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Stability and bifurcation analysis of a discrete predator-prey model with Holling type III functional response

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Abstract

The paper studies the dynamical behaviors of a discrete predator-prey system with Holling type III functional response. More precisely, we investigate the local stability of equilibriums, flip bifurcation and Neimark-Sacker bifurcation of the model by using the center manifold theorem and the bifurcation theory. And analyze the dynamic characteristics of the system in two-dimensional parameter-spaces, one can observe the "cluster" phenomenon. Numerical simulations not only illustrate our results, but also exhibit the complex dynamical behaviors of the model. The results show that we can more clearly and directly observe the chaotic phenomenon, period-adding and Neimark-Sacker bifurcation from two-dimensional parameter-spaces and the optimal parameters matching interval can also be found easily.

Keywords: Predator-prey system, stability, flip bifurcation, Neimark-Sacker bifurcation, cluster, parameter-space.
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1. Introduction

The predator-prey models play an important and fundamental role among the relationships of the biological populations, which causes great attention in ecological and biological fields. In the biology field,

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the predator-prey model is widely applied to deal with the problem in real world. The predator-prey model can also show the relationships between two predator-prey species. The size of the population is not only influenced by the species competition and predation but also influenced by the parasitic infection. There are many results on the nonlinear dynamics of predator-prey models with harvesting, such as permanence, extinction, stability of equilibrium, Hopf bifurcation, limit cycle, chaotic behavior, and so on. There are many scholars who investigated the nonlinear dynamic characteristics of continuous system, however, the research about discrete systems is relatively rare. Compared with the continuous system, the discrete system posses its unique dynamic characteristics. And in the real life, many practical problems can be depicted by the discrete systems. At the same time, in order to calculate the exact solution which is hard to obtain of the equation, we can also discrete the continuous systems to obtain numerical solution by the difference methods. Therefore, the study of discrete system achieved great development among mathematics, physics and engineering. And due to many infectious diseases' data is collected to day, week, month and year. Hence, we can establish the discrete model to show the infectious diseases' character. At the same time, the discrete model is more tied to the life than the continuous model. Chen et al. [3] applied the forward Euler method to the ratio-dependent predator-prey model, and then investigated the dynamical behaviors of discrete system by using the center manifold theorem. Zhang et al. [26] investigated the dynamical behaviors of the discrete-time predator-prey biological economic system by using new normal form of differential-algebraic system. Wang and Li [24] revisited a discrete predator-prey model and proposed a very meaningful lemma which can be used to study the system's stability and bifurcation. Elabbasy et al. [5] derived the existence and stability of the fixed points of a discrete reduced Lorenz system by using the center manifold theorem and bifurcation theory. Ghaziani et al. [6] studied the resonance and bifurcation in a discrete-time predator-prey system with Holling functional response. Zhao et al. [27] focused on a reaction-diffusion neural network with delays and studied the stability and bifurcation of the networks.

It is well-known that the single-species discrete Logistic models possess a lot of rich dynamic behaviors [16, 21]. Sangapate [23] presented new sufficient conditions for the asymptotic stability of discrete time-delay systems. Li [17] studied the bifurcation of a prey-predator system with sex-structure and sexual favoritism. Hu and Cao [12] studied the bifurcation and chaos in a discrete-time predator-prey system of Holling and Leslie type, and derived the existence conditions of flip bifurcation and Hopf bifurcation. Hu et al. [13] discussed the dynamical behaviors of a discrete-time SIR epidemic model and studied the local stability of the disease-free equilibrium and endemic equilibrium. Misra et al. [19] analyzed the stability and bifurcation of a prey-predator model with age based predation, and obtained all the feasible equilibriums of the discrete system. Banerjee et al. [1] presented a delay differential equation model of immunotherapy for tumor-immune response, and estimated the length of delay to preserve the stability of an equilibrium state of biological significance. Jana [14] presented the existence conditions of stability, flip bifurcation and Neimark-Sacker bifurcation, and found very rich dynamic characteristics through theoretical and numerical analysis. The predator-prey system has a theoretical and practical significance. And in recent years, the study of dynamic behavior of predator-prey models has attracted much attention of many scholars [4, 7, 8, 10, 11, 15, 18]. Qiu et al. [20] discussed equilibrium and limit cycle of an autonomy predator-prey model by using qualitative stability theory of differential equation, and when positive equilibrium is unstable, the sufficient conditions of the existence and uniqueness of limit cycle is obtained. In this paper, we firstly apply the forward Euler scheme to the autonomy predator-prey model to get the discrete predator-prey model. Then, we investigate the local stability of equilibriums, flip bifurcation and Neimark-Sacker bifurcation of the model by using the bifurcation theory [9, 22, 25] and the center manifold theorem [2, 22, 25]. And we also analyze the dynamic characteristics of the discrete predator-prey model in two-dimensional parameter-spaces, the numerical results show that the model exist many very interesting dynamic characteristics.

The paper is organized as following. We obtain a discrete predator-prey system with Holling type III functional response system in Section 2. In Section 3, we investigate the existence and stability of the fixed points. We discuss the flip bifurcation and Neimark-Sacker bifurcation of (x_0, y_0) in Section 4. In Section 5, we give some numerical simulations, which not only illustrate our results with the theoretical analysis, but also exhibit the complex dynamical behaviors such as orbits with period 5, 10, cascades of period-doubling bifurcation in orbits with period 2, 4, 8, 16, quasi-periodic orbits and chaotic sets. We present several panels displaying parameter-space plots with respect to different pairs of parameters in Section 6. In Section 7, we give a conclusion.

2. Model formulation

Qiu et al. [20] proposed a continuous-time predator-prey model

$$\begin{cases} \dfrac{dx}{dt} = x(a - bx^3) - \dfrac{\alpha x^2 y}{1+\omega x^2}, \\ \dfrac{dy}{dt} = -d_1 y + \dfrac{k\alpha x^2 y}{1+\omega x^2}, \end{cases} \tag{2.1}$$

where x and y denote the prey and predator densities, respectively, and $a, b, \alpha, \omega, d_1, k$ are positive constants.

For the convenience, we substitute $t = \frac{\tau}{b}, a_1^3 = \frac{a}{b}, a_2 = \frac{d_1}{b}, a_3 = \frac{k\alpha}{b}, m = \frac{\alpha}{b},\ \bar{y} = my,\ \bar{y} = y$ into system (2.1). Then we obtain the following system

$$\begin{cases} \dfrac{dx}{dt} = x(a_1^3 - x^3) - \dfrac{x^2 y}{1+\omega x^2}, \\ \dfrac{dy}{dt} = y(-a_2 + \dfrac{a_3 x^2}{1+\omega x^2}), \end{cases} \tag{2.2}$$

Applying the forward Euler scheme to system (2.2), we can get the discrete predator-prey system as follows:

$$\begin{cases} x \to x + \delta x(a_1^3 - x^3) - \dfrac{\delta x^2 y}{1+\omega x^2}, \\ y \to y + \delta y(-a_2 + \dfrac{a_3 x^2}{1+\omega x^2}), \end{cases} \tag{2.3}$$

where δ is the step size, $a_1, a_2, a_3, \omega > 0$. The purpose of this paper is to investigate the stability of the equilibriums and bifurcation of system (2.3) in $R_+ = \{(x,y)|x > 0, y > 0\}$.

3. The existence and stability of fixed points

Obviously, the fixed points of system (2.3) are satisfied the following equations:

$$\begin{cases} x = x + \delta x(a_1^3 - x^3) - \dfrac{\delta x^2 y}{1+\omega x^2}, \\ y = y + \delta y(-a_2 + \dfrac{a_3 x^2}{1+\omega x^2}). \end{cases} \tag{3.1}$$

Without loss of generality, we assume that $\omega = 1$ and the system (3.1) is reduced to:

$$\begin{cases} x = x + \delta x(a_1^3 - x^3) - \dfrac{\delta x^2 y}{1+x^2}, \\ y = y + \delta y(-a_2 + \dfrac{a_3 x^2}{1+x^2}), \end{cases} \tag{3.2}$$

Lemma 3.1.

