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A multi-dimensional functional equation having cubic forms as solutions

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Abstract

In this paper, we obtain some results on the m-variable cubic functional equation

$$\begin{aligned} &f(2x_1 + y_1, \cdots, 2x_m + y_m) + f(2x_1 - y_1, \cdots, 2x_m - y_m) \\ &\quad = 2f(x_1 + y_1, \cdots, x_m + y_m) + 2f(x_1 - y_1, \cdots, x_m - y_m) + 12f(x_1, \cdots, x_m). \end{aligned}$$

The cubic form $f(x_1, \cdots, x_m) = \sum_{1 \le i \le j \le k \le m} a_{ijk} x_i x_j x_k$ is a solution of the above functional equation.

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1. Introduction

In this paper, let X and Y be real vector spaces. A mapping f is called a *cubic form* (*homogeneous polynomial of degree* 3) if there exists $a_{ijk} \in \mathbb{R}$ $(1 \le i \le j \le k \le m)$ such that

$$f(x_1, \cdots, x_m) = \sum_{1 \le i \le j \le k \le m} a_{ijk} x_i x_j x_k \tag{1.1}$$

for all $x_1, \cdots, x_m \in X$. For a mapping $f : X^m \to Y$, consider the m-variable cubic functional equation:

$$\begin{aligned} &f(2x_1 + y_1, \cdots, 2x_m + y_m) + f(2x_1 - y_1, \cdots, 2x_m - y_m) \\ &\quad = 2f(x_1 + y_1, \cdots, x_m + y_m) + 2f(x_1 - y_1, \cdots, x_m - y_m) + 12f(x_1, \cdots, x_m). \end{aligned} \tag{1.2}$$

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When $X = Y = \mathbb{R}$, the cubic form $f : \mathbb{R}^m \to \mathbb{R}$ given by (1.1) is a solution of (1.2).

For a mapping $g : X \to Y$, consider the cubic functional equation:

$$g(2x+y) + g(2x-y) = 2g(x+y) + 2g(x-y) + 12g(x). \tag{1.3}$$

In 2002, Jun and Kim [4] solved the equation (1.3). Later, many different cubic functional equations were solved by numerous authors ([2–5]).

In 2008, the authors [1] investigated the solution and stability of (1.2) for the case $m = 2$. In this paper, we investigate the relation between (1.2) and (1.3) and some sufficient conditions that satisfy the equation (1.2), and prove the generalized Hyers-Ulam stability of (1.2).

2. Results

The m-variable cubic functional equation (1.2) induces the cubic functional equation (1.3) as follows.

Theorem 2.1. *Let $f : X^m \to Y$ be a mapping satisfying (1.2) and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, \cdots, x) \tag{2.1}$$

for all $x \in X$, then g satisfies (1.3).

Proof. By (1.2) and (2.1), we have

$$\begin{aligned} g(2x+y) + g(2x-y) &= f(2x+y, \cdots, 2x+y) + f(2x-y, \cdots, 2x-y) \\ &= 2f(x+y, \cdots, x+y) + 2f(x-y, \cdots, x-y) + 12f(x, \cdots, x) \\ &= 2g(x+y) + 2g(x-y) + 12g(x) \end{aligned}$$

for all $x, y \in X$. □

The cubic functional equation (1.3) induces the m-variable cubic functional equation (1.2) with an additional condition.

Theorem 2.2. *Let $a_{ijk} \in \mathbb{R}$ $(1 \le i \le j \le k \le m)$ and $g : X \to Y$ be a mapping satisfying (1.3). If $f : X^m \to Y$ is the mapping given by*

$$\begin{aligned} f(x_1, \cdots, x_m) := \sum_{i=1}^m a_{iii} g(x_i) + \frac{1}{24} \sum_{1 \le i < j \le m} \Big(& a_{iij}[g(2x_i + x_j) - g(2x_i - x_j) - 2g(x_j)] \\ & + a_{ijj}[g(x_i + 2x_j) + g(x_i - 2x_j) - 2g(x_i)] \Big) \\ & + \frac{1}{6} \sum_{1 \le i < j < k \le m} a_{ijk}[7g(x_i + x_j + x_k) + 2g(x_i) + 2g(x_j) + 2g(x_k) \\ & - g(2x_i + x_j + x_k) - g(x_i + 2x_j + x_k) - g(x_i + x_j + 2x_k)] \end{aligned} \tag{2.2}$$

for all $x_1, \cdots, x_m \in X$, then f satisfies (1.2).

