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A hybrid extragradient method for bilevel pseudomonotone variational inequalities with multiple solutions

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Abstract

In this paper, we introduce and analyze a hybrid extragradient algorithm for solving bilevel pseudomonotone variational inequalities with multiple solutions in a real Hilbert space. The proposed algorithm is based on Korpelevich's extragradient method, Mann's iteration method, hybrid steepest-descent method, and viscosity approximation method (including Halpern's iteration method). Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is derived.

Keywords: Bilevel variational inequality, hybrid extragradient algorithm, pseudomonotonicity, Lipschitz continuity, global convergence.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. If $\{x_n\}$ is a sequence in H, then we denote by $x_n \to x$ (respectively, $x_n \rightharpoonup x$) the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the

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set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\| \quad \forall x, y \in C.$$

In particular, if $L = 1$, then S is called a nonexpansive mapping; if $L \in [0, 1)$ then S is called a contraction.

Let $\mathcal{A} : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle \mathcal{A}x, y - x\rangle \geq 0 \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, \mathcal{A})$.

The VIP (1.1) was first discussed by Lions [23]. There are many applications of VIP (1.1) in various fields; see e.g., [6, 7, 9, 32]. In 1976, Korpelevich [22] proposed an iterative algorithm for solving the VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_k = P_C(x_k - \tau A x_k), \\ x_{k+1} = P_C(x_k - \tau A y_k), \qquad \forall k \geq 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [1, 4–13, 17, 25, 33, 38] and references therein.

Let $A : C \to H$ and $B : H \to H$ be two mappings. Consider the following bilevel variational inequality problem (BVIP):

Problem 1.1 (**Problem AKM).** We find $x^* \in \mathrm{VI}(C, B)$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0 \quad \forall x \in \mathrm{VI}(C, B), \tag{1.2}$$

where $\mathrm{VI}(C, B)$ denotes the set of solutions of the VIP: Find $y^* \in C$ such that

$$\langle By^*, y - y^*\rangle \geq 0 \quad \forall y \in C. \tag{1.3}$$

In particular, whenever $H = \mathbf{R}^n$, the BVIP was recently studied by Anh, Kim and Muu [1].

Bilevel variational inequalities are special classes of quasivariational inequalities (see [2, 3, 14, 31]) and of equilibrium with equilibrium constraints considered in [19, 24]. However it covers some classes of mathematical programs with equilibrium constraints (see [24]), bilevel minimization problems (see [26]), variational inequalities (see [16, 21, 35, 37]), and complementarity problems.

In what follows, suppose that A and B satisfy the following conditions:

(C1) B is pseudomonotone on H and A is β-strongly monotone on C;

(C2) A is L_1-Lipschitz continuous on C;

(C3) B is L_2-Lipschitz continuous on H;

(C4) $\mathrm{VI}(C, B) \neq \emptyset$.

It is remarkable that under conditions (C1)-(C4), Problem AKM has only a solution because A is β-strongly monotone and L_1-Lipschitz continuous on C. In 2012, Anh, Kim and Muu [1] introduced the following extragradient iterative algorithm for solving the above bilevel variational inequality.

Algorithm 1.2 (**Algorithm AKM).** ([1, Algorithm 2.1]). *Initialization. Choose* $u \in \mathbf{R}^n$, $x_0 \in C$, $k = 0$, $0 < \lambda \leq \frac{2\beta}{L_1^2}$, *positive sequences* $\{\delta_k\}, \{\lambda_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, *and* $\{\bar{\epsilon}_k\}$ *such that*

$$\begin{cases} \lim\limits_{k\to\infty} \delta_k = 0, \ \sum\limits_{k=0}^{\infty} \bar{\epsilon}_k < \infty, \\ \alpha_k + \beta_k + \gamma_k = 1 \ \forall k \geq 0, \ \sum\limits_{k=0}^{\infty} \alpha_k = \infty, \\ \lim\limits_{k\to\infty} \alpha_k = 0, \ \lim\limits_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}], \ \lim\limits_{k\to\infty} \lambda_k = 0, \ \lambda_k \leq \frac{1}{L_2} \qquad \forall k \geq 0. \end{cases}$$

- *Step 1. Compute*

$$\begin{cases} y_k := P_C(x_k - \lambda_k B x_k), \\ z_k := P_C(x_k - \lambda_k B y_k). \end{cases}$$

- *Step 2. Inner loop $j = 0, 1, \ldots.$ Compute*

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_C(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_C(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{\mathrm{VI}(C,B)} x_{k,0}\| \le \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

- *Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.*

Theorem 1.3 (Theorem AKM). ([1, Theorem 3.1]). *Suppose that the assumptions (C1)-(C4) hold. Then the two sequences $\{x_k\}$ and $\{z_n\}$ in Algorithm AKM converges to the same point x^* which is a solution of the BVIP.*

It is well known that an important approach to the BVIP is the Tikhonov regularization method. The main idea of this method for monotone variational inequalities is to add a strongly monotone operator depending on a parameter to the cost operator to obtain a parameterized strongly monotone variational inequality, which is uniquely solved. By letting the parameter to a suitable limit, the sequence of the solutions of the regularized problems will tend to the solution of the original problem. This result allows that the Tikhonov regularization method can be used to solve bilevel monotone variational inequalities. Recently, in [18–20] the Tikhonov method with generalized regularization operators and bifunctions is extended to pseudomonotone variational inequalities and equilibrium problems, respectively. However in this case, the regularized subproblems, may fail to be strongly monotone, even pseudomonotone, since the sum of a strongly monotone operator and a pseudomonotone operator, in general, is not pseudomonotone. In our opinion, the existing methods that require some monotonicity properties cannot be applied to solve the regularized subvariational inequalities. Therefore the above Theorem AKM shows that the Algorithm AKM (i.e., an extragradient-tyle algorithm) is an efficient approach for directly solving bilevel pseudomonotone variational inequalities.

Motivated and inspired by the above facts, we introduce and analyze a hybrid extragradient algorithm for solving bilevel pseudomonotone variational inequalities with multiple solutions in a real Hilbert space. The proposed algorithm is based on Korpelevich's extragradient method (see [22]), Mann's iteration method, hybrid steepest-descent method (see [30, 32]) and viscosity approximation method (see[33, 36]) (including Halpern's iteration method). Under some mild conditions, the strong convergence of the iteration sequences generated by the proposed algorithm is derived. Our results improve and extend the corresponding results announced by some others, e.g., Anh, Kim amd Muu [1, Theorem 3.1].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \ge 0 \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2 \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2 \quad \forall x, y \in C.$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_Cx \in C$ satisfying the property

$$\|x - P_Cx\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 ([15])**.** *For given $x \in H$ and $z \in C$:*

(i) $z = P_Cx \Leftrightarrow \langle x - z, y - z\rangle \leq 0\ \forall y \in C$;

(ii) $z = P_Cx \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2\ \forall y \in C$;

(iii) $\langle P_Cx - P_Cy, x - y\rangle \geq \|P_Cx - P_Cy\|^2\ \forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda A)u - (I - \lambda A)v\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \tag{2.1}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 2.2. A mapping $T : H \to H$ is said to be:

(a) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in H;$$

(b) firmly nonexpansive, if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2 \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.3 ([29])**.** *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \quad \forall x, y \in X.$$

It is not hard to prove the following lemmas which will be used in the sequel. Here we omit their proofs.