(i) *The system* (3.2) *has two fixed points* $(0,0)$ *and* $(a_1, 0)$ *for all parameters.*

(ii) *The system* (3.2) *has the fixed point* (x_0, y_0) *if and only if* $a_3 a_1^2 > a_2(1 + a_1^2)$, *where* x_0, y_0 *satisfy*

$$\begin{cases} a_1^3 - x_0^3 = \dfrac{x_0 y_0}{1+x_0^2}, \\ a_2 = \dfrac{a_3 x_0^2}{1+x_0^2}. \end{cases}$$

Next, we will study the stability of the fixed points. Note that the local stability of a fixed point (x, y) is determined by the modules of eigenvalues of the characteristic equation at the fixed point.

The Jacobian matrix J of the system (3.2) at fixed point (x, y) is given by

$$J(x,y) = \begin{pmatrix} 1+\delta(a_1^3-4x^3)-\frac{2\delta xy}{(1+x^2)^2} & -\frac{\delta x^2}{1+x^2} \\ \frac{2\delta a_3 xy}{(1+x^2)^2} & 1+\delta(-a_2+\frac{a_3x^2}{1+x^2}) \end{pmatrix},$$

and we have

$$J(0,0) = \begin{pmatrix} 1+\delta a_1^3 & 0 \\ 0 & 1-\delta a_2 \end{pmatrix},$$

$$J(a_1,0) = \begin{pmatrix} 1-3\delta a_1^3 & -\frac{\delta a_1^2}{1+a_1^2} \\ 0 & 1+\delta(-a_2+\frac{a_3a_1^2}{1+a_1^2}) \end{pmatrix},$$

$$J(x,y) = \begin{pmatrix} 1+\delta(a_1^3-4x_0^3)-\frac{2\delta x_0y_0}{(1+x_0^2)^2} & -\frac{\delta x_0^2}{1+x_0^2} \\ \frac{2\delta a_3 x_0y_0}{(1+x_0^2)^2} & 1+\delta(-a_2+\frac{a_3x_0^2}{1+x_0^2}) \end{pmatrix}.$$

For the sake of analyze the stability of fixed points of system (3.2), we first introduce a lemma as follows:

Lemma 3.2. *Let $F(\lambda) = \lambda^2 + B\lambda + C$. Suppose that $F(1) > 0$, and λ_1, λ_2 be the two roots of $F(\lambda) = 0$, then*

(i) *$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;*
(ii) *$|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;*
(iii) *$|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C < 1$;*
(iv) *$\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;*
(v) *λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.*

Let λ_1 and λ_2 be two roots of the characteristic equation of Jacobian matrix J. The fixed point (x, y) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and the sink is locally asymptotic stable. (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and the source is locally unstable. (x, y) is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). And (x, y) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

From Lemma 3.2, we can get the following results:

Proposition 3.3. *The eigenvalues of the fixed point $(0, 0)$ are $\lambda_1 = 1 + \delta a_1^3, \lambda_2 = 1 - \delta a_2$.*

(i) *the fixed point $(0, 0)$ is a saddle if $0 < \delta a_2 < 2$;*
(ii) *the fixed point $(0, 0)$ is a source if $\delta a_2 > 2$;*
(iii) *the fixed point $(0, 0)$ is non-hyperbolic if $\delta a_2 = 2$.*

Proposition 3.4. *The eigenvalues of the fixed point $(a_1, 0)$ are $\lambda_1 = 1 - 3\delta a_1^3, \lambda_2 = 1 + \delta(-a_2 + \frac{a_3a_1^2}{1+a_1^2})$.*

(i) *the fixed point $(a_1, 0)$ is a saddle if $0 < \delta a_1^3 < \frac{2}{3}$;*
(ii) *the fixed point $(a_1, 0)$ is a source if $\delta a_1^3 > \frac{2}{3}$;*
(iii) *the fixed point $(a_1, 0)$is non-hyperbolic if $\delta a_1^3 = \frac{2}{3}$.*

The characteristic equation of the Jacobian matrix J of the system (3.2) evaluated at (x_0, y_0) can be written as

$$\lambda^2 + p(x_0, y_0)\lambda + q(x_0, y_0) = 0, \tag{3.3}$$

where

$$p(x_0, y_0) = -2 - (-a_2 + a_1^3 - 4x_0^3 + \frac{a_3x_0^2}{1+x_0^2} - \frac{2x_0y_0}{(1+x_0^2)^2})\delta,$$

$$q(x_0,y_0)=1+(-a_2+a_1^3-4x_0^3+\frac{a_3x_0^2}{1+x_0^2}-\frac{2x_0y_0}{(1+x_0^2)^2})\delta$$
$$+(-a_1^3a_2+4a_2x_0^3+\frac{a_1^4x_0^2}{1+x_0^2}-\frac{4x_0^5}{1+x_0^2}+\frac{2a_2x_0y_0}{(1+x_0^2)^2})\delta^2.$$

Let

$$\xi=-a_2+a_1^3-4x_0^3+\frac{a_3x_0^2}{1+x_0^2}-\frac{2x_0y_0}{(1+x_0^2)^2},\quad \eta=-a_1^3a_2+4a_2x_0^3+\frac{a_1^4x_0^2}{1+x_0^2}-\frac{4x_0^5}{1+x_0^2}+\frac{2a_2x_0y_0}{(1+x_0^2)^2},$$

then Eq. (3.3) can be written as

$$\lambda^2-(2+\xi\delta)\lambda+(1+\xi\delta+\eta\delta^2)=0.$$

And let $F(\lambda)=\lambda^2-(2+\xi\delta)\lambda+(1+\xi\delta+\eta\delta^2)$, then we have

$$F(1)=\eta\delta^2>0, F(-1)=4+2\xi\delta+\eta\delta^2.$$

Proposition 3.5. *Assume that (x_0,y_0) is the positive fixed point of system (3.2):*

(i) *the positive fixed point (x_0,y_0) is a sink if one of the following conditions holds:*
 (i.1) $-2\sqrt{\eta}\le\xi<0$ *and* $0<\delta<-\frac{\xi}{\eta}$;
 (i.2) $\xi<-2\sqrt{\eta}$ *and* $0<\delta<\frac{-\xi-\sqrt{\xi^2-4\eta}}{\eta}$;

(ii) *the positive fixed point (x_0,y_0) is a source if one of the following conditions holds:*
 (ii.1) $-2\sqrt{\eta}\le\xi<0$ *and* $\delta>-\frac{\xi}{\eta}$;
 (ii.2) $\xi<-2\sqrt{\eta}$ *and* $\delta>\frac{-\xi+\sqrt{\xi^2-4\eta}}{\eta}$;
 (ii.3) $\xi\ge0$;

(iii) *the positive fixed point (x_0,y_0) is a saddle if the following condition holds:*
 $\xi<-2\sqrt{\eta}$ *and* $\frac{-\xi-\sqrt{\xi^2-4\eta}}{\eta}<\delta<\frac{-\xi+\sqrt{\xi^2-4\eta}}{\eta}$;

(iv) *the positive fixed point (x_0,y_0) is non-hyperbolic if one of the following conditions holds:*
 (iv.1) $\xi<-2\sqrt{\eta}$ *and* $\delta=\frac{-\xi\pm\sqrt{\xi^2-4\eta}}{\eta}$ *and* $\delta\ne-\frac{2}{\xi},-\frac{4}{\xi}$;
 (iv.2) $-2\sqrt{\eta}<\xi<0$ *and* $\delta=-\frac{\delta}{\eta}$.

From Lemma 3.2, we can see that if (iv.2) of Proposition 3.4 holds, then one of the eigenvalues of the positive fixed point (x_0,y_0) is -1 and the other is neither 1 nor -1.

Let

$$F_{B1}=\left\{(a_1,a_2,a_3,\delta):\delta=\frac{-\xi-\sqrt{\xi^2-4\eta}}{\eta},\xi<-2\sqrt{\eta},a_1,a_2,a_3,\delta>0\right\},$$

or

$$F_{B2}=\left\{(a_1,a_2,a_3,\delta):\delta=\frac{-\xi+\sqrt{\xi^2-4\eta}}{\eta},\xi<-2\sqrt{\eta},a_1,a_2,a_3,\delta>0\right\},$$

the flip bifurcation at fixed point (x_0,y_0) will appear if the parameters vary in the small neighborhood F_{B1} or F_{B2}.