Furthermore, (2.1) holds if $\sum_{i=1}^m a_{iii} + \sum_{1 \le i < j \le m} (a_{iij} + a_{ijj}) + \frac{1}{2} \sum_{1 \le i < j < k \le m} a_{ijk} = 1$.

Proof. By (1.3) and (2.2), we obtain

$$\begin{aligned} & f(2x_1 + y_1, \cdots, 2x_m + y_m) + f(2x_1 - y_1, \cdots, 2x_m - y_m) \\ & \quad = \sum_{i=1}^m a_{iii} \big[g(2x_i + y_i) + g(2x_i - y_i) \big] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{24} \sum_{1 \le i < j \le m} \Big(a_{iij} \big[g(4x_i + 2x_j + 2y_i + y_j) - g(4x_i - 2x_j + 2y_i - y_j) \\
&- 2g(2x_j + y_j) + g(4x_i + 2x_j - 2y_i - y_j) - g(4x_i - 2x_j - 2y_i + y_j) - 2g(2x_j - y_j) \big] \\
&+ a_{ijj} \big[g(2x_i + 4x_j + y_i + 2y_j) + g(2x_i - 4x_j + y_i - 2y_j) \\
&- 2g(2x_i + y_i) + g(2x_i + 4x_j - y_i - 2y_j) + g(2x_i - 4x_j - y_i + 2y_j) - 2g(2x_i - y_i) \big] \Big) \\
&+ \frac{1}{6} \sum_{1 \le i < j < k \le m} a_{ijk} \Big[7g(2x_i + 2x_j + 2x_k + y_i + y_j + y_k) + 2g(2x_i + y_i) \\
&+ 2g(2x_j + y_j) + 2g(2x_k + y_k) - g(4x_i + 2x_j + 2x_k + 2y_i + y_j + y_k) \\
&- g(2x_i + 4x_j + 2x_k + y_i + 2y_j + y_k) - g(2x_i + 2x_j + 4x_k + y_i + y_j + 2y_k) \\
&+ 7g(2x_i + 2x_j + 2x_k - y_i - y_j - y_k) + 2g(2x_i - y_i) \\
&+ 2g(2x_j - y_j) + 2g(2x_k - y_k) - g(4x_i + 2x_j + 2x_k - 2y_i - y_j - y_k) \\
&- g(2x_i + 4x_j + 2x_k - y_i - 2y_j - y_k) - g(2x_i + 2x_j + 4x_k - y_i - y_j - 2y_k) \Big] \\
= &\left(\sum_{i=1}^{m} a_{iii} - \frac{1}{12} \sum_{1 \le i < j \le m} a_{ijj} + \frac{1}{3} \sum_{1 \le i < j < k \le m} a_{ijk} \right) \big[g(2x_i + y_i) + g(2x_i - y_i) \big] \\
&- \left(\frac{1}{12} \sum_{1 \le i < j \le m} a_{iij} - \frac{1}{3} \sum_{1 \le i < j < k \le m} a_{ijk} \right) \big[g(2x_j + y_j) + g(2x_j - y_j) \big] \\
&+ \frac{1}{3} \sum_{1 \le i < j < k \le m} a_{ijk} \big[g(2x_k + y_k) + g(2x_k - y_k) \big] \\
&+ \frac{1}{24} \sum_{1 \le i < j \le m} \Big[a_{iij} \Big(g\big[2(2x_i + x_j) + (2y_i + y_j)\big] + g\big[2(2x_i + x_j) - (2y_i + y_j)\big] \\
&- g\big[2(2x_i - x_j) + (2y_i - y_j)\big] - g\big[2(2x_i - x_j) - (2y_i - y_j)\big] \Big) \\
&+ a_{ijj} \Big(g\big[2(x_i + 2x_j) + (y_i + 2y_j)\big] + g\big[2(x_i - 2x_j) + (y_i - 2y_j)\big] \\
&+ g\big[2(x_i + 2x_j) - (y_i + 2y_j)\big] + g\big[2(x_i - 2x_j) - (y_i - 2y_j)\big] \Big) \Big] + \frac{1}{6} \sum_{1 \le i < j < k \le m} a_{ijk} \\
&\Big(7g\big[2(x_i + x_j + x_k) + (y_i + y_j + y_k)\big] + 7g\big[2(x_i + x_j + x_k) - (y_i + y_j + y_k)\big] \\
&- g\big[2(2x_i + x_j + x_k) + (2y_i + y_j + y_k)\big] - g\big[2(2x_i + x_j + x_k) - (2y_i + y_j + y_k)\big] \\
&- g\big[2(x_i + 2x_j + x_k) + (y_i + 2y_j + y_k)\big] - g\big[2(x_i + 2x_j + x_k) - (y_i + 2y_j + y_k)\big] \\