Lemma 2.4. *Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$ and $V : H \to H$ be an l-Lipschitzian mapping. If $\mu\eta - \gamma l > 0$ for $\mu, \gamma \geq 0$, then $\mu F - \gamma V$ is $(\mu\eta - \gamma l)$-strongly monotone, that is,*

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2 \quad \forall x, y \in H.$$

Lemma 2.5. *Let H be a real Hilbert space. Then the followings hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$;

(c) *If $\{x_k\}$ is a sequence in H such that $x_k \rightharpoonup x$, it follows that*

$$\limsup_{k\to\infty} \|x_k - y\|^2 = \limsup_{k\to\infty} \|x_k - x\|^2 + \|x - y\|^2 \quad \forall y \in H.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $S : C \to H$, we define the mapping $S^\lambda : C \to H$ by

$$S^\lambda x := Sx - \lambda\mu F(Sx) \quad \forall x \in C,$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in H$.

Lemma 2.6 ([30], Lemma 3.1)**.** *S^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|S^\lambda x - S^\lambda y\| \leq (1 - \lambda\tau)\|x - y\| \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 2.7 ([28], Demiclosedness principle)**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with $\mathrm{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.8 ([30], Lemma 2.1)**.** *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \alpha_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1 - \alpha_k) = 0;$$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, *or* $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Iterative algorithm and convergence criteria

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this section, we always assume the following:

- $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, and $V : H \to H$ is an l-Lipschitzian mapping;
- $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $A : C \to H$ and $B : H \to H$ are two mappings such that the hypotheses (H1)-(H4) hold:
  - (H1) B is pseudomonotone on H;
  - (H2) A is β-inverse-strongly monotone on C;
  - (H3) B is L-Lipschitz continuous on H;
  - (H4) $\mathrm{VI}(C, B) \neq \emptyset$.

Next, we introduce and consider the following BVIP, which may have multiple solutions.

Problem 3.1. We find $x^* \in \mathrm{VI}(C, B)$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in \mathrm{VI}(C, B), \tag{3.1}$$

where $\mathrm{VI}(C, B)$ denotes the set of solutions of the VIP: Find $y^* \in C$ such that

$$\langle By^*, y - y^* \rangle \geq 0 \quad \forall y \in C. \tag{3.2}$$

Algorithm 3.2. *Initialization. Choose $u \in H$, $x_0 \in H$, $k = 0$, $0 < \lambda \leq 2\beta$, positive sequences $\{\delta_k\}, \{\lambda_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, and $\{\bar{\epsilon}_k\}$ such that*

$$\begin{cases} \lim_{k\to\infty} \delta_k = 0, \ \sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty, \\ \alpha_k + \beta_k + \gamma_k = 1 \ \forall k \geq 0, \ \sum_{k=0}^{\infty} \alpha_k = \infty, \\ \lim_{k\to\infty} \alpha_k = 0, \ \lim_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}], \ \lim_{k\to\infty} \lambda_k = 0, \ \lambda_k \leq \frac{1}{L} \ \forall k \geq 0. \end{cases}$$

- *Step 1.*

$$\begin{cases} v_k = \alpha_k \gamma V x_k + \gamma_k x_k + ((1 - \gamma_k)I - \alpha_k \mu F)x_k, \\ y_k := P_C(v_k - \lambda_k B v_k), \\ z_k := P_C(v_k - \lambda_k B y_k). \end{cases}$$

- *Step 2. Inner loop $j = 0, 1, \ldots$. Compute*

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_C(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_C(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{\mathrm{VI}(C,B)} x_{k,0}\| \leq \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

- *Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.*

Let C be a nonempty closed convex subset of H, $B : C \to H$ be monotone and L-Lipschitz continuous on C, and $S : C \to C$ be a nonexpansive mapping such that $\mathrm{VI}(C, B) \cap \mathrm{Fix}(S) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_k = P_C(x_k - \delta_k B x_k), \\ x_{k+1} = \alpha_k x_0 + \beta_k x_k + \gamma_k S P_C(x_k - \delta_k B y_k) \quad \forall k \geq 0, \end{cases}$$

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, and $\{\delta_k\}$ satisfy the following conditions:

$$\begin{cases} \delta_k > 0 \ \ \forall k \geq 0, \ \lim_{k\to\infty} \delta_k = 0, \\ \alpha_k + \beta_k + \gamma_k = 1 \ \ \forall k \geq 0, \\ \sum_{k=0}^{\infty} \alpha_k = \infty, \ \lim_{k\to\infty} \alpha_k = 0, \\ 0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1. \end{cases}$$

Under these conditions, Yao, Liou and Yao [34] proved that the sequences $\{x_k\}$ and $\{y_k\}$ converge to the same point $P_{\mathrm{VI}(C,B)\cap \mathrm{Fix}(S)} x_0$.

Applying these iteration sequences with S being the identity mapping, we have the following lemma.

Lemma 3.3. *Suppose that the hypotheses* (H1)-(H4) *hold. Then the sequence* $\{x^{k,j}\}$ *generated by Algorithm* 3.2 *converges strongly to the point* $P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)$ *as* $j \to \infty$. *Consequently, we have*

$$\|h_k - P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| \leq \bar{\epsilon}_k \quad \forall k \geq 0.$$

In the sequel we always suppose that the inner loop in the Algorithm 3.2 terminates after a finite number of steps. This assumption, by Lemma 3.3, is satisfied when B is monotone on C.

Lemma 3.4. *Let sequences* $\{v_k\}, \{y_k\}$ *and* $\{z_k\}$ *be generated by Algorithm* 3.2, *B be L-Lipschitzian and pseudomonotone on H, and* $p \in \mathrm{VI}(C, B)$. *Then, we have*

$$\|z_k - p\|^2 \leq \|v_k - p\|^2 - (1 - \lambda_k L)\|v_k - y_k\|^2 - (1 - \lambda_k L)\|y_k - z_k\|^2. \tag{3.3}$$

Proof. Let $p \in \mathrm{VI}(C, B)$. That means

$$\langle Bp, x - p\rangle \geq 0 \quad \forall x \in C.$$

Then, for each $\lambda_k > 0$, p satisfies the fixed point equation

$$p = P_C(p - \lambda_k B p).$$

Since B is pseudomonotone on H and $p \in \mathrm{VI}(C, B)$, we have

$$\langle Bp, y_k - p\rangle \geq 0 \ \Rightarrow \ \langle B y_k, y_k - p\rangle \geq 0.$$

Then, applying Proposition 2.1 (ii) with $v_k - \lambda_k B y_k$ and p, we obtain

$$\begin{aligned} \|z_k - p\|^2 &\leq \|v_k - \lambda_k B y_k - p\|^2 - \|v_k - \lambda_k B y_k - z_k\|^2 \\ &= \|v_k - p\|^2 - 2\lambda_k \langle B y_k, v_k - p\rangle + \lambda_k^2 \|B y_k\|^2 - \|v_k - z_k\|^2 \\ &\quad - \lambda_k^2 \|B y_k\|^2 + 2\lambda_k \langle B y_k, v_k - z_k\rangle \\ &= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k \langle B y_k, p - z_k\rangle \\ &= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k \langle B y_k, p - y_k\rangle + 2\lambda_k \langle B y_k, y_k - z_k\rangle \\ &\leq \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k \langle B y_k, y_k - z_k\rangle. \end{aligned} \tag{3.4}$$