Let

$$H_B=\left\{(a_1,a_2,a_3,\delta):\delta=-\frac{\xi}{\eta},-2\sqrt{\eta}<\xi<0,a_1,a_2,a_3,\delta>0\right\},$$

the Neimark-Sacker bifurcation at fixed point (x_0,y_0) will appear if the parameters vary in the small neighborhood H_B.

4. Flip bifurcation and Neimark-Sacker bifurcation

Next, we are choosing δ as the bifurcation parameter and study the flip bifurcation and Neimark-Sacker bifurcation. We first discuss the flip bifurcation of the system (3.2) at (x_0, y_0) when the parameters vary in the small neighborhood F_{B1}. For the case of F_{B2}, we can give a similar arguments.

Taking parameters $(a_1, a_2, a_3, \delta_1) \in F_{B1}$, we consider the following system:

$$\begin{cases} x \to x + \delta_1 x(a_1^3 - x^3) - \dfrac{\delta_1 x^2 y}{1+x^2}, \\ y \to y + \delta_1 y(-a_2 + \dfrac{a_3 x^2}{1+x^2}). \end{cases} \tag{4.1}$$

The system (4.1) has a unique positive fixed point (x_0, y_0), and its eigenvalues are $\lambda_1 = -1, \lambda_2 = 3 + \xi\delta_1$ with $|\lambda_2| \neq 1$.

Since $(a_1, a_2, a_3, \delta_1) \in F_{B1}, \delta_1 = \left(-\xi - \sqrt{\xi^2 - 4\eta}\right)\Big/\eta$. We choose δ^* as the bifurcation parameter, and consider a perturbation of (4.1) as follows:

$$\begin{cases} x \to x + (\delta_1 + \delta^*)[x(a_1^3 - x^3) - \dfrac{x^2 y}{1+x^2}], \\ y \to y + (\delta_1 + \delta^*)[y(-a_2 + \dfrac{a_3 x^2}{1+x^2})], \end{cases} \tag{4.2}$$

where $|\delta^*| << 1$.

Let $u = x - x_0, v = y - y_0$, and then (x_0, y_0) of system (4.2) can be transformed into the origin, and we have

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + a_{15}v^2 + b_1 u\delta^* + b_2 v\delta^* + e_1 u^3 \\ +e_2 u^2 v + e_3 uv^2 + e_4 v^3 + b_3 u^2\delta^* + b_4 uv\delta^* + b_5 v^2\delta^* + O((|u| + |v| + |\delta^*|)^4) \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + a_{25}v^2 + c_1 u\delta^* + c_2 v\delta^* + d_1 u^3 \\ +d_2 u^2 v + d_3 uv^2 + d_4 v^3 + c_3 u^2\delta^* + c_4 uv\delta^* + c_5 v^2\delta^* + O((|u| + |v| + |\delta^*|)^4) \end{pmatrix}, \tag{4.3}$$

where

$$\begin{aligned} &a_{11} = 1 + \delta a_1^3 - 4\delta x_0^3 - \frac{2\delta x_0 y_0}{(1+x_0^2)^2}, \quad a_{12} = -\frac{\delta x_0^2}{1+x_0^2}, \quad a_{13} = -12\delta x_0^2 - \frac{2\delta y_0(1 - 3x_0^2 - 4x_0^4)}{(1+x_0^2)^4}, \\ &a_{14} = -\frac{2\delta x_0}{(1+x_0^2)^2}, \quad a_{15} = 0, \quad b_1 = a_1^3 - 4x_0^3 - \frac{2x_0 y_0}{(1+x_0^2)^2}, \quad b_2 = -\frac{x_0^2}{1+x_0^2}, \\ &e_1 = -24\delta x_0 - \frac{4\delta x_0 y_0(7 - x_0^2 - 8x_0^4)}{(1+x_0^2)^5}, \quad e_2 = -\frac{2\delta(1 + 3x_0^2 + 4x_0^4)}{(1+x_0^2)^4}, \quad e_3 = 0, e_4 = 0, \\ &b_3 = -12x_0^2 - \frac{2y_0(1 - 3x_0^2 - 4x_0^4)}{(1+x_0^2)^4}, \quad b_4 = -\frac{2x_0}{(1+x_0^2)^2}, \quad b_5 = 0, a_{21} = \frac{2\delta a_3 x_0 y_0}{(1+x_0^2)^2}, \\ &a_{22} = 1 - \delta a_2 + \frac{\delta a_3 x_0^2}{1+x_0^2}, \quad a_{23} = \frac{2\delta a_3 y_0(1 - 2x_0^2 - 3x_0^4)}{(1+x_0^2)^4}, \quad a_{24} = \frac{2\delta a_3 x_0}{(1+x_0^2)^2}, \\ &a_{25} = 0, \quad c_1 = \frac{2a_3 x_0 y_0}{(1+x_0^2)^2}, \quad c_2 = -a_2 + \frac{a_3 x_0^2}{1+x_0^2}, \quad d_1 = -\frac{24\delta a_3 x_0 y_0(1 - x_0^4)}{(1+x_0^2)^5}, \\ &d_2 = \frac{2\delta a_3(1 - 2x_0^2 - 3x_0^4)}{(1+x_0^2)^4}, \quad d_3 = 0, \ d_4 = 0, \ c_3 = \frac{2a_3 y_0(1 - 2x_0^2 - 3x_0^4)}{(1+x_0^2)^4}, \quad c_4 = \frac{2a_3 x_0}{(1+x_0^2)^2}, \ c_5 = 0, \end{aligned} \tag{4.4}$$

and $\delta = \delta_1$.

We construct the following invertible matrix:

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix},$$

and use the following transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix},$$

then system (4.3) becomes into

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} f(X,Y,\delta) \\ g(X,Y,\delta) \end{pmatrix}, \tag{4.5}$$

where

$$\begin{aligned}
f(X,Y,\delta^*) = {} & \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]}{a_{12}(\lambda_2+1)}u^2 + \frac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}]}{a_{12}(\lambda_2+1)}uv + \frac{[b_1(\lambda_2 - a_{11}) - a_{12}c_1]}{a_{12}(\lambda_2+1)}u\delta^* \\
& + \frac{[b_2(\lambda_2 - a_{11}) - a_{12}c_2]}{a_{12}(\lambda_2+1)}v\delta^* + \frac{[e_1(\lambda_2 - a_{11}) - a_{12}d_1]}{a_{12}(\lambda_2+1)}u^3 + \frac{[e_2(\lambda_2 - a_{11}) - a_{12}d_2]}{a_{12}(\lambda_2+1)}u^2v \\
& + \frac{[b_3(\lambda_2 - a_{11}) - a_{12}c_3]}{a_{12}(\lambda_2+1)}u^2\delta^* + \frac{[b_4(\lambda_2 - a_{11}) - a_{12}c_4]}{a_{12}(\lambda_2+1)}uv\delta^* + O((|u|+|v|+|\delta^*|)^4), \\
g(X,Y,\delta^*) = {} & \frac{[a_{13}(1 + a_{11}) + a_{12}a_{23}]}{a_{12}(\lambda_2+1)}u^2 + \frac{[a_{14}(1 + a_{11}) + a_{12}a_{24}]}{a_{12}(\lambda_2+1)}uv + \frac{[b_1(1 + a_{11}) + a_{12}c_1]}{a_{12}(\lambda_2+1)}u\delta^* \\
& + \frac{[b_2(1 + a_{11}) + a_{12}c_2]}{a_{12}(\lambda_2+1)}v\delta^* + \frac{[e_1(1 + a_{11}) + a_{12}d_1]}{a_{12}(\lambda_2+1)}u^3 + \frac{[e_2(1 + a_{11}) + a_{12}d_2]}{a_{12}(\lambda_2+1)}u^2v \\
& + \frac{[b_3(1 + a_{11}) + a_{12}c_3]}{a_{12}(\lambda_2+1)}u^2\delta^* + \frac{[b_4(1 + a_{11}) + a_{12}c_4]}{a_{12}(\lambda_2+1)}uv\delta^* + O((|u|+|v|+|\delta^*|)^4),
\end{aligned}$$

and

$$\begin{aligned}
u &= a_{12}(X+Y), v = -(1+a_{11})X + (\lambda_2 - a_{11})Y, \\
uv &= a_{12}[-(1+a_{11})X^2 + (\lambda_2 - 1 - 2a_{11})XY + (\lambda_2 - a_{11})Y^2], \\
u^2 &= a_{12}^2(X^2 + XY + Y^2), \quad u^3 = a_{12}^3(X^3 + 3X^2Y + 3XY^3 + Y^3), \\
u^2v &= a_{12}^2[-(1+a_{11})X^3 + (\lambda_2 - 2 - 3a_{11})X^2Y + (2\lambda_2 - 1 - 3a_{11})XY^3 + (\lambda_2 - a_{11})Y^3].
\end{aligned}$$