&- g\big[2(x_i + x_j + 2x_k) + (y_i + y_j + 2y_k)\big] - g\big[2(x_i + x_j + 2x_k) - (y_i + y_j + 2y_k)\big] \Big) \\
= &\left(2\sum_{i=1}^{m} a_{iii} - \frac{1}{6} \sum_{1 \le i < j \le m} a_{ijj} + \frac{2}{3} \sum_{1 \le i < j < k \le m} a_{ijk} \right) \big[g(x_i + y_i) + g(x_i - y_i) + 6g(x_i) \big] \\
&- \left(\frac{1}{6} \sum_{1 \le i < j \le m} a_{iij} - \frac{2}{3} \sum_{1 \le i < j < k \le m} a_{ijk} \right) \big[g(x_j + y_j) + g(x_j - y_j) + 6g(x_j) \big] \\
&+ \frac{2}{3} \sum_{1 \le i < j < k \le m} a_{ijk} \big[g(x_k + y_k) + g(x_k - y_k) + 6g(x_k) \big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{12}\sum_{1\le i<j\le m}\Bigg[a_{iij}\bigg(g\big[(2x_i+x_j)+(2y_i+y_j)\big]+g\big[(2x_i+x_j)-(2y_i+y_j)\big]\\
&+6g(2x_i+x_j)-g\big[(2x_i-x_j)+(2y_i-y_j)\big]-g\big[(2x_i-x_j)-(2y_i-y_j)\big]-6g(2x_i-x_j)\bigg)\\
&+a_{ijj}\bigg(g\big[(x_i+2x_j)+(y_i+2y_j)\big]+g\big[(x_i+2x_j)-(y_i+2y_j)\big]+6g(x_i+2x_j)\\
&+g\big[(x_i-2x_j)+(y_i-2y_j)\big]+g\big[(x_i-2x_j)-(y_i-2y_j)\big]+6g(x_i-2x_j)\bigg)\Bigg]\\
&+\frac{1}{3}\sum_{1\le i<j<k\le m}a_{ijk}\bigg(7g\big[(x_i+x_j+x_k)+(y_i+y_j+y_k)\big]+7g\big[(x_i+x_j+x_k)-(y_i+y_j+y_k)\big]\\
&+42g(x_i+x_j+x_k)-g\big[(2x_i+x_j+x_k)+(2y_i+y_j+y_k)\big]\\
&-g\big[(2x_i+x_j+x_k)-(2y_i+y_j+y_k)\big]-6g(2x_i+x_j+x_k)\\
&-g\big[(x_i+2x_j+x_k)+(y_i+2y_j+y_k)\big]-g\big[(x_i+2x_j+x_k)-(y_i+2y_j+y_k)\big]\\
&-6g(x_i+2x_j+x_k)-g\big[(x_i+x_j+2x_k)+(y_i+y_j+2y_k)\big]\\
&-g\big[(x_i+x_j+2x_k)-(y_i+y_j+2y_k)\big]-6g(x_i+x_j+2x_k)\bigg)\\
=\;&2\sum_{i=1}^{m}a_{iii}[g(x_i+y_i)+g(x_i-y_i)+6g(x_i)]\\
&+\frac{1}{12}\sum_{1\le i<j\le m}\bigg(a_{iij}\Big[g(2x_i+x_j+2y_i+y_j)-g(2x_i-x_j+2y_i-y_j)\\
&-2g(x_j+y_j)+g(2x_i+x_j-2y_i-y_j)-g(2x_i-x_j-2y_i+y_j)\\
&-2g(x_j-y_j)+6g(2x_i+x_j)-6g(2x_i-x_j)-12g(x_j)\Big]\\
&+a_{ijj}\Big[g(x_i+2x_j+y_i+2y_j)+g(x_i-2x_j+y_i-2y_j)\\
&-2g(x_i+y_i)+g(x_i+2x_j-y_i-2y_j)+g(x_i-2x_j-y_i+2y_j)\\
&-2g(x_i-y_i)+6g(x_i+2x_j)+6g(x_i-2x_j)-12g(x_i)\Big]\bigg)\\
&+\frac{1}{3}\sum_{1\le i<j<k\le m}a_{ijk}[7g(x_i+x_j+x_k+y_i+y_j+y_k)+2g(x_i+y_i)+2g(x_j+y_j)+2g(x_k+y_k)\\
&-g(2x_i+x_j+x_k+2y_i+y_j+y_k)-g(x_i+2x_j+x_k+y_i+2y_j+y_k)\\
&-g(x_i+x_j+2x_k+y_i+y_j+2y_k)+7g(x_i+x_j+x_k-y_i-y_j-y_k)\\
&+2g(x_i-y_i)+2g(x_j-y_j)+2g(x_k-y_k)\\
&-g(2x_i+x_j+x_k-2y_i-y_j-y_k)-g(x_i+2x_j+x_k-y_i-2y_j-y_k)\\
&-g(x_i+x_j+2x_k-y_i-y_j-2y_k)+42g(x_i+x_j+x_k)+12g(x_i)\\
&+12g(x_j)+12g(x_k)-6g(2x_i+x_j+x_k)-6g(x_i+2x_j+x_k)-6g(x_i+x_j+2x_k)]\\
=\;&2f(x_1+y_1,\cdots,x_m+y_m)+2f(x_1-y_1,\cdots,x_m-y_m)+12f(x_1,\cdots,x_m)
\end{aligned}
$$