Applying Proposition 2.1 (i) with $v_k - \lambda_k B v_k$ and z_k, we also have

$$\langle v_k - \lambda_k B v_k - y_k, z_k - y_k\rangle \leq 0.$$

Combining this inequality with (3.4) and observing that B is L-Lipschitz continuous on H, we obtain

$$\begin{aligned}\|z_k-p\|^2 &\le \|v_k-p\|^2-\|(v_k-y_k)+(y_k-z_k)\|^2+2\lambda_k\langle By_k,y_k-z_k\rangle\\ &=\|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2-2\langle v_k-y_k,y_k-z_k\rangle\\ &\quad+2\lambda_k\langle By_k,y_k-z_k\rangle\\ &=\|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2-2\langle v_k-\lambda_kBy_k-y_k,y_k-z_k\rangle\\ &=\|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2-2\langle v_k-\lambda_kBv_k-y_k,y_k-z_k\rangle\\ &\quad+2\lambda_k\langle Bv_k-By_k,z_k-y_k\rangle\\ &\le \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+2\lambda_k\langle Bv_k-By_k,z_k-y_k\rangle\\ &\le \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+2\lambda_k\|Bv_k-By_k\|\|z_k-y_k\|\\ &\le \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+2\lambda_kL\|v_k-y_k\|\|z_k-y_k\|\\ &\le \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+\lambda_kL(\|v_k-y_k\|^2+\|z_k-y_k\|^2)\\ &\le \|v_k-p\|^2-(1-\lambda_kL)\|v_k-y_k\|^2-(1-\lambda_kL)\|y_k-z_k\|^2.\end{aligned} \tag{3.5}$$

This completes the proof. □

Lemma 3.5. *Suppose that the hypotheses* (H1)-(H4) *hold and that* $\mathrm{VI}(\mathrm{VI}(C,B),A)\neq\emptyset$. *Then the sequence* $\{x_k\}$ *generated by Algorithm* 3.2 *is bounded.*

Proof. Since $\lim_{k\to\infty}\alpha_k=0$, $\lim_{k\to\infty}\beta_k=\xi\in(0,\frac{1}{2}]$ and $\alpha_k+\beta_k+\gamma_k=1$, we get

$$\lim_{k\to\infty}(1-\gamma_k)=\lim_{k\to\infty}(\alpha_k+\beta_k)=\xi.$$

Hence, we may assume, without loss of generality, that $0<\frac{\alpha_k}{1-\gamma_k}\le 1$ for all $k\ge 0$.

Take an arbitrary $p\in\mathrm{VI}(\mathrm{VI}(C,B),A)$. Then we have

$$\langle Ap,x-p\rangle\ge 0\quad \forall x\in \mathrm{VI}(C,B),$$

which implies

$$p=P_{\mathrm{VI}(C,B)}(p-\lambda Ap).$$

Then, it follows from (2.1), Proposition 2.1 (iii), β-inverse strong monotonicity of A, and $0<\lambda\le 2\beta$ that

$$\begin{aligned}\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-p\|^2 &=\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-P_{\mathrm{VI}(C,B)}(p-\lambda Ap)\|^2\\ &\le \|z_k-\lambda Az_k-p+\lambda Ap\|^2\\ &\le \|z_k-p\|^2+\lambda(\lambda-2\beta)\|Az_k-Ap\|^2\\ &\le \|z_k-p\|^2.\end{aligned} \tag{3.6}$$

Furthermore, from Algorithm 3.2 and Lemma 2.6, we have

$$\begin{aligned}\|v_k-p\| &=\|\alpha_k\gamma Vx_k+\gamma_kx_k+((1-\gamma_k)I-\alpha_k\mu F)x_k-p\|\\ &=\|\alpha_k(\gamma Vx_k-\mu Fp)+\gamma_k(x_k-p)+((1-\gamma_k)I-\alpha_k\mu F)x_k\\ &\quad-((1-\gamma_k)I-\alpha_k\mu F)p\|\\ &\le \alpha_k\gamma\|Vx_k-Vp\|+\alpha_k\|(\gamma V-\mu F)p\|+\gamma_k\|x_k-p\|\\ &\quad+\|((1-\gamma_k)I-\alpha_k\mu F)x_k-((1-\gamma_k)I-\alpha_k\mu F)p\|\\ &\le \alpha_k\gamma l\|x_k-p\|+\alpha_k\|(\gamma V-\mu F)p\|+\gamma_k\|x_k-p\|\\ &\quad+(1-\gamma_k)\|(I-\tfrac{\alpha_k\mu}{1-\gamma_k}F)x_k-(I-\tfrac{\alpha_k\mu}{1-\gamma_k}F)p\|\\ &\le \alpha_k\gamma l\|x_k-p\|+\alpha_k\|(\gamma V-\mu F)p\|+\gamma_k\|x_k-p\|\\ &\quad+(1-\gamma_k)(1-\tfrac{\alpha_k}{1-\gamma_k}\tau)\|x_k-p\|\\ &=\alpha_k\gamma l\|x_k-p\|+\alpha_k\|(\gamma V-\mu F)p\|+\gamma_k\|x_k-p\|\\ &\quad+(1-\gamma_k-\alpha_k\tau)\|x_k-p\|\\ &=(1-\alpha_k(\tau-\gamma l))\|x_k-p\|+\alpha_k\|(\gamma V-\mu F)p\|\\ &=(1-\alpha_k(\tau-\gamma l))\|x_k-p\|+\alpha_k(\tau-\gamma l)\tfrac{\|(\gamma V-\mu F)p\|}{\tau-\gamma l}\\ &\le \max\{\|x_k-p\|,\tfrac{\|(\gamma V-\mu F)p\|}{\tau-\gamma l}\}.\end{aligned} \tag{3.7}$$

Utilizing (3.5)-(3.7) and the assumptions $0 < \lambda \le 2\beta,\ \sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$ we obtain that

$$\begin{aligned}
\|x_{k+1} - p\| &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\| \\
&\le \alpha_k\|u-p\| + \beta_k\|x_k - p\| + \gamma_k\|h_k - p\| \\
&\le \alpha_k\|u-p\| + \beta_k\|x_k - p\| + \gamma_k\|h_k - P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| \\
&\quad + \gamma_k\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k) - p\| \\
&\le \alpha_k\|u-p\| + \beta_k\|x_k - p\| + \gamma_k\bar{\epsilon}_k + \gamma_k\|z_k - p\| \\
&\le \alpha_k\|u-p\| + \beta_k\|x_k - p\| + \gamma_k\bar{\epsilon}_k + \gamma_k\|v_k - p\| \\
&\le \alpha_k\|u-p\| + \beta_k\|x_k - p\| + \gamma_k\bar{\epsilon}_k + \gamma_k \max\{\|x_k - p\|, \tfrac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} \\
&\le \alpha_k\|u-p\| + (\beta_k + \gamma_k)\max\{\|x_k - p\|, \tfrac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \bar{\epsilon}_k \\
&= \alpha_k\|u-p\| + (1-\alpha_k)\max\{\|x_k - p\|, \tfrac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \bar{\epsilon}_k \\
&\le \max\{\|x_k - p\|, \|u-p\|, \tfrac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \bar{\epsilon}_k \\
&\le \max\{\|x_0 - p\|, \|u-p\|, \tfrac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \sum_{j=0}^{k}\bar{\epsilon}_j \\
&\le \max\{\|x_0 - p\|, \|u-p\|, \tfrac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \sum_{k=0}^{\infty}\bar{\epsilon}_k \\
&< \infty,
\end{aligned}$$

which shows that the sequence $\{x_k\}$ is bounded, and so are the sequences $\{v_k\}, \{y_k\}$, and $\{z_k\}$. □

Lemma 3.6 ([27]). *Let $\{x_k\}$ and $\{y_k\}$ be two bounded sequences in a real Banach space X. Let $\{\beta_k\}$ be a sequence in $[0,1]$. Suppose that*

$$\begin{cases} 0 < \liminf\limits_{k\to\infty}\beta_k \le \limsup\limits_{k\to\infty}\beta_k < 1, \\ x_{k+1} = (1-\beta_k)y_k + \beta_k x_k, \\ \limsup\limits_{k\to\infty}(\|y_{k+1} - y_k\| - \|x_{k+1} - x_k\|) \le 0. \end{cases}$$

Then, $\lim_{k\to\infty}\|y_k - x_k\| = 0$.