By the center manifold theorem, we can obtain a center manifold $W^c(0,0,0)$, which can be approximately represented as follows:

$$W^c(0,0,0) = \left\{(X,Y,\delta*) \in R^3 : Y = a_1X^2 + a_2X\delta^* + a_3\delta^{*2} + O((|X|+|\delta^*|)^3)\right\},$$

where $O((|X|+|\delta^*|)^3)$ is a function with order at least 3, and

$$\begin{aligned}
a_1 &= \frac{a_{12}[a_{13}(1+a_{11}) + a_{12}a_{23}] + (1+a_{11})[-a_{14}(1+a_{11}) - a_{12}a_{23}]}{1-\lambda_2^2}, \\
a_2 &= \frac{(1+a_{11})[b_2(1+a_{11}) + a_{12}c_2] - a_{12}[b_1(1+a_{11}) + a_{12}c_1]}{a_{12}(1+\lambda_2)^2}, \\
a_3 &= 0.
\end{aligned}$$

Therefore, the system (4.5) which is restricted to the center manifold $W^c(0,0,0)$ has the following form:

$$F_1 : X \to -X + h_1X^2 + h_2X\delta^* + h_3X^2\delta^* + h_4X\delta^{*2} + h_5X^3 + O((|X|+|\delta^*|)^3), \tag{4.6}$$

where

$$h_1 = \frac{1}{\lambda_2+1}\{a_{12}[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}] - (1+a_{11})[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}]\},$$

$$\begin{aligned}
h_2 &= \frac{1}{a_{12}(\lambda_2+1)}\{a_{12}[b_1(\lambda_2-a_{11})-a_{12}c_1]-(1+a_{11})[b_2(\lambda_2-a_{11})-a_{12}c_2]\},\\
h_3 &= \frac{a_2}{\lambda_2+1}\{2a_{12}[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]+(\lambda_2-1-2a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]\}\\
&\quad+\frac{a_1}{a_{12}(\lambda_2+1)}\{a_{12}[b_1(\lambda_2-a_{11})-a_{12}c_1]+(\lambda_2-a_{11})[b_2(\lambda_2-a_{11})-a_{12}c_2]\}\\
&\quad+\frac{1}{\lambda_2+1}\{a_{12}[b_3(\lambda_2-a_{11})-a_{12}c_3]-(1+a_{11})[b_4(\lambda_2-a_{11})-a_{12}c_4]\},\\
h_4 &= \frac{a_2}{a_{12}(\lambda_2+1)}\{a_{12}[b_1(\lambda_2-a_{11})-a_{12}c_1]+(\lambda_2-a_{11})[b_2(\lambda_2-a_{11})-a_{12}c_2]\},\\
h_5 &= \frac{1}{\lambda_2+1}\{2a_{12}a_1[a_{13}(\lambda_2-a_{11})-a_{12}a_{23}]+a_1(\lambda_2-1-2a_{11})[a_{14}(\lambda_2-a_{11})-a_{12}a_{24}]\\
&\quad+a_{12}^2[e_1(\lambda_2-a_{11})-a_{12}d_1]-a_2(1+a_{11})[e_2(\lambda_2-a_{11})-a_{12}d_2]\}.
\end{aligned}$$

In order to undergo a flip bifurcation for system (4.6), we require that two discriminatory quantities α_1 and α_2 are not equal to zero, where

$$\alpha_1 = (\frac{\partial^2 F_1}{\partial X\partial\delta^*}+\frac{1}{2}\frac{\partial F_1}{\partial\delta^*}\frac{\partial^2 F_1}{\partial X^2})\big|_{(0,0)} = h_2, \quad \alpha_2 = (\frac{1}{6}\frac{\partial^3 F_1}{\partial X^3}+(\frac{1}{2}\frac{\partial^2 F_1}{\partial X^2})^2)\big|_{(0,0)} = h_5+h_1^2.$$

Based on the above analysis, we can get the following theorem:

Theorem 4.1. *If $\alpha_2 \neq 0$, then system (2.3) undergoes a flip bifurcation at fixed point (x_0, y_0) when δ varies in the small neighborhood of δ_1. Moreover, if $\alpha_2 > 0$ (resp., $\alpha_2 < 0$), then the period-2 orbits that bifurcate from (x_0, y_0) are stable (resp., unstable).*

Next, we study the Neimark-Sacker bifurcation of (x_0, y_0) when the parameters $(a_1, a_2, a_3, \delta_2)$ vary in the small neighborhood of H_B. Taking parameters $(a_1, a_2, a_3, \delta_2) \in H_B$, we consider the following system

$$\begin{cases} x \to x+\delta_2[x(a_1^3-x^3)-\dfrac{x^2y}{1+x^2}],\\ y \to y+\delta_2[y(-a_2+\dfrac{a_3x^2}{1+x^2})]. \end{cases} \tag{4.7}$$

The system (4.7) has a unique positive fixed point (x_0, y_0).

Since $(a_1, a_2, a_3, \delta_2) \in H_B, \delta_2 = -\xi/\eta$. We choose the parameter $\bar{\delta}^*$ as the bifurcation parameter, and consider a perturbation of system (4.7) as follows:

$$\begin{cases} x \to x+(\delta_2+\delta^*)[x(a_1^3-x^3)-\dfrac{x^2y}{1+x^2}],\\ y \to y+(\delta_2+\delta^*)[y(-a_2+\dfrac{a_3x^2}{1+x^2})], \end{cases}$$

where $|\delta^*| << 1$.

Let $u = x-x_0, v = y-y_0$, then (x_0, y_0) can be transformed into the origin, and we can get

$$\begin{pmatrix} u\\ v \end{pmatrix} \to \begin{pmatrix} a_{11}u+a_{12}v+a_{13}u^2+a_{14}uv+e_1u^3+e_2u^2v+O((|u|+|v|)^4)\\ a_{21}u+a_{22}v+a_{23}u^2+a_{24}uv+d_1u^3+d_2u^2v+O((|u|+|v|)^4) \end{pmatrix}, \tag{4.8}$$

Substituting $\delta = \delta_2+\bar{\delta}^*$ into Eq. (4.4), we can get $a_{11}, a_{12}, a_{13}, a_{14}, e_1, e_2, a_{21}, a_{22}, a_{23}, a_{24}, d_1, d_2$.

The characteristic equation of system (4.8) at $(u, v) = (0, 0)$ is as follows:

$$\lambda^2+p(\bar{\delta}^*)\lambda+q(\bar{\delta}^*) = 0,$$

where

$$p(\bar{\delta}^*) = -2 - \xi(\delta_2 + \bar{\delta}^*), \quad q(\bar{\delta}^*) = 1 + \xi(\delta_2 + \bar{\delta}^*) + \eta(\delta_2 + \bar{\delta}^*)^2.$$

Since parameters $(a_1, a_2, a_3, \delta_2) \in H_B$, the roots of the characteristic equation are

$$\lambda, \bar{\lambda} = -\frac{p(\bar{\delta}^*)}{2} \pm \frac{i}{2}\sqrt{4q(\bar{\delta}^*) - p^2(\bar{\delta}^*)} = 1 + \frac{\xi(\delta_2 + \bar{\delta}^*)}{2} \pm \frac{i(\delta_2 + \bar{\delta}^*)}{2}\sqrt{4\eta - \xi^2},$$

and we can get

$$|\lambda| = \sqrt{q(\bar{\delta}^*)}, \quad l = \frac{d\,|\lambda|}{d\bar{\delta}^*}\,|_{\bar{\delta}^*=0} = -\frac{\xi}{2} > 0.$$

In addition, it is required that $\bar{\delta}^* = 0, \lambda^m, \bar{\lambda}^m \neq 1$ $(m = 1, 2, 3, 4)$ which is equivalent to $p(0) \neq -2, 0, 1, 2$. Because $(a_1, a_2, a_3, \delta_2) \in H_B$, thus $p(0) \neq -2, 2$. We only require $p(0) \neq 0, 1$, so that

$$\xi^2 \neq 2\eta, 3\eta. \tag{4.9}$$

Therefore, the eigenvalues $\lambda, \bar{\lambda}$ of fixed point $(0, 0)$ of system (4.8) do not lay in the intersection of the unit circle with the coordinate axes when $\bar{\delta}^* = 0$ and the condition (4.9) holds.