for all $x_1,\cdots,x_m,y_1,\cdots,y_m\in X$. Letting $x=y=0$ in (1.3), we get $g(0)=0$. Putting $y=0$ in (1.3), we obtain $g(2x)=8g(x)$ for all $x\in X$. And putting $y=x$ in (1.3), we infer $g(3x)=27g(x)$ for all $x\in X$. Setting $x=0$ in (1.3), we have that g is an odd mapping. If $\sum_{i=1}^{m}a_{iii}+\sum_{1\le i<j\le m}(a_{iij}+a_{ijj})+\frac{1}{2}\sum_{1\le i<j<k\le m}a_{ijk}=$

1, by (2.2), we find

$$\begin{aligned} f(x,\cdots,x) &= \sum_{i=1}^{m} a_{iii}g(x) + \frac{1}{24}\sum_{1\le i<j\le m}\Big(a_{iij}[g(3x)-3g(x)] + a_{ijj}[g(3x)+g(-x)-2g(x)]\Big) \\ &\quad + \frac{1}{6}\sum_{1\le i<j<k\le m} a_{ijk}[7g(3x)+6g(x)-3g(4x)] \\ &= \sum_{i=1}^{m} a_{iii}g(x) + \sum_{1\le i<j\le m}\big[a_{iij}g(x)+a_{ijj}g(x)\big] + \frac{1}{2}\sum_{1\le i<j<k\le m} a_{ijk}g(x) \\ &= \Big[\sum_{i=1}^{m} a_{iii} + \sum_{1\le i<j\le m}(a_{iij}+a_{ijj}) + \frac{1}{2}\sum_{1\le i<j<k\le m} a_{ijk}\Big]g(x) \\ &= g(x) \end{aligned}$$

for all $x \in X$. □

A mapping $S : X^3 \to Y$ is called *symmetric* if

$$S(x,y,z) = S(x,z,y) = S(y,x,z) = S(y,z,x) = S(z,x,y) = S(z,y,x)$$

for all $x, y, z \in X$.