Lemma 3.7. *Suppose that the hypotheses* (H1)-(H4) *and that the sequences $\{v_k\}, \{y_k\}$, and $\{z_k\}$ are generated by Algorithm* 3.2. *Then, we have*

$$\|z_{k+1} - z_k\| \le (1 + \lambda_{k+1}L)\|v_{k+1} - v_k\| + \lambda_k\|By_k\| + \lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|). \tag{3.8}$$

Moreover

$$\lim_{k\to\infty}\|z_{k+1} - z_k\| = \lim_{k\to\infty}\|v_{k+1} - v_k\| = 0.$$

Proof. Since B is L-Lipschitzian on H, for each $x, y \in H$, we have

$$\begin{aligned}
\|(I - \lambda_k B)x - (I - \lambda_k B)y\| &= \|x - y - \lambda_k(Bx - By)\| \\
&\le \|x - y\| + \lambda_k\|Bx - By\| \\
&\le (1 + \lambda_k L)\|x - y\|.
\end{aligned}$$

Combining this inequality with Proposition 2.1 (iii), we have

$$\begin{aligned}
\|z_{k+1} - z_k\| &= \|P_C(v_{k+1} - \lambda_{k+1}By_{k+1}) - P_C(v_k - \lambda_k By_k)\| \\
&\le \|(v_{k+1} - \lambda_{k+1}By_{k+1}) - v_k + \lambda_k By_k\| \\
&= \|(v_{k+1} - \lambda_{k+1}Bv_{k+1}) - (v_k - \lambda_{k+1}Bv_k) \\
&\quad + \lambda_{k+1}(Bv_{k+1} - By_{k+1} - Bv_k) + \lambda_k By_k\| \\
&\le (1 + \lambda_{k+1}L)\|v_{k+1} - v_k\| + \lambda_k\|By_k\| \\
&\quad + \lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|).
\end{aligned} \tag{3.9}$$

This is the desired result (3.8). □

Now we denote $x_{k+1} = (1-\beta_k)w_k + \beta_k x_k$. Then, we have

$$\begin{aligned} w_{k+1} - w_k &= \frac{\alpha_{k+1}u+\gamma_{k+1}h_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k u+\gamma_k h_k}{1-\beta_k} \\ &= (\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k})u + (\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k})h_k + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(h_{k+1}-h_k). \end{aligned} \tag{3.10}$$

Note that, for $0 < \lambda \le 2\beta$, we have from (2.1) that

$$\begin{aligned} &\|P_{\mathrm{VI}(C,B)}(z_{k+1} - \lambda A z_{k+1}) - P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\|^2 \\ &\le \|(z_{k+1} - \lambda A z_{k+1}) - (z_k - \lambda A z_k)\|^2 \\ &\le \|z_{k+1} - z_k\|^2 + \lambda(\lambda - 2\beta)\|Az_{k+1} - Az_k\|^2 \\ &\le \|z_{k+1} - z_k\|^2. \end{aligned}$$

Then, by using (3.10) we get

$$\begin{aligned} \|w_{k+1} - w_k\| &- \|x_{k+1} - x_k\| \\ &\le |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|\|h_k\| + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}\|h_{k+1} - h_k\| - \|x_{k+1} - x_k\| \\ &\le |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}\|P_{\mathrm{VI}(C,B)}(z_{k+1} - \lambda A z_{k+1}) - P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\|P_{\mathrm{VI}(C,B)}(z_{k+1} - \lambda A z_{k+1}) - h_{k+1}\| \\ &\quad + \|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k) - h_k\|) - \|x_{k+1} - x_k\| \\ &\le |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}\|z_{k+1} - z_k\| + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) - \|x_{k+1} - x_k\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|w_{k+1} - w_k\| &- \|x_{k+1} - x_k\| \\ &\le |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}}\|v_{k+1} - v_k\| + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|) \\ &\quad - \|x_{k+1} - x_k\| \\ &\le |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}}\|v_{k+1} - v_k\| - \|x_{k+1} - x_k\| + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|). \end{aligned} \tag{3.11}$$

On the other hand, we define $\tilde{w}_k = \frac{v_k - \gamma_k x_k}{1-\gamma_k}$, which implies that $v_k = (1-\gamma_k)\tilde{w}_k + \gamma_k x_k$. Simple calculations show that

$$\begin{aligned} \tilde{w}_{k+1} - \tilde{w}_k &= \tfrac{v_{k+1} - \gamma_{k+1}x_{k+1}}{1-\gamma_{k+1}} - \tfrac{v_k - \gamma_k x_k}{1-\gamma_k} \\ &= \tfrac{\alpha_{k+1}\gamma V x_{k+1} + ((1-\gamma_{k+1})I - \alpha_{k+1}\mu F)x_{k+1}}{1-\gamma_{k+1}} - \tfrac{\alpha_k \gamma V x_k + ((1-\gamma_k)I - \alpha_k\mu F)x_k}{1-\gamma_k} \\ &= \tfrac{\alpha_{k+1}}{1-\gamma_{k+1}}\gamma V x_{k+1} - \tfrac{\alpha_k}{1-\gamma_k}\gamma V x_k + x_{k+1} - x_k + \tfrac{\alpha_k}{1-\gamma_k}\mu F x_k - \tfrac{\alpha_{k+1}}{1-\gamma_{k+1}}\mu F x_{k+1} \\ &= \tfrac{\alpha_{k+1}}{1-\gamma_{k+1}}(\gamma V x_{k+1} - \mu F x_{k+1}) + \tfrac{\alpha_k}{1-\gamma_k}(\mu F x_k - \gamma V x_k) + x_{k+1} - x_k. \end{aligned}$$

So, it follows that

$$\begin{aligned} &\|\tilde{w}_{k+1} - \tilde{w}_k\| \\ &\quad \le \frac{\alpha_{k+1}}{1-\gamma_{k+1}}(\|\gamma V x_{k+1}\| + \|\mu F x_{k+1}\|) + \frac{\alpha_k}{1-\gamma_k}(\|\mu F x_k\| + \|\gamma V x_k\|) + \|x_{k+1} - x_k\| \\ &\quad \le \|x_{k+1} - x_k\| + (\frac{\alpha_{k+1}}{1-\gamma_{k+1}} + \frac{\alpha_k}{1-\gamma_k})M_0, \end{aligned} \tag{3.12}$$

where $\sup_{k\geq 0}\{\|\mu F x_k\| + \|\gamma V x_k\|\} \leq M_0$ for some $M_0 > 0$. In the meantime, from $v_k = (1-\gamma_k)\tilde{w}_k + \gamma_k x_k$, together with (3.12), we get

$$\begin{aligned}\|v_{k+1} - v_k\| &= \|\gamma_{k+1}x_{k+1} + (1-\gamma_{k+1})\tilde{w}_{k+1} - (\gamma_k x_k + (1-\gamma_k)\tilde{w}_k)\| \\ &= \|(1-\gamma_{k+1})(\tilde{w}_{k+1} - \tilde{w}_k) - (\gamma_{k+1} - \gamma_k)\tilde{w}_k \\ &\quad + \gamma_{k+1}(x_{k+1} - x_k) + (\gamma_{k+1} - \gamma_k)x_k\| \\ &\leq (1-\gamma_{k+1})\|\tilde{w}_{k+1} - \tilde{w}_k\| + \gamma_{k+1}\|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\| \\ &\leq (1-\gamma_{k+1})[\|x_{k+1} - x_k\| + (\tfrac{\alpha_{k+1}}{1-\gamma_{k+1}} + \tfrac{\alpha_k}{1-\gamma_k})M_0] \\ &\quad + \gamma_{k+1}\|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\| \\ &\leq \|x_{k+1} - x_k\| + (\tfrac{\alpha_{k+1}}{1-\gamma_{k+1}} + \tfrac{\alpha_k}{1-\gamma_k})M_0 + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\|.\end{aligned} \tag{3.13}$$