Next, we study the normal form of system (4.8) at $\bar{\delta}^* = 0$. Let

$$\bar{\delta}^* = 0, \quad \mu = 1 + \frac{\xi\delta_2}{2}, \quad \omega = \frac{\delta_2}{2}\sqrt{4\eta - \xi^2}, \quad T = \begin{pmatrix} a_{12} & 0 \\ \mu - a_{11} & -\omega \end{pmatrix},$$

and use the following translation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix},$$

then the system (4.8) becomes into the following form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \mu & -\omega \\ \omega & -\mu \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \bar{f}(X, Y) \\ \bar{g}(X, Y) \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned}
\bar{f}(X, Y) &= \frac{a_{13}}{a_{12}}u^2 + \frac{a_{14}}{a_{12}}uv + \frac{e_1}{a_{12}}u^3 + \frac{e_2}{a_{12}}u^2v + O((|X| + |Y|)^4), \\
\bar{g}(X, Y) &= \frac{[a_{13}(\mu - a_{11}) - a_{12}a_{23}]}{a_{12}\omega}u^2 + \frac{[a_{14}(\mu - a_{11}) - a_{12}a_{24}]}{a_{12}\omega}uv \\
&\quad + \frac{[e_1(\mu - a_{11}) - a_{12}d_1]}{a_{12}\omega}u^3 + \frac{[e_2(\mu - a_{11}) - a_{12}d_2]}{a_{12}\omega}u^2v + O((|X| + |Y|)^4), \\
u^2 &= a_{12}^2X^2, \quad uv = a_{12}(\mu - a_{11})X^2 - a_{12}\omega XY, u^3 = a_{12}^3X^3, \quad u^2v = a_{12}^2(\mu - a_{11})X^3 - a_{12}^2\omega X^2Y,
\end{aligned}$$

and

$$\begin{aligned}
\bar{f}_{XX} &= 2a_{13}a_{12} + 2a_{14}(\mu - a_{11}), \quad \bar{f}_{XY} = -a_{14}\omega, \quad \bar{f}_{YY} = 0, \\
\bar{f}_{XXX} &= 6a_{12}[e_1a_{12} + e_2(\mu - a_{11})], \quad \bar{f}_{XXY} = -2a_{12}e_2\omega, \bar{f}_{XYY} = \bar{f}_{YYY} = 0, \\
\bar{g}_{XX} &= \frac{2}{\omega}\{a_{12}[a_{13}(\mu - a_{11}) - a_{12}a_{23}] + (\mu - a_{11})[a_{14}(\mu - a_{11}) - a_{12}a_{24}]\}, \\
\bar{g}_{XY} &= a_{12}a_{24} - a_{14}(\mu - a_{11}), \bar{g}_{YY} = 0, \\
\bar{g}_{XXX} &= \frac{6a_{12}}{\omega}\{a_{12}[e_1(\mu - a_{11}) - a_{12}d_1] + (\mu - a_{11})[e_2(\mu - a_{11}) - a_{12}d_2]\}, \\
\bar{g}_{XXY} &= 2a_{12}[a_{12} - 2e_2(\mu - a_{11})], \quad \bar{g}_{XYY} = \bar{g}_{YYY} = 0.
\end{aligned}$$

In order to undergo Neimark-Sacker bifurcation for system (4.10), we require that the following discriminatory quantity is not equal to zero [4, 11, 15, 18]:

$$a = [-Re(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{20}\xi_{11}) - \frac{1}{2}\left|\xi_{11}\right|^2 - \left|\xi_{02}\right|^2 + Re(\bar{\lambda}\xi_{21})]\,|_{\bar{\delta}^*=0}\ ,$$

where

$$\begin{aligned}
\xi_{20} &= \frac{1}{8}[(\bar{f}_{XX} - \bar{f}_{YY} + 2\bar{g}_{XY}) + i(\bar{g}_{XX} - \bar{g}_{YY} - 2\bar{f}_{XY})],\\
\xi_{11} &= \frac{1}{4}[(\bar{f}_{XX} + \bar{f}_{YY}) + i(\bar{g}_{XX} + \bar{g}_{YY})],\\
\xi_{02} &= \frac{1}{8}[(\bar{f}_{XX} - \bar{f}_{YY} - 2\bar{g}_{XY}) + i(\bar{g}_{XX} - \bar{g}_{YY} + 2\bar{f}_{XY})],\\
\xi_{21} &= \frac{1}{16}[(\bar{f}_{XXX} + \bar{f}_{XYY} + \bar{g}_{XXY} + \bar{g}_{YYY}) + i(\bar{g}_{XXX} + \bar{g}_{XYY} - \bar{f}_{XXY} - \bar{f}_{YYY})].
\end{aligned}$$

Based on the above analysis, we can obtain the following theorem:

Theorem 4.2. *If $a \neq 0$ and the condition (4.9) holds, then the system (2.3) undergoes Neimark-Sacker bifurcation at the fixed point (x_0, y_0) when the parameter δ varies in the small neighborhood of δ_2. Moreover, if $a < 0$ (resp., $a > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for $\delta > \delta_2$ (resp., $\delta < \delta_2$).*

5. Numerical simulations

In this section, we present the bifurcation diagrams and phase portraits of system (2.3) to confirm the above theoretical analysis and show the complex dynamical behaviors by using numerical simulations. The bifurcation parameters are considered in the following two cases:

(i) We fix $a_1 = 2, a_2 = 0.3, a_3 = 2$ and let the parameter δ vary in the range $[0.25, 0.6]$. And we can get the positive fixed point $(0.42, 22.2)$ of system (2.3). By calculating we know that, the flip bifurcation occurs at $(0.42, 22.2)$ when $\delta = 0.405$ with $\alpha_1 = -4.95, \alpha_2 = -0.19$ and $(a_1, a_2, a_3, \delta) = (2, 0.3, 2, 0.405) \in F_{B1}$. From Fig. 1 (a) and (b), we see that the fixed point $(0.42, 22.2)$ is stable for $\delta < 0.405$, and loses its stability when $\delta = 0.405$. And we can see the period 2, 4, 8, 16 appear in the range $\delta \in (0.405, 0.538)$. The phase portraits which are associated with Fig. 1 are displayed in Fig. 2.

(ii) We fix $a_1 = 0.6, a_2 = 0.3, a_3 = 2.2$ and let the parameter δ vary in the range $[3.2, 5.2]$. And now the positive fixed point of system (2.3) is $(0.397, 0.446)$. By calculating we know that the Neimark-Sacker bifurcation occurs at $(0.397, 0.446)$ when $\delta = 3.77$, and its eigenvalues are $\lambda_{\pm} = -2.08 \pm i1.8473$. For $\delta = 3.77$, we have $|\lambda_{\pm}| = 1, a = -1.4672$, $l = 0.818 > 0$, and $(a_1, a_2, a_3, \delta) = (0.6, 0.3, 2.2, 3.77) \in H_B$. From Fig. 3 (a) and (c), we observe that the fixed point $(0.397, 0.446)$ of system (2.3) is stable when $\delta < 3.77$, and loses its stability when $\delta = 3.77$, also an invariant circle appears when δ passes through 3.77. The local amplifications of Fig. 3 (a) and (c) are Fig. 3 (b) and (d), respectively. The phase portraits which are associated with Fig. 3 are displayed in Fig. 4, which clearly depicts how a smooth invariant circle bifurcates from the stable fixed point $(0.397, 0.446)$. There appears a circular curve enclosing the fixed point $(0.397, 0.446)$ when δ passes through 3.77, and its radius gradually becomes larger with the growth of δ. The circle disappears and a period-5 orbit appears when $\delta = 4.45$. There are period-5, period-10, quasi-periodic orbits and attracting chaotic sets as shown in Fig. 4.