In the following theorem, we find out some sufficient conditions that satisfy the equation (1.2).

Theorem 2.3. *A mapping $f : X^m \to Y$ satisfies (1.2) if there exist symmetric multi-additive mappings $S_1,\cdots,S_m : X^3 \to Y$ and multi-additive mappings $L_{ij}, N_{ij} : X^3 \to Y (1 \le i < j \le m)$, $M_{ijk} : X^3 \to Y (1 \le i < j < k \le m)$ such that*

$$f(x_1,\cdots,x_m) = \sum_{i=1}^{m} S_i(x_i,x_i,x_i) + \sum_{1\le i<j\le m}[L_{ij}(x_i,x_i,x_j) + N_{ij}(x_i,x_j,x_j)] + \sum_{1\le i<j<k\le m} M_{ijk}(x_i,x_j,x_k),$$

$L_{ij}(x,y,z) = L_{ij}(y,x,z)$ and $N_{ij}(x,y,z) = N_{ij}(x,z,y)$ for all $x,y,z,x_1,\cdots,x_m \in X$.

Proof. We assume that there exist symmetric multi-additive mappings $S_1,\cdots,S_m : X^3 \to Y$ and multi-additive mappings $L_{ij}, N_{ij} : X^3 \to Y (1 \le i < j \le m)$, $M_{ijk} : X^3 \to Y (1 \le i < j < k \le m)$ such that

$$f(x_1,\cdots,x_m) = \sum_{i=1}^{m} S_i(x_i,x_i,x_i) + \sum_{1\le i<j\le m}[L_{ij}(x_j,x_i,x_i) + N_{ij}(x_i,x_j,x_j)] + \sum_{1\le i<j<k\le m} M_{ijk}(x_i,x_j,x_k),$$

$L_{ij}(x,y,z) = L_{ij}(y,x,z)$ and $N_{ij}(x,y,z) = N_{ij}(x,z,y)$ for all $x,y,z,x_1,\cdots,x_m \in X$. Since M_{ij} $(1 \le i < j \le m)$ are multi-additive and $S_1,\cdots,S_m$ are symmetric multi-additive,