Combining (3.11) and (3.13) we have

$$\begin{aligned}\|w_{k+1} - w_k\| &- \|x_{k+1} - x_k\| \\ &\leq |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}}[\|x_{k+1} - x_k\| + (\tfrac{\alpha_{k+1}}{1-\gamma_{k+1}} + \tfrac{\alpha_k}{1-\gamma_k})M_0 \\ &\quad + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\|] - \|x_{k+1} - x_k\| + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|) \\ &= |\tfrac{\alpha_{k+1}}{1-\beta_{k+1}} - \tfrac{\alpha_k}{1-\beta_k}|\|u\| + |\tfrac{\gamma_{k+1}}{1-\beta_{k+1}} - \tfrac{\gamma_k}{1-\beta_k}|(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}}[(\tfrac{\alpha_{k+1}}{1-\gamma_{k+1}} + \tfrac{\alpha_k}{1-\gamma_k})M_0 + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\|] \\ &\quad + (\tfrac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}} - 1)\|x_{k+1} - x_k\| + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\ &\quad + \tfrac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|).\end{aligned} \tag{3.14}$$

From the assumptions $\alpha_k + \beta_k + \gamma_k = 1$, $\lim_{k\to\infty}\beta_k = \xi \in (0, \frac{1}{2}]$, $\lim_{k\to\infty}\alpha_k = 0$ and $\lim_{k\to\infty}\lambda_k = 0$, it follows that $\lim_{k\to\infty}|\gamma_{k+1} - \gamma_k| = 0$,

$$\lim_{k\to\infty}|\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}| = \lim_{k\to\infty}|\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}| = 0 \quad \text{and} \quad \lim_{k\to\infty}\frac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}} = 1.$$

Combining these equalities with (3.14), we obtain from Lemma 3.5 and $\lim_{k\to\infty}\bar{\epsilon}_k = 0$ that

$$\limsup_{k\to\infty}(\|w_{k+1} - w_k\| - \|x_{k+1} - x_k\|) \leq 0.$$

Now applying Lemma 3.6, we have

$$\lim_{k\to\infty}\|w_k - x_k\| = 0.$$

Hence by $x_{k+1} = (1-\beta_k)w_k + \beta_k x_k$, we deduce that

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = \lim_{k\to\infty}(1-\beta_k)\|w_k - x_k\| = 0, \tag{3.15}$$

which together with $\lim_{k\to\infty}\lambda_k = 0$, (3.8) and (3.13), implies that

$$\lim_{k\to\infty}\|v_{k+1} - v_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|z_{k+1} - z_k\| = 0. \tag{3.16}$$

Lemma 3.8. *Suppose that the hypotheses* (H1)-(H4) *hold and that* $\mathrm{VI}(\mathrm{VI}(C,B),A) \neq \emptyset$. *Then for any* $p \in \mathrm{VI}(\mathrm{VI}(C,B),A)$ *we have*

$$\begin{aligned}\|x_{k+1} - p\|^2 &\leq \alpha_k\|u - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k - p\| \\ &\quad + \gamma_k\bar{\epsilon}_k^2 - \gamma_k(1-\lambda_k L)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2).\end{aligned} \tag{3.17}$$

Moreover,

$$\begin{cases}\lim_{k\to\infty}\|P_{\mathrm{VI}(C,B)}(z_k - \lambda_k A z_k) - z_k\| = 0, \\ \lim_{k\to\infty}\|P_{\mathrm{VI}(C,B)}(y_k - \lambda_k A y_k) - y_k\| = 0.\end{cases}$$

Proof. By Lemma 3.3, we know that

$$\lim_{j\to\infty} x_{k,j} = P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k),$$

which together with $0 < \lambda \le 2\beta$, inequality (3.3), $\lim_{k\to\infty}\beta_k = \xi \in (0, \frac{1}{2}]$, and $p \in \mathrm{VI}(\mathrm{VI}(C,B), A)$, implies that

$$\begin{aligned}
\|x_{k+1}-p\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\|^2 \\
&\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|h_k-p\|^2 \\
&\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k(\|P_{\mathrm{VI}(C,B)}(z_k-\lambda A z_k)-p\| + \bar\epsilon_k)^2 \\
&= \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 \\
&\quad + \gamma_k(\|P_{\mathrm{VI}(C,B)}(z_k-\lambda A z_k) - P_{\mathrm{VI}(C,B)}(p-\lambda Ap)\| + \bar\epsilon_k)^2 \\
&\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k(\|(z_k-\lambda Az_k)-(p-\lambda Ap)\| + \bar\epsilon_k)^2 \\
&\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k(\|z_k-p\| + \bar\epsilon_k)^2 \\
&= \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|z_k-p\|^2 + 2\gamma_k\bar\epsilon_k\|z_k-p\| + \gamma_k\bar\epsilon_k^2 \\
&\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + 2\gamma_k\bar\epsilon_k\|z_k-p\| + \gamma_k\bar\epsilon_k^2 \\
&\quad + \gamma_k(\|v_k-p\|^2 - (1-\lambda_k L)\|v_k-y_k\|^2 - (1-\lambda_k L)\|y_k-z_k\|^2) \\
&= \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|v_k-p\|^2 + 2\gamma_k\bar\epsilon_k\|z_k-p\| + \gamma_k\bar\epsilon_k^2 \\
&\quad - \gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2).
\end{aligned} \tag{3.18}$$

On the other hand, from Algorithm 3.2 we have

$$\begin{aligned}
\|v_k-p\|^2 &= \|\alpha_k\gamma V x_k + \gamma_k x_k + ((1-\gamma_k)I - \alpha_k\mu F)x_k - p\|^2 \\
&= \|\alpha_k(\gamma V x_k - \mu F p) + \gamma_k(x_k-p) + ((1-\gamma_k)I - \alpha_k\mu F)x_k \\
&\quad - ((1-\gamma_k)I - \alpha_k\mu F)p\|^2 \\
&= \|\alpha_k\gamma(Vx_k - Vp) + \alpha_k(\gamma Vp - \mu Fp) + \gamma_k(x_k-p) \\
&\quad + ((1-\gamma_k)I - \alpha_k\mu F)x_k - ((1-\gamma_k)I - \alpha_k\mu F)p\|^2 \\
&\le \|\alpha_k\gamma(Vx_k - Vp) + \gamma_k(x_k-p) + ((1-\gamma_k)I - \alpha_k\mu F)x_k \\
&\quad - ((1-\gamma_k)I - \alpha_k\mu F)p\|^2 + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle \\
&\le [\alpha_k\gamma\|Vx_k - Vp\| + \gamma_k\|x_k-p\| + \|((1-\gamma_k)I - \alpha_k\mu F)x_k \\
&\quad - ((1-\gamma_k)I - \alpha_k\mu F)p\|]^2 + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle \\
&\le [\alpha_k\gamma l\|x_k-p\| + \gamma_k\|x_k-p\| + (1-\gamma_k-\alpha_k\tau)\|x_k-p\|]^2 \\
&\quad + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle \\
&= [\alpha_k\tau\tfrac{\gamma l}{\tau}\|x_k-p\| + \gamma_k\|x_k-p\| + (1-\gamma_k-\alpha_k\tau)\|x_k-p\|]^2 \\
&\quad + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle \\
&\le \alpha_k\tau\tfrac{(\gamma l)^2}{\tau^2}\|x_k-p\|^2 + \gamma_k\|x_k-p\|^2 + (1-\gamma_k-\alpha_k\tau)\|x_k-p\|^2 \\
&\quad + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle \\
&= (1-\alpha_k\tfrac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2 + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle \\
&\le \|x_k-p\|^2 + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle.
\end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19), we get