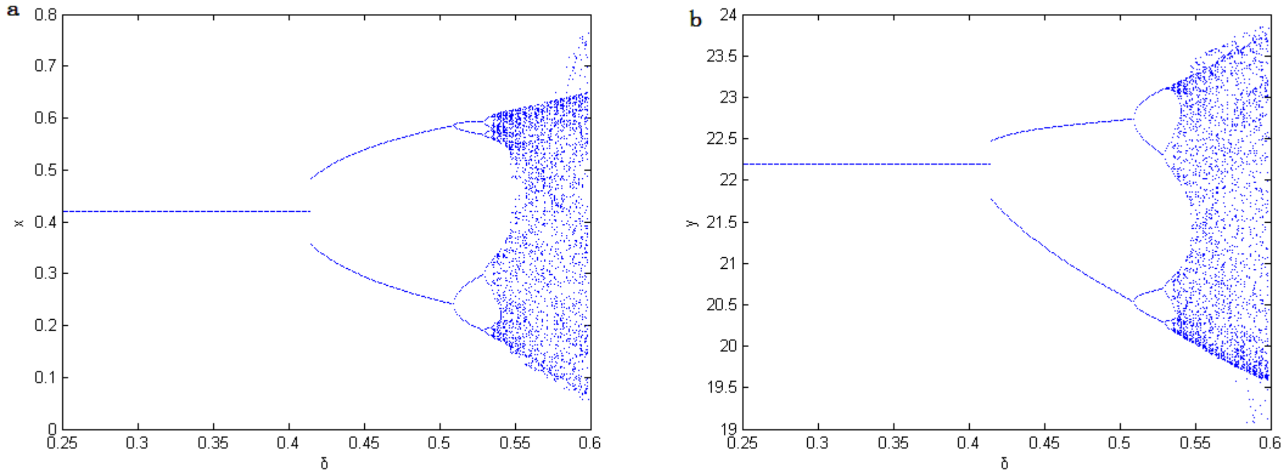

Figure 1: (a) Bifurcation diagram of system (2.3) in the (δ, x) plane for $a_1 = 2, a_2 = 0.3, a_3 = 2$, the initial value is $(0.01, 0.01)$. (b) Bifurcation diagram of system (2.3) in the (δ, y) plane.

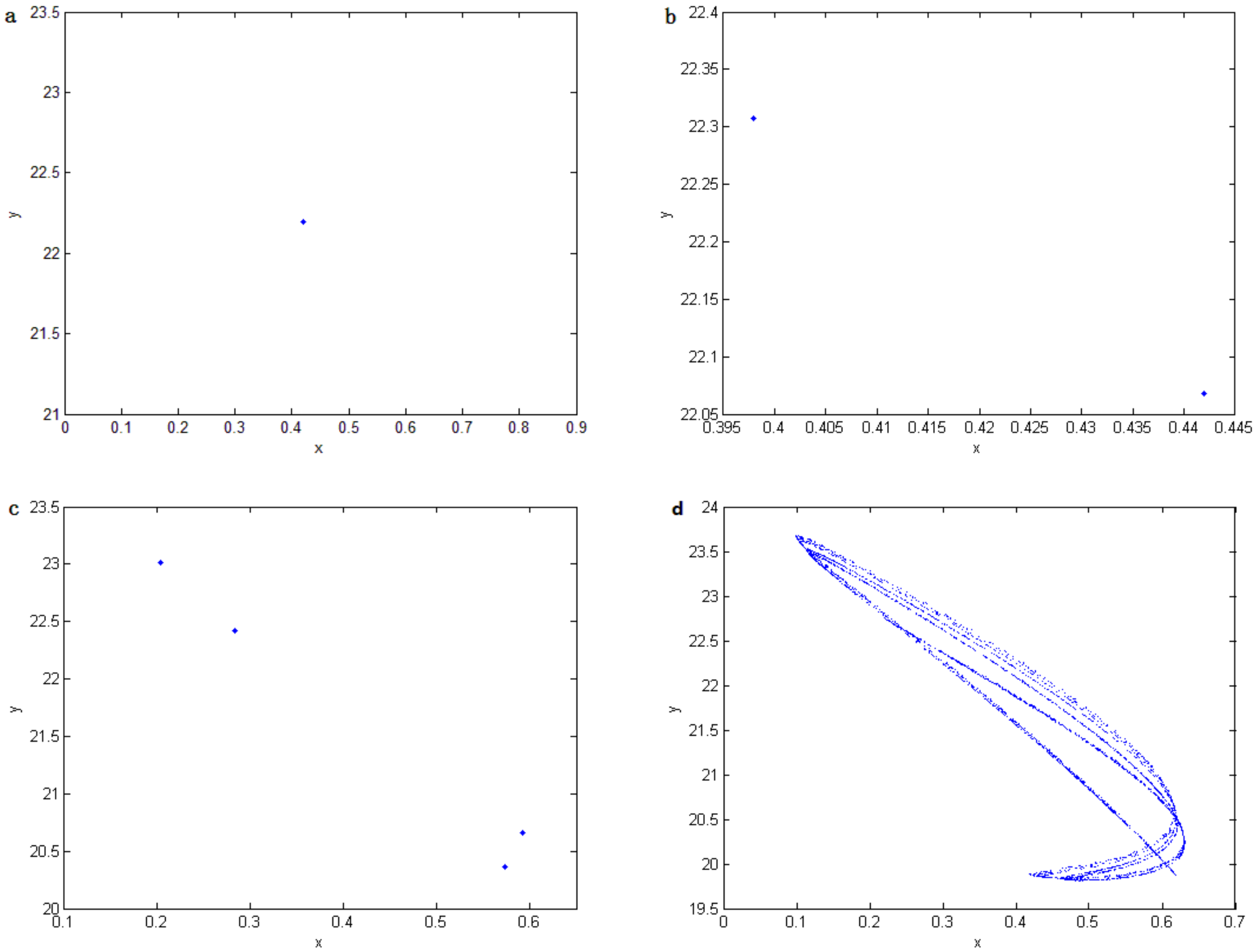

Figure 2: Phase portraits for various values of δ corresponding to Fig. 1 (a), (a) $\delta = 0.3$, (b) $\delta = 0.405$, (c) $\delta = 0.52$, (d) $\delta = 0.57$.

