by the second part of (H$_2$).

Let $\Omega = \{x \in X | \|x\|_\infty < \max\{M_3, \frac{B}{\eta^{\alpha-1}}\} + 1\}$. Since L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$, then by the previous proof, we have

(i) $Lx \ne \lambda Nx$, for every $(x, \lambda) \in [(\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0,1)$.

(ii) $Nx \notin \mathrm{Im}L$, for every $x \in \mathrm{Ker}L \cap \partial\Omega$.

Let

$$H(x, \lambda) = \pm\lambda J(x) + (1-\lambda)QNx.$$

We can obtain $H(x, \lambda) \ne 0$ for $x \in \mathrm{Ker}L \cap \partial\Omega$. Therefore,

$$\begin{aligned}\deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot, 0), \Omega \cap \mathrm{Ker}L, 0)\\ &= \deg(H(\cdot, 1), \Omega \cap \mathrm{Ker}L, 0)\\ &= \deg(\pm J, \Omega \cap \mathrm{Ker}L, 0) \ne 0.\end{aligned}$$

Hence, the conditions of Lemma 2.3 are satisfied. Therefore, we can obtain that $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$. Then BVP (1.1) has at least one solution. □

Example 3.4. Consider the following BVP:

$$\begin{cases} D_{0^+}^{\frac{2}{3}}\varphi_{(t^2+2)}(D_{0^+}^{\frac{3}{2}}x(t)) + \frac{1}{10} - \frac{1}{10}x(t) = 0, & t \in (0,1), \\ x(0) = 0, \quad D_{0^+}^{\frac{1}{2}}x(1) = \gamma I_{0^+}^{\frac{1}{2}}x(\eta), \quad D_{0^+}^{\frac{1}{2}}x(0) = 0, \end{cases}$$

where $p(t) = t^2 + 2$, $\alpha = \frac{3}{2}$, $\beta = \frac{2}{3}$, $\theta = 2$, $f(t, x(t)) = \frac{1}{10} - \frac{1}{10}|x(t)|$ $a = b = \frac{1}{10}$. Clearly, (H_1) holds. By choosing $\gamma = 2$, $\eta = \frac{1}{2}$, $B = 20$, it is easy to verify $\Gamma(2\alpha - 1) = \gamma\eta^{2\alpha-2}$. Moreover, if $x(t) > 20$, we have

$$-f(t, x(t)) = -\frac{1}{10} + \frac{1}{10}|x(t)| > 0.$$

Thus, we obtain $Nx(t) = -\varphi^{-1}_{(t^2+2)}(I_{0^+}^{\frac{2}{3}}f(t, x(t))) > 0$ and

$$\begin{aligned} QNx(t) &= \Delta[\int_0^1 Nx(s)ds - \frac{2}{\Gamma(2)}\int_0^{\frac{1}{2}}(\frac{1}{2} - s)Nx(s)ds] \\ &= \Delta[\int_0^1 Nx(s)ds - \int_0^{\frac{1}{2}}(1 - 2s)Nx(s)ds] \\ &= \Delta\{\int_{\frac{1}{2}}^1 Nx(s)ds + \int_0^{\frac{1}{2}}[1 - (1 - 2s)]Nx(s)ds\} > 0. \end{aligned}$$

Hence, we have

$$\mathrm{sgn}\{x(t)\}QN(x(t)) > 0. \tag{3.5}$$

If $x(t) < -20$, it can be found that (3.5) is also true. Thus, the first part of (H_2) holds. In view of (3.2), we have

$$\frac{2 + \sqrt{2}}{5\Gamma(\frac{5}{3})\Gamma(\frac{5}{2})} < 1.$$

Therefore, it has at least one solution.

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