$$\begin{aligned}
\|x_{k+1}-p\|^2 &\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|v_k-p\|^2 + 2\gamma_k\bar\epsilon_k\|z_k-p\| \\
&\quad + \gamma_k\bar\epsilon_k^2 - \gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2) \\
&\le \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k[\|x_k-p\|^2 + 2\alpha_k\langle(\gamma V - \mu F)p, v_k-p\rangle] \\
&\quad + 2\gamma_k\bar\epsilon_k\|z_k-p\| + \gamma_k\bar\epsilon_k^2 - \gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2) \\
&\le \alpha_k\|u-p\|^2 + \|x_k-p\|^2 + 2\alpha_k\|(\gamma V - \mu F)p\|\|v_k-p\| + 2\gamma_k\bar\epsilon_k\|z_k-p\| \\
&\quad + \gamma_k\bar\epsilon_k^2 - \gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2),
\end{aligned}$$

which immediately yields

$$\begin{aligned}&\gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2+\|y_k-z_k\|^2)\\&\le \alpha_k\|u-p\|^2+\|x_k-p\|^2-\|x_{k+1}-p\|^2+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|\\&\quad+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2\\&\le \alpha_k\|u-p\|^2+\|x_k-x_{k+1}\|(\|x_k-p\|+\|x_{k+1}-p\|)+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|\\&\quad+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2.\end{aligned}\tag{3.20}$$

Since $\alpha_k+\beta_k+\gamma_k=1$, $\alpha_k\to 0$, $\beta_k\to\xi\in(0,\frac{1}{2}]$, $\bar{\epsilon}_k\to 0$, $\lambda_k\to 0$ and $\|x_k-x_{k+1}\|\to 0$, from the boundedness of $\{x_k\},\{v_k\}$, and $\{z_k\}$ we obtain

$$\lim_{k\to\infty}\|v_k-y_k\|=0\quad\text{and}\quad\lim_{k\to\infty}\|y_k-z_k\|=0.$$

In addition, it is clear from $\alpha_k\to 0$ that as $k\to\infty$,

$$\begin{aligned}\|v_k-x_k\|&=\|\alpha_k\gamma Vx_k+\gamma_k x_k+((1-\gamma_k)I-\alpha_k\mu F)x_k-x_k\|\\&=\|\alpha_k(\gamma Vx_k-\mu Fx_k)\|\to 0.\end{aligned}$$

That is,

$$\lim_{k\to\infty}\|v_k-x_k\|=0.\tag{3.21}$$

Taking into consideration that $\|v_k-z_k\|\le\|v_k-y_k\|+\|y_k-z_k\|$ and $\|z_k-x_k\|\le\|z_k-v_k\|+\|v_k-x_k\|$, we deduce from (3.20) and (3.21) that

$$\lim_{k\to\infty}\|v_k-z_k\|=0\quad\text{and}\quad\lim_{k\to\infty}\|z_k-x_k\|=0.\tag{3.22}$$

It is clear from (3.20) and (3.21) that

$$\lim_{k\to\infty}\|x_k-y_k\|=0.\tag{3.23}$$

Since A is β-inverse-strongly monotone, it is known from (2.1) that $I-\lambda A$ is a nonexpansive mapping for $0<\lambda\le 2\beta$. Again by Proposition 2.1 (iii) and Lemma 3.3 we have

$$\begin{aligned}&\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-x_{k+1}\|\\&\le\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)\|\\&\quad+\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-x_{k+1}\|\\&\le\|(I-\lambda A)y_k-(I-\lambda A)z_k\|+\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-x_{k+1}\|\\&\le\|y_k-z_k\|+\alpha_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|\\&\quad+\beta_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-x_k\|+\gamma_k\bar{\epsilon}_k\\&\le\|y_k-z_k\|+\alpha_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|+\bar{\epsilon}_k\\&\quad+\beta_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)\|\\&\quad+\beta_k\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\beta_k\|y_k-x_k\|\\&\le\|y_k-z_k\|+\alpha_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|\\&\quad+\bar{\epsilon}_k+\beta_k\|(I-\lambda A)z_k-(I-\lambda A)y_k\|\\&\quad+\beta_k\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\beta_k\|y_k-x_k\|\\&\le\|y_k-z_k\|+\alpha_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|+\bar{\epsilon}_k\\&\quad+\beta_k\|z_k-y_k\|+\beta_k\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\beta_k\|y_k-x_k\|.\end{aligned}\tag{3.24}$$

Consequently, from (3.24), we have

$$\begin{aligned}\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|&\le\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-x_{k+1}\|+\|x_{k+1}-x_k\|+\|x_k-y_k\|\\&\le\|y_k-z_k\|+\alpha_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|+\bar{\epsilon}_k\\&\quad+\beta_k\|z_k-y_k\|+\beta_k\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\beta_k\|y_k-x_k\|\\&\quad+\|x_{k+1}-x_k\|+\|x_k-y_k\|\\&=(1+\beta_k)\|y_k-z_k\|+\alpha_k\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|+\bar{\epsilon}_k\\&\quad+\beta_k\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+(1+\beta_k)\|y_k-x_k\|+\|x_{k+1}-x_k\|,\end{aligned}$$

which immediately yields

$$\begin{aligned}\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\| &\le \tfrac{1+\beta_k}{1-\beta_k}\|y_k-z_k\|+\tfrac{\alpha_k}{1-\beta_k}\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-u\|+\tfrac{\bar\epsilon_k}{1-\beta_k}\\ &\quad+\tfrac{1+\beta_k}{1-\beta_k}\|y_k-x_k\|+\tfrac{1}{1-\beta_k}\|x_{k+1}-x_k\|.\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k=1,\ \alpha_k\to 0,\ \beta_k\to\xi\in(0,\frac12],\ \bar\epsilon_k\to 0,\ \|y_k-z_k\|\to 0,\ \|y_k-x_k\|\to 0$, and $\|x_{k+1}-x_k\|\to 0$ (due to (3.15), (3.20) and (3.23)), we conclude that

$$\lim_{k\to\infty}\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|=0. \tag{3.25}$$

From (2.1) and Proposition 2.1 (iii), it follows that

$$\begin{aligned}\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-z_k\| &\le \|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)\|\\ &\quad+\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\|y_k-z_k\|\\ &\le \|(I-\lambda A)z_k-(I-\lambda A)y_k\|+\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\|y_k-z_k\|\\ &\le \|z_k-y_k\|+\|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+\|y_k-z_k\|\\ &\le \|P_{\mathrm{VI}(C,B)}(y_k-\lambda Ay_k)-y_k\|+2\|y_k-z_k\|.\end{aligned}$$

Utilizing the last inequality we obtain from (3.20) and (3.25) that

$$\lim_{k\to\infty}\|P_{\mathrm{VI}(C,B)}(z_k-\lambda Az_k)-z_k\|=0. \tag{3.26}$$

This completes the proof. □

Theorem 3.9. *Suppose that the hypotheses* (H1)-(H4) *hold and that* $\mathrm{VI}(\mathrm{VI}(C,B),A)\neq\emptyset$. *Then the two sequences* $\{x_k\}$ *and* $\{z_k\}$ *in Algorithm* 3.2 *converge strongly to the same point* $x^*\in\mathrm{VI}(\mathrm{VI}(C,B),A)$ *provided* $\|x_{k+1}-x_k\|+\bar\epsilon_k=o(\alpha_k)$, *which is a unique solution to the VIP*

$$\langle (I+\bar\xi\mu F-\bar\xi\gamma V)x^*-u,p-x^*\rangle\ge 0\quad\forall p\in\mathrm{VI}(\mathrm{VI}(C,B),A), \tag{3.27}$$

where $\bar\xi=1-\xi\in[\frac12,1)$.