$$\begin{aligned} & f(2x_1+y_1,\cdots,2x_m+y_m) + f(2x_1-y_1,\cdots,2x_m-y_m) \\ &= \sum_{i=1}^{m} S_i(2x_i+y_i,2x_i+y_i,2x_i+y_i) \\ &\quad + \sum_{1\le i<j\le m}[L_{ij}(2x_j+y_j,2x_i+y_i,2x_i+y_i) + N_{ij}(2x_i+y_i,2x_j+y_j,2x_j+y_j)] \\ &\quad + \sum_{1\le i<j<k\le m} M_{ijk}(2x_i+y_i,2x_j+y_j,2x_k+y_k) + \sum_{i=1}^{m} S_i(2x_i-y_i,2x_i-y_i,2x_i-y_i) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{1\le i<j\le m}[L_{ij}(2x_j-y_j,2x_i-y_i,2x_i-y_i)+N_{ij}(2x_i-y_i,2x_j-y_j,2x_j-y_j)]\\
&+\sum_{1\le i<j<k\le m}M_{ijk}(2x_i-y_i,2x_j-y_j,2x_k-y_k)\\
=&\sum_{i=1}^{m}[S_i(2x_i+y_i,2x_i+y_i,2x_i+y_i)+S_i(2x_i-y_i,2x_i-y_i,2x_i-y_i)]\\
&+4\sum_{1\le i<j\le m}[4L_{ij}(x_j,x_i,x_i)+L_{ij}(x_j,y_i,y_i)+L_{ij}(y_j,x_i,y_i)+L_{ij}(y_j,y_i,x_i)\\
&+4N_{ij}(x_i,x_j,x_j)+N_{ij}(x_i,y_j,y_j)+N_{ij}(y_i,x_j,y_j)+N_{ij}(y_i,y_j,x_j)]\\
&+4\sum_{1\le i<j<k\le m}[4M_{ijk}(x_i,x_j,x_k)+M_{ijk}(x_i,y_j,y_k)+M_{ijk}(y_i,x_j,y_k)+M_{ijk}(y_i,y_j,x_k)]\\
=&\,2\Big(\sum_{i=1}^{m}S_i(x_i+y_i,x_i+y_i,x_i+y_i)+\sum_{1\le i<j\le m}[L_{ij}(x_j+y_j,x_i+y_i,x_i+y_i)\\
&+N_{ij}(x_i+y_i,x_j+y_j,x_j+y_j)]+\sum_{1\le i<j<k\le m}M_{ijk}(x_i+y_i,x_j+y_j,x_k+y_k)\Big)\\
&+2\Big(\sum_{i=1}^{m}S_i(x_i-y_i,x_i-y_i,x_i-y_i)+\sum_{1\le i<j\le m}[L_{ij}(x_j-y_j,x_i-y_i,x_i-y_i)\\
&+N_{ij}(x_i-y_i,x_j-y_j,x_j-y_j)]+\sum_{1\le i<j<k\le m}M_{ijk}(x_i-y_i,x_j-y_j,x_k-y_k)\Big)\\
&+12\Big(\sum_{i=1}^{m}S_i(x_i,x_i,x_i)+\sum_{1\le i<j\le m}[L_{ij}(x_j,x_i,x_i)+N_{ij}(x_i,x_j,x_j)]+\sum_{1\le i<j<k\le m}M_{ijk}(x_i,x_j,x_k)\Big)\\
=&\,2f(x_1+y_1,\cdots,x_m+y_m)+2f(x_1-y_1,\cdots,x_m-y_m)+12f(x_1,\cdots,x_m)
\end{aligned}
$$

for all $x_1,\cdots,x_m,y_1,\cdots,y_m\in X$. □

From now on, let Y be complete and let $\varphi : X^{2m}\to[0,\infty)$ be a function satisfying

$$\tilde{\varphi}(x_1,\cdots,x_m,y_1,\cdots,y_m):=\sum_{j=0}^{\infty}\frac{1}{2\cdot 8^{j+1}}\varphi(2^jx_1,\cdots,2^jx_m,2^jy_1,\cdots,2^jy_m)<\infty \tag{2.3}$$

for all $x_1,\cdots,x_m,y_1,\cdots,y_m\in X$.

Theorem 2.4. *Let $f : X^m\to Y$ be a mapping such that*

$$
\begin{aligned}
&\|f(2x_1+y_1,\cdots,2x_m+y_m)+f(2x_1-y_1,\cdots,2x_m-y_m)\\
&\quad-2f(x_1+y_1,\cdots,x_m+y_m)-2f(x_1-y_1,\cdots,x_m-y_m)-12f(x_1,\cdots,x_m)\|\\
&\le\varphi(x_1,\cdots,x_m,y_1,\cdots,y_m)
\end{aligned}\tag{2.4}
$$

for all $x_1,\cdots,x_m,y_1,\cdots,y_m\in X$. Then there exists a unique mapping $F : X^m\to Y$ satisfying (1.2) such that

$$\|f(x_1,\cdots,x_m)-F(x_1,\cdots,x_m)\|\le\tilde{\varphi}(x_1,\cdots,x_m,0,\cdots,0) \tag{2.5}$$

for all $x_1,\cdots,x_m\in X$, where the mapping F is given by

$$F(x_1,\cdots,x_m):=\lim_{j\to\infty}\frac{1}{8^j}f(2^jx_1,\cdots,2^jx_m)$$

for all $x_1,\cdots,x_m\in X$.