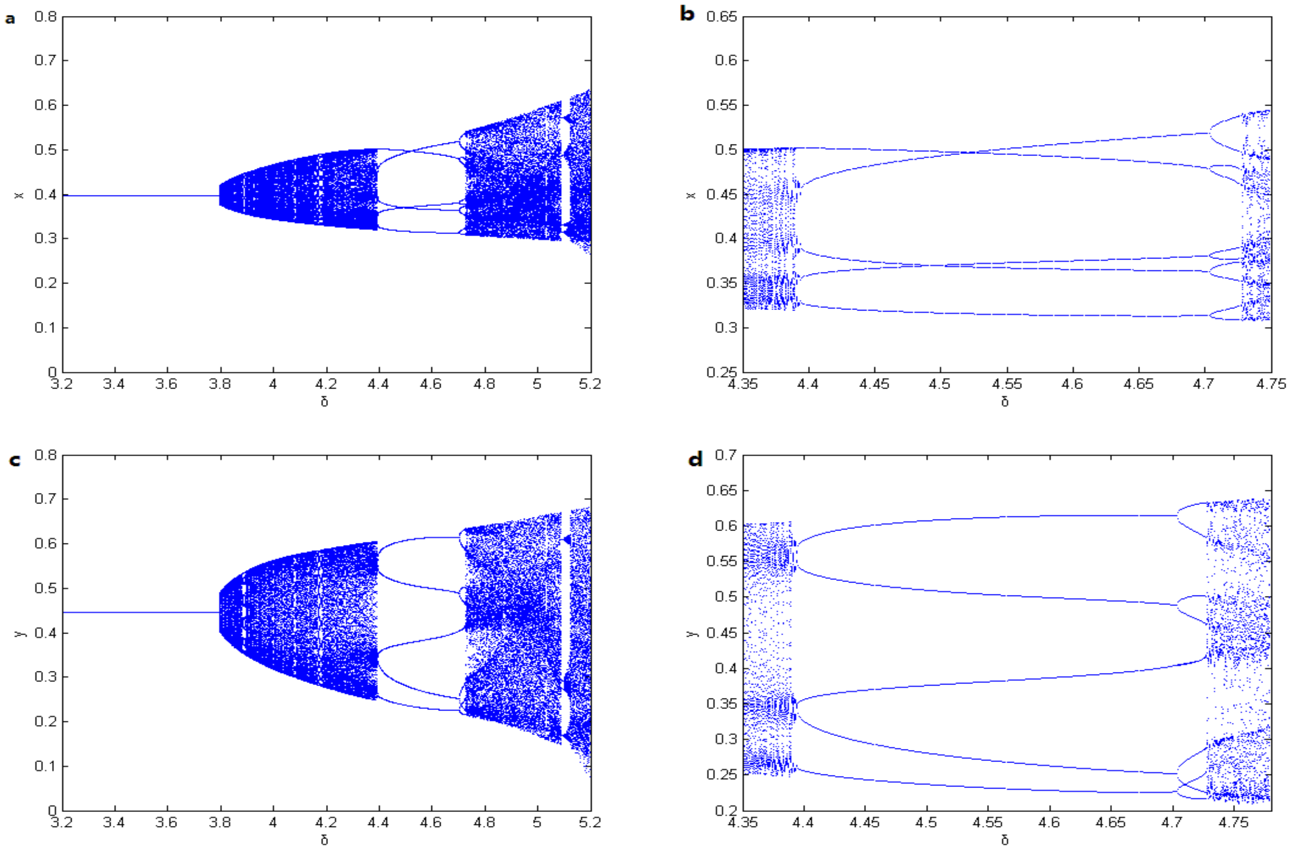

Figure 3: (a) Bifurcation diagram of system (2.3) in the (δ, x) plane for $a_1 = 0.6, a_2 = 0.3, a_3 = 2.2$, the initial value is $(0.01, 0.01)$. (b) Local amplification corresponding to (a) for $\delta \in [4.35, 4.75]$. (c) Bifurcation diagram of system (2.3) in the (δ, y) plane. (d) Local amplification corresponding to (c) for $\delta \in [4.35, 4.75]$.

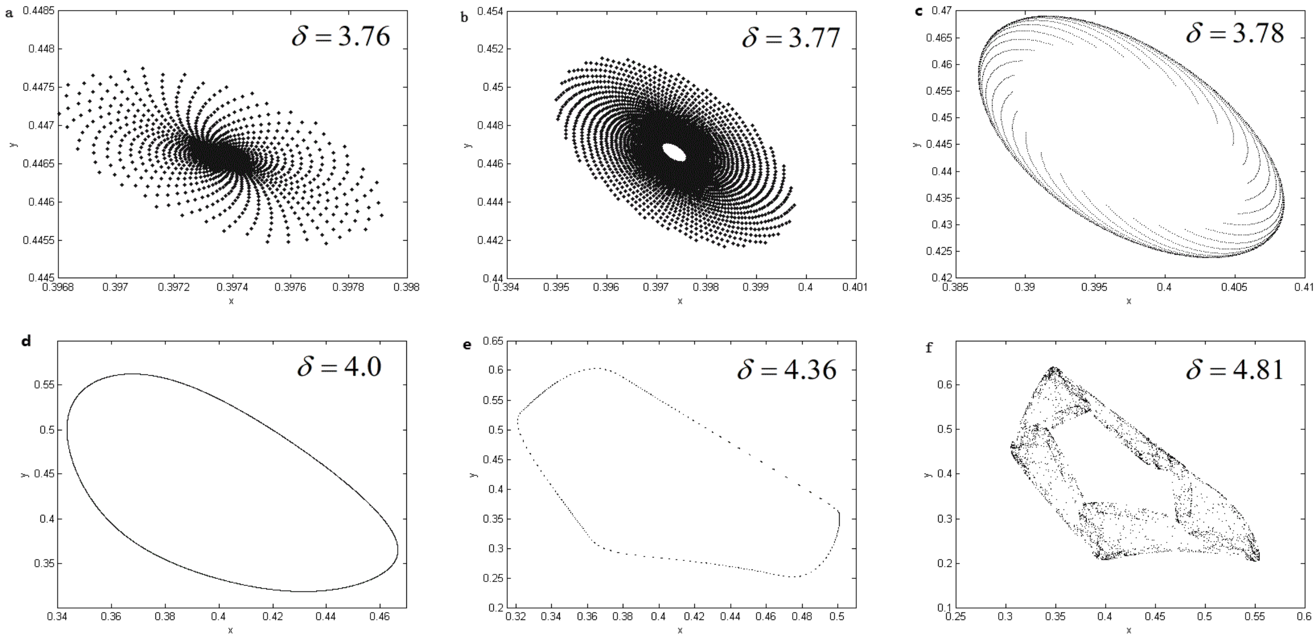

Figure 4: Phase portraits for various values of δ corresponding to Fig. 3 (a).

6. The effect of two-parameter variation on dynamic behavior of the system

In this section, we present several parameter-space diagrams to display the dynamic behavior of system (2.3). With the parameter changes, the system appears period-adding, and there are many periodic windows are embedded in chaos area. The periodic solutions are plotted in different colors, and marked by the corresponding numbers (such as the number 2 represents period-2, the number 10 represents period-10, and the chaotic region appears when the number is equal or greater than 20).

Firstly, we let a_1 vary in the range $[0.1, 0.9]$, and δ vary in the range $[3.2, 5.2]$, by calculate and draw the simulation diagram in two-dimensional parameter-spaces, as is shown in Fig. 5. With the parameter changes, the periodic regions are organized in the parameter plane. Obviously, the system has period-doubling and many periodic windows are embedded in chaos area.

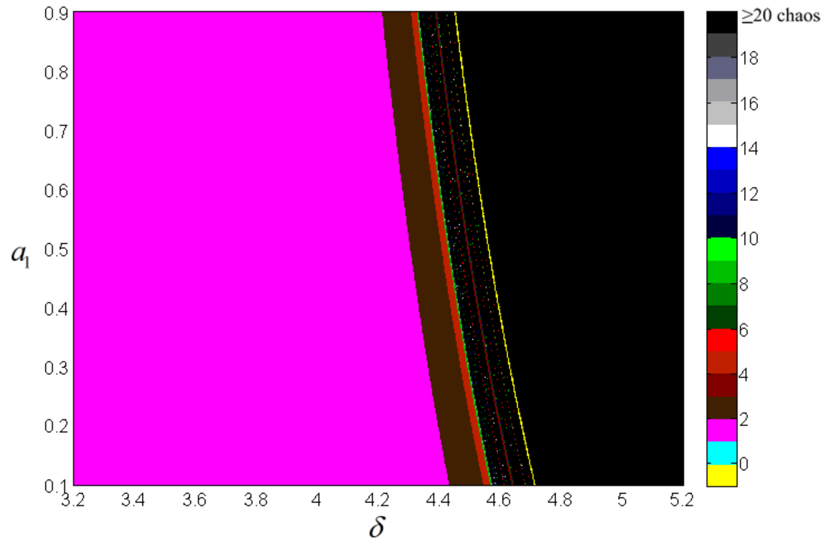

Figure 5: $a_1 \times \delta$ parameter-space plot of system (2.3), different color intensities represent to different periodic values, the legend at right side specifies the correspondence between colors and cycle periods. Chaotic phases are shown in black.

Next, we use other combinations of two parameters to plot the bifurcation diagrams of the system, as shown in Fig. 6, where a_2 and δ are taken as variables, with a_2 varies in the range $[0.1, 0.9]$, and δ varies in the range $[3.2, 5.2]$. As demonstrated in Fig. 6, the distributions of the periodic window are messy, and the periodic windows are surrounded by the chaotic region in black. Obviously, we can find a cluster which adjacent to the yellow parabola (in the upper portion of the yellow parabola) in the left part of Fig. 6, and the Neimark-Sacker bifurcation will be occured when the parameters a_2 and δ take values on the yellow parabolic. Then the system appears the cluster phenomenon (the periodic region such as period-6, period-7, and period-10, etc.) when the parameters pass through yellow parabolic, that is the different period island connect to the yellow parabolic and enters the chaotic region when the system passes through the different period island.