Proof. Note that Lemma 3.5 shows the boundedness of $\{x_k\}$. Since H is reflexive, there is at least a weak convergence subsequence of $\{x_k\}$. First, let us assert that $\omega_w(x_k)\subset\mathrm{VI}(\mathrm{VI}(C,B),A)$. As a matter of fact, take an arbitrary $w\in\mathrm{VI}(\mathrm{VI}(C,B),A)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i}\rightharpoonup w$. From (3.23), we know that $y_{k_i}\rightharpoonup w$. It is easy to see that the mapping $P_{\mathrm{VI}(C,B)}(I-\lambda A):C\to\mathrm{VI}(C,B)\subset C$ is nonexpansive because $P_{\mathrm{VI}(C,B)}$ is nonexpansive and $I-\lambda A$ is nonexpansive for β-inverse-strongly monotone mapping A with $0<\lambda\le 2\beta$. So, utilizing Lemma 2.7 and (3.25), we obtain

$$w=P_{\mathrm{VI}(C,B)}(w-\lambda Aw),$$

which leads to $w\in\mathrm{VI}(\mathrm{VI}(C,B),A)$. Thus, the assertion is valid.

Also, note that $0\le\gamma l<\tau$ and

$$\begin{aligned}\mu\eta\ge\tau &\Leftrightarrow \mu\eta\ge 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}\\ &\Leftrightarrow \sqrt{1-\mu(2\eta-\mu\kappa^2)}\ge 1-\mu\eta\\ &\Leftrightarrow 1-2\mu\eta+\mu^2\kappa^2\ge 1-2\mu\eta+\mu^2\eta^2\\ &\Leftrightarrow \kappa^2\ge\eta^2\\ &\Leftrightarrow \kappa\ge\eta.\end{aligned}$$

It is clear that

$$\langle(\mu F-\gamma V)x-(\mu F-\gamma V)y,x-y\rangle\ge(\mu\eta-\gamma l)\|x-y\|^2\quad\forall x,y\in H.$$

Hence, it follows from $0 \le \gamma l < \tau \le \mu\eta$ that $\mu F - \gamma V$ is $(\mu\eta - \gamma l)$-strongly monotone. In the meantime, it is clear that $\mu F - \gamma V$ is Lipschitzian with constant $\mu\kappa + \gamma l > 0$. We define the mapping $\Gamma : H \to H$ as below

$$\Gamma x = (\mu F - \gamma V)x + \frac{1}{\bar{\xi}}(x - u) \quad \forall x \in H,$$

where $u \in H$ and $\bar{\xi} = 1 - \xi \in [\frac{1}{2}, 1)$. Then it is easy to see that Γ is $(\mu\eta - \gamma l + \frac{1}{\bar{\xi}})$-strongly monotone and Lipschitzian with constant $\mu\kappa + \gamma l + \frac{1}{\bar{\xi}} > 0$. Thus, there exists a unique solution $x^* \in \mathrm{VI}(\mathrm{VI}(C, B), A)$ to the VIP

$$\langle (\mu F - \gamma V)x^* + \frac{1}{\bar{\xi}}(x^* - u), p - x^*\rangle \ge 0 \quad \forall p \in \mathrm{VI}(\mathrm{VI}(C, B), A). \tag{3.28}$$

Next, let us show that $x_k \rightharpoonup x^*$. Indeed, take an arbitrary $p \in \mathrm{VI}(\mathrm{VI}(C, B), A)$. In terms of Algorithm 3.2 and Lemma 2.3, we conclude from (3.3), (3.5), and the β-inverse-strong monotonicity of A with $\lambda \le 2\beta$, that

$$\begin{aligned}
&\|x_{k+1} - p\|^2\\
&= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\|^2\\
&\le \|\beta_k(x_k - p) + \gamma_k(h_k - p)\|^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k\|h_k - p\|^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k) - p\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&= \beta_k\|x_k - p\|^2 + \gamma_k(\|P_{\mathrm{VI}(C,B)}(z_k - \lambda A z_k) - P_{\mathrm{VI}(C,B)}(p - \lambda A p)\| + \bar{\epsilon}_k)^2\\
&\quad + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k(\|(I - \lambda A)z_k - (I - \lambda A)p\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k(\|z_k - p\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&= \beta_k\|x_k - p\|^2 + \gamma_k\|z_k - p\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - p, x_{k+1} - p\rangle.
\end{aligned} \tag{3.29}$$

Combining (3.19) and (3.29), we get

$$\begin{aligned}
&\|x_{k+1} - p\|^2\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&\le \beta_k\|x_k - p\|^2 + \gamma_k[(1 - \alpha_k\tfrac{\tau^2 - (\gamma l)^2}{\tau})\|x_k - p\|^2 + 2\alpha_k\langle(\gamma V - \mu F)p, v_k - p\rangle]\\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - p, x_{k+1} - p\rangle\\
&= (\beta_k + \gamma_k - \alpha_k\gamma_k\tfrac{\tau^2 - (\gamma l)^2}{\tau})\|x_k - p\|^2 + 2\alpha_k\gamma_k\langle(\gamma V - \mu F)p, v_k - x_{k+1}\rangle\\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\gamma_k\langle(\gamma V - \mu F)p + \tfrac{1}{\gamma_k}(u - p), x_{k+1} - p\rangle\\
&\le (1 - \alpha_k\gamma_k\tfrac{\tau^2 - (\gamma l)^2}{\tau})\|x_k - p\|^2 + 2\alpha_k\|(\gamma V - \mu F)p\|(\|v_k - x_k\| + \|x_k - x_{k+1}\|)\\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\gamma_k\langle(\gamma V - \mu F)p + \tfrac{1}{\gamma_k}(u - p), x_{k+1} - p\rangle\\
&\le \|x_k - p\|^2 + 2\alpha_k\|(\gamma V - \mu F)p\|(\|v_k - x_k\| + \|x_k - x_{k+1}\|)\\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\gamma_k\langle(\gamma V - \mu F)p + \tfrac{1}{\gamma_k}(u - p), x_{k+1} - p\rangle,
\end{aligned} \tag{3.30}$$

which immediately yields

$$\begin{aligned}
&\langle(\mu F - \gamma V)p + \tfrac{1}{\gamma_k}(p - u), x_{k+1} - p\rangle\\
&\quad\le \tfrac{1}{2\alpha_k\gamma_k}(\|x_k - p\|^2 - \|x_{k+1} - p\|^2) + \tfrac{1}{\gamma_k}\|(\gamma V - \mu F)p\|(\|v_k - x_k\| + \|x_k - x_{k+1}\|)\\
&\qquad + \tfrac{\bar{\epsilon}_k}{2\alpha_k}(2\|z_k - p\| + \bar{\epsilon}_k)\\
&\quad\le \tfrac{\|x_k - x_{k+1}\|}{2\alpha_k\gamma_k}(\|x_k - p\| + \|x_{k+1} - p\|) + \tfrac{1}{\gamma_k}\|(\gamma V - \mu F)p\|(\|v_k - x_k\| + \|x_k - x_{k+1}\|)\\
&\qquad + \tfrac{\bar{\epsilon}_k}{2\alpha_k}(2\|z_k - p\| + \bar{\epsilon}_k).
\end{aligned} \tag{3.31}$$