Proof. Letting $y_1 = \cdots = y_m = 0$ in (2.4), we have

$$\left\| f(x_1, \cdots, x_m) - \frac{1}{8} f(2x_1, \cdots, 2x_m) \right\| \le \frac{1}{16} \varphi(x_1, \cdots, x_m, 0, \cdots, 0)$$

for all $x_1, \cdots, x_m \in X$. Thus we obtain

$$\left\| \frac{1}{8^j} f(2^j x_1, \cdots, 2^j x_m) - \frac{1}{8^{j+1}} f(2^{j+1} x_1, \cdots, 2^{j+1} x_m) \right\| \le \frac{1}{2 \cdot 8^{j+1}} \varphi(2^j x_1, \cdots, 2^j x_m, 0, \cdots, 0)$$

for all $x_1, \cdots, x_m \in X$ and all j. For given integers k, l $(0 \le k < l)$, we get

$$\left\| \frac{1}{8^k} f(2^k x_1, \cdots, 2^k x_m) - \frac{1}{8^l} f(2^l x_1, \cdots, 2^l x_m) \right\| \le \sum_{j=k}^{l-1} \frac{1}{2 \cdot 8^{j+1}} \varphi(2^j x_1, \cdots, 2^j x_m, 0, \cdots, 0) \tag{2.6}$$

for all $x_1, \cdots, x_m \in X$. By (2.6), the sequence $\{\frac{1}{8^j} f(2^j x_1, \cdots, 2^j x_m)\}$ is a Cauchy sequence for all $x_1, \cdots, x_m \in X$. Since Y is complete, the sequence $\{\frac{1}{8^j} f(2^j x_1, \cdots, 2^j x_m)\}$ converges for all $x_1, \cdots, x_m \in X$. Define $F : X^m \to Y$ by

$$F(x_1, \cdots, x_m) := \lim_{j \to \infty} \frac{1}{8^j} f(2^j x_1, \cdots, 2^j x_m)$$

for all $x_1, \cdots, x_m \in X$. By (2.4), we have

$$\begin{aligned} &\Bigg\| \frac{1}{8^j} \Big[f\big(2^j(2x_1 + y_1), \cdots, 2^j(2x_m + y_m)\big) + f\big(2^j(2x_1 - y_1), \cdots, 2^j(2x_m - y_m)\big) \Big] \\ &\quad - \frac{2}{8^j} \Big[f\big(2^j(x_1 + y_1), \cdots, 2^j(x_m + y_m)\big) - f\big(2^j(x_1 - y_1), \cdots, 2^j(x_m - y_m)\big) \Big] + 6f(x_1, \cdots, x_m) \Bigg\| \\ &\quad \le \frac{1}{8^j} \varphi(2^j x_1, \cdots, 2^j x_m, 2^j y_1, \cdots, 2^j y_m) \end{aligned}$$

for all $x_1, \cdots, x_m, y_1, \cdots, y_m \in X$ and all j. Letting $j \to \infty$ and using (2.3), we see that F satisfies (1.2). Setting $k = 0$ and taking $l \to \infty$ in (2.6), one can obtain the inequality (2.5). If $G : X^m \to Y$ is another mapping satisfying (1.2) and (2.5), we obtain

$$\begin{aligned} \|F(x_1, \cdots, x_m) - G(x_1, \cdots, x_m)\| &= \frac{1}{8^j} \|F(2^j x_1, \cdots, 2^j x_m) - G(2^j x_1, \cdots, 2^j x_m)\| \\ &\le \frac{1}{8^j} \|F(2^j x_1, \cdots, 2^j x_m) - f(2^j x_1, \cdots, 2^j x_m)\| \\ &\quad + \frac{1}{8^j} \|f(2^j x_1, \cdots, 2^j x_m) - G(2^j x_1, \cdots, 2^j x_m)\| \\ &\le \frac{2}{8^j} \tilde{\varphi}(2^j x_1, \cdots, 2^j x_m, 0, \cdots, 0) \to 0 \quad \text{as} \quad j \to \infty \end{aligned}$$

for all $x_1, \cdots, x_m \in X$. Hence the mapping F is the unique mapping satisfying (1.2), as desired. □

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