Take a_1 and a_2 as the variables to draw the two-dimensional parameter-spaces of the model, as shown in Fig. 7, where with a_1 varies in the range $[0.1, 0.6]$, and a_2 varies in the range $[0.3, 0.8]$. Obviously, the system has a periodic-2 solution when the parameter a_2 takes a small value. The system appears a cluster which is adjacent to the yellow parabola (in the upper portion of the yellow parabola) with increase of the parameter a_2, and the Neimark-Sacker bifurcation will be occurred when the parameters a_1 and a_2 take values on the yellow parabolic. Then the cluster phenomenon appears when the parameters pass through the yellow parabolic, that is the different period island connect to the yellow parabolic enters the chaotic region when the system passes through the different period island. And in the upper portion of the yellow parabola of Fig. 7, we can see the distributions of the periodic window are messy and the system have a large part of chaotic region when the parameter a_2 has a large value.

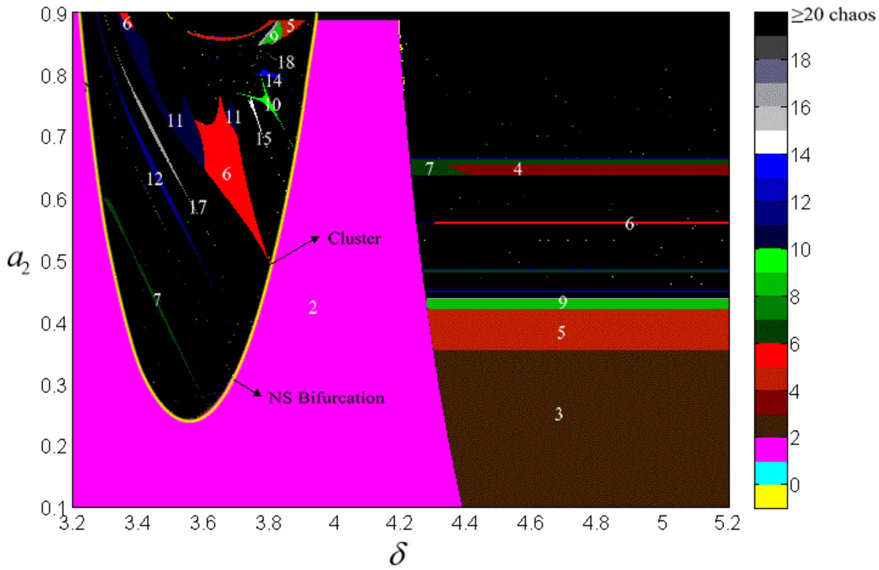

Figure 6: $a_2 \times \delta$parameter-space plot of system (2.3), different color intensities represent to different periodic values, the legend at right side specifies the correspondence between colors and cycle periods. Chaotic phases are shown in black.

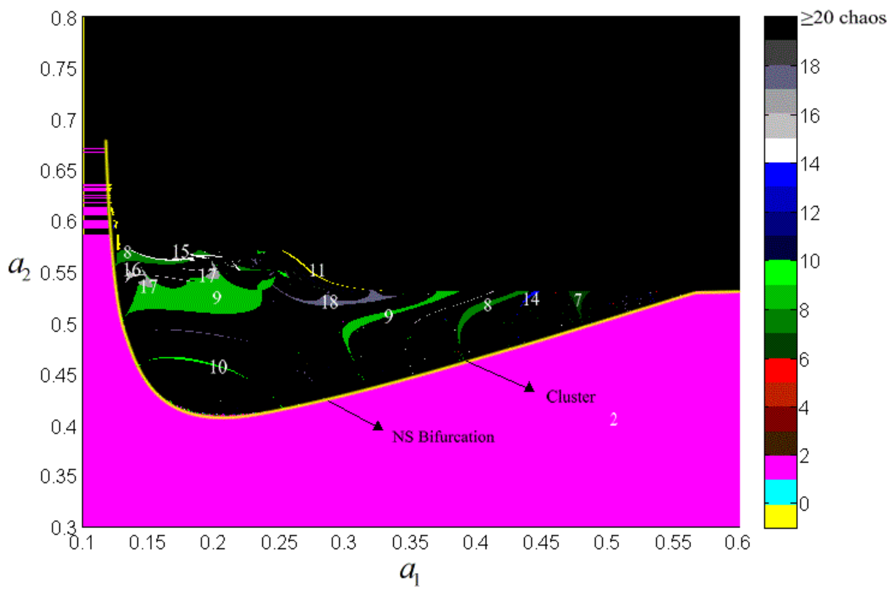

Figure 7: $a_1 \times a_2$parameter-space plot of system (2.3), different color intensities represent to different periodic values, the legend at right side specifies the correspondence between colors and cycle periods. Chaotic phases are shown in black.

Let a_1 varies in the range $[0.1, 0.9]$, and a_3 varies in the range $[0.1, 4]$, and Fig. 8 shows the corresponding parameter-space diagrams. The system has a wide range of chaotic region in the lower right part of Fig. 8, and a periodic-2 region embedded in chaotic region. Obviously, we can find a yellow parabola above the periodic-2 region and the system will be occurred Neimark-Sacker bifurcation when the parameters a_1 and a_3 take values on the yellow parabolic. Then the system appears the cluster phenomenon when the parameters pass through yellow parabolic, that is the different period island connect to the yellow parabolic enters the chaotic region when the system passes through the different period island.

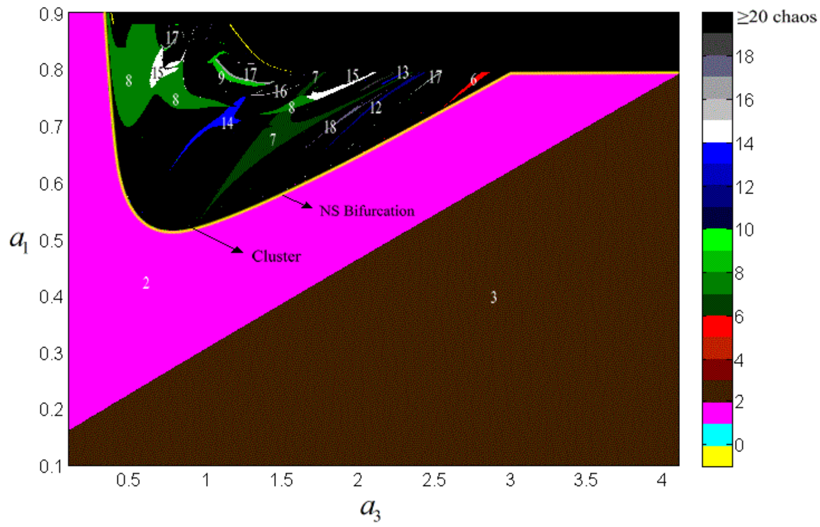

Figure 8: $a_1 \times a_3$parameter-space plot of system (2.3), different color intensities represent to different periodic values, the legend at right side specifies the correspondence between colors and cycle periods. Chaotic phases are shown in black.

7. Conclusion

In this paper, the complex dynamic behaviors of a discrete predator-prey system with Holling type III functional response are analysis in detail. Based on the center manifold theorem and the bifurcation theory, the local stability of equilibriums, flip bifurcation and Neimark-Sacker bifurcation of the model are studied. Moreover, the dynamic characteristics of the model in two-dimensional parameter-spaces are analyzed. Numerical simulation shows that the "cluster" phenomenon has emerged when the parameters pass through Neimark-Sacker bifurcation curve. Through the two-dimensional parameter-spaces, the rich and complex dynamic behavior can be observed as well. And numerical simulations not only illustrate our results, but also exhibit the complex dynamical behaviors of the model. The results show that we can more clearly and directly observe the chaotic phenomenon, period-adding and Neimark-Sacker bifurcation from two-dimensional parameter-spaces and the optimal parameters matching interval can also be found easily. Apparently there are more interesting problems about this chaotic system which deserves further investigation.

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