Since for any $w \in \omega_w(x_k)$ there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup w$, we deduce from (3.21), (3.31), $\frac{1}{\gamma_k} \to \frac{1}{\bar{\xi}}$, and $\|x_{k+1} - x_k\| + \bar{\epsilon}_k = o(\alpha_k)$ that

$$\begin{aligned}\langle(\mu F-\gamma V)p+\tfrac{1}{\bar{\xi}}(p-u),w-p\rangle &= \lim_{i\to\infty}\langle(\mu F-\gamma V)p+\tfrac{1}{\gamma_{k_i}}(p-u),x_{k_i}-p\rangle\\ &\le \limsup_{k\to\infty}\langle(\mu F-\gamma V)p+\tfrac{1}{\gamma_k}(p-u),x_k-p\rangle\\ &= \limsup_{k\to\infty}\langle(\mu F-\gamma V)p+\tfrac{1}{\gamma_k}(p-u),x_{k+1}-p\rangle\\ &\le \limsup_{k\to\infty}\tfrac{\|x_k-x_{k+1}\|}{2\alpha_k\gamma_k}(\|x_k-p\|+\|x_{k+1}-p\|)\\ &\quad+\limsup_{k\to\infty}\tfrac{1}{\gamma_k}\|(\gamma V-\mu F)p\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\ &\quad+\limsup_{k\to\infty}\tfrac{\bar{\epsilon}_k}{2\alpha_k}(2\|z_k-p\|+\bar{\epsilon}_k)\\ &= 0.\end{aligned}$$

So, it follows that

$$\langle(\mu F-\gamma V)p+\frac{1}{\bar{\xi}}(p-u),p-w\rangle\ge 0 \quad \forall p\in \mathrm{VI}(\mathrm{VI}(C,B),A).$$

Since $w\in\omega_w(x_k)\subset \mathrm{VI}(\mathrm{VI}(C,B),A)$, by Minty's lemma [7] we have

$$\langle(\mu F-\gamma V)w+\frac{1}{\bar{\xi}}(w-u),p-w\rangle\ge 0 \quad \forall p\in \mathrm{VI}(\mathrm{VI}(\Omega,B),A);$$

that is, w is a solution of VIP (3.28). Utilizing the uniqueness of solutions of VIP (3.28), we get $w=x^*$, which hence implies that $\omega_w(x_k)=\{x^*\}$. Therefore, it is known that $\{x_k\}$ converges weakly to the unique solution $x^*\in \mathrm{VI}(\mathrm{VI}(C,B),A)$ of VIP (3.28).

Finally, let us show that $\|x_k-x^*\|\to 0$ as $k\to\infty$. Indeed, utilizing (3.30) with $p=x^*$, we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le (1-\alpha_k\gamma_k\tfrac{\tau^2-(\gamma l)^2}{\tau})\|x_k-x^*\|^2+2\alpha_k\|(\gamma V-\mu F)x^*\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\ &\quad+\gamma_k\bar{\epsilon}_k(2\|z_k-x^*\|+\bar{\epsilon}_k)+2\alpha_k\gamma_k\langle(\gamma V-\mu F)x^*+\tfrac{1}{\gamma_k}(u-x^*),x_{k+1}-x^*\rangle\\ &= (1-\alpha_k\gamma_k\tfrac{\tau^2-(\gamma l)^2}{\tau})\|x_k-x^*\|^2+(\alpha_k\gamma_k\tfrac{\tau^2-(\gamma l)^2}{\tau})\\ &\quad\times\tfrac{\tau}{\tau^2-(\gamma l)^2}[\tfrac{2}{\gamma_k}\|(\gamma V-\mu F)x^*\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\ &\quad+\tfrac{\bar{\epsilon}_k}{\alpha_k}(2\|z_k-x^*\|+\bar{\epsilon}_k)+2\langle(\gamma V-\mu F)x^*+\tfrac{1}{\gamma_k}(u-x^*),x_{k+1}-x^*\rangle].\end{aligned} \tag{3.32}$$

Since $\alpha_k\to 0$, $\alpha_k+\beta_k+\gamma_k=1$, $\sum_{k=0}^{\infty}\alpha_k=\infty$, $\beta_k\to\xi\in(0,\frac{1}{2}]$, $\bar{\epsilon}_k=o(\alpha_k)$, and $x_k\rightharpoonup x^*$, from (3.15) and (3.21) we conclude that $\sum_{k=0}^{\infty}\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau}=\infty$ and

$$\begin{aligned}\limsup_{k\to\infty}\{&\tfrac{\tau}{\tau^2-(\gamma l)^2}[\tfrac{2}{\gamma_k}\|(\gamma V-\mu F)x^*\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\ &+\tfrac{\bar{\epsilon}_k}{\alpha_k}(2\|z_k-x^*\|+\bar{\epsilon}_k)+2\langle(\gamma V-\mu F)x^*+\tfrac{1}{\gamma_k}(u-x^*),x_{k+1}-x^*\rangle]\}\le 0.\end{aligned}$$

Therefore, applying Lemma 2.8 to (3.32), we obtain that $\|x_k-x^*\|\to 0$ as $k\to\infty$. Utilizing (3.23) we also obtain that $\|z_k-x^*\|\to 0$ as $k\to\infty$. This completes the proof. □

Remark 3.10. Theorem 3.9 extends, improves, supplements, and develops Anh, Kim and Muu [1, Theorem 3.1] in the following aspects.

(i) The problem of finding a solution x^* of Problem 3.1 (BVIP) in Theorem 3.9 is very different from the problem of finding a solution x^* of Problem AKM because our BVIP generalizes Anh, Kim and Muu's BVIP in [1, Theorem 3.1] from the space $\mathbf{R}^n$ to the general Hilbert space, and extends Anh, Kim and Muu's BVIP with only a solution to the setting of the BVIP with multiple solutions.

(ii) The Algorithm 2.1 in [1] is extended to develop Algorithm 3.2 by virtue of Mann's iteration method, hybrid steepest descent method and viscosity approximation method. The Algorithm 3.2 is more advantageous and more flexible than Algorithm 2.1 in [1] because it involves solving the BVIP with multiple solutions, that is, Problem 3.1.

(iii) The proof of our Theorem 3.9 is very different from the proof of Anh, Kim and Muu's Theorem 3.1 [1] because the proof of our Theorem 3.9 makes use of the nonexpansivity of the combination mapping $I - \lambda A$ for inverse-strongly monotone mapping A (see inequality (2.1)), the contraction coefficient estimate for the composite mapping S^{λ} (see Lemma 2.6), the demiclosedness principle for nonexpansive mappings (see Lemma 2.7), the convergence criteria for nonnegative real sequences (see Lemma 2.8), and Suzuki's lemma (see Lemma 3.6).

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