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# Some fixed point results of multi-valued nonlinear $F$-contractions without the Hausdorff metric

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**Abstract**

Fixed point results for several multi-valued nonlinear $F$-contractions without the Hausdorff metric are given and three examples are included. The results obtained in this paper differ from the corresponding results in the literature. 

*Keywords:* Multi-valued nonlinear $F$-contraction, fixed point, iterative approximation.
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## 1. Introduction and preliminaries

Throughout this article, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Let $(X, d)$ be a metric space, $CL(X)$, $CB(X)$ and $C(X)$ denote the families of all nonempty closed, all nonempty bounded closed and all nonempty compact subsets of $X$, respectively. For $T : X \to CL(X)$, $A, B \in X$ and $x \in X$, put

$$d(x, B) = \inf\{d(x, y), y \in B\}, \quad f(x) = d(x, Tx),$$

$$H(A, B) = \begin{cases} \max\left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, & \text{if the maximum exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

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Such a mapping $H$ is called a *generalized Hausdorff metric* induced by $d$ in $CL(X)$. A sequence $\{x_n\}_{n\in\mathbb{N}_0} \subseteq X$ is said to be an *orbit* of $T$ if $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}_0$. A function $h : X \to \mathbb{R}^+$ is said to be $T$*-orbitally lower semi-continuous* at $z \in X$ if $h(z) \le \liminf_{n\to\infty} h(x_n)$ for any orbit $\{x_n\}_{n\in\mathbb{N}_0} \subseteq X$ of $T$ with $\lim_{n\to\infty} x_n = z$.

It is well-known that the Banach contraction principle has a lot of generalizations and applications, (see [2, 6, 7, 9, 10, 12, 17–19, 25]). In 1969, Nadler [19] obtained the following fixed point theorem for the multi-valued contraction mappings.

**Theorem 1.1** ([19])**.** *Let $(X, d)$ be a complete metric space and $T$ a mapping from $X$ to $CB(X)$ such that*

$$H(Tx, Ty) \le cd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

*where $c \in [0, 1)$ is a constant. Then $T$ has a fixed point.*

Later, many researchers generalized Theorem 1.1 in various directions (see [1, 3–6, 9, 10, 13, 14, 16, 18–24]). In 1972, Reich [22] extended Theorem 1.1 and proved the following fixed point theorem for the multi-valued contraction mapping which maps points into compact sets.

**Theorem 1.2** ([22])**.** *Let $(X, d)$ be a complete metric space and $T : X \to C(X)$ satisfies*

$$H(Tx, Ty) \le \varphi(d(x, y))d(x, y), \quad \forall x, y \in X, \tag{1.2}$$

*where*

$$\varphi : (0, +\infty) \to [0, 1) \quad \text{with} \quad \limsup_{r\to t^+} \varphi(r) < 1, \quad \forall t \in (0, +\infty). \tag{1.3}$$

*Then $T$ has a fixed point.*

In 1989, Mizoguchi and Takahashi [18] responded to the conjecture which has been asked whether Reich's theorem [22] can be extended to multi-valued mappings whose range consists of bounded and closed sets and proved the following result.

**Theorem 1.3** ([18])**.** *Let $(X, d)$ be a complete metric space and $T : X \to CB(X)$ satisfy that*

$$H(Tx, Ty) \le \varphi(d(x, y))d(x, y), \quad \forall x, y \in X \text{ with } x \ne y, \tag{1.4}$$

*where*

$$\varphi : (0, +\infty) \to [0, 1) \quad \text{with} \quad \limsup_{r\to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+. \tag{1.5}$$

*Then $T$ has a fixed point.*

In 2006, Feng and Liu [10] generalized Theorem 1.1 to a new type of multi-valued nonlinear contraction mapping without using the Hausdorff metric. Ćirić [5, 6], and Klim and Wardowski [14] extended the result of Feng and Liu [10] and showed the existence of fixed points for some new set-valued contraction mappings. Pathak and Shahzad [21] introduced a new concept of generalized contraction of set-valued mappings and got fixed point theorems for such mappings.

In 2012, Wardowski [25] introduced the concept of $F$-contractions for single-valued mappings and proved a fixed point theorem for the $F$-contraction, which is a generalization of the Banach contraction principle.

**Definition 1.4** ([25])**.** Let $F : (0, +\infty) \to \mathbb{R}$ be a mapping satisfying:

(F1) $F$ is strictly increasing;

(F2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to0^+} \alpha^k F(\alpha) = 0$.

Denote by $\mathcal{F}$ the family of all functions $F$ that satisfy (F1)-(F3).

**Definition 1.5** ([25]). Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad \forall x, y \in X \quad \text{with} \quad d(Tx, Ty) > 0.$$

**Theorem 1.6.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be an $F$-contraction. Then $T$ has a unique fixed point $u \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to $u$.*

Recently, the researchers have been attracted to study new classes of $F$-contractions and to prove the existence of fixed point theorems for these $F$-contractions (see [1, 2, 8, 11, 15, 17, 20, 23, 25]). In particular, Minak et al. [17] and Cosentino and Vetro [8] introduced Ćirić type generalized $F$-contractions and Hardy-Rogers type $F$-contraction mappings and proved some fixed point results for the $F$-contractions.

The purpose of this paper is to introduce some new multi-valued nonlinear $F$-contractions without using the Hausdorff metric and to establish the existence and iterative approximations of fixed points for these multi-valued nonlinear $F$-contractions in complete metric spaces. Three examples are included.

## 2. Main results

In this section, we establish four fixed point theorems for the multi-valued nonlinear $F$-contractions (a1), (a3), (a4), and (a6) in complete metric spaces.

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space, $T : X \to CL(X)$ be a multi-valued mapping such that*

(a1) *for any $x \in X - Tx$ there is $y \in Tx - Ty$ with*

$$F(d(x, y)) \leq F(f(x)) + \tau, \quad F(f(y)) + \tau + \eta(f(x)) \leq F(d(x, y)),$$

*where $F \in \mathcal{F}$, $\tau > 0$ and $\eta : (0, +\infty) \to (0, +\infty)$ satisfies that*

(a2) $\liminf_{s\to t^+} \eta(s) > 0, \forall t \in \mathbb{R}^+$.

*Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of $T$ and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z$ is a fixed point of $T$ in $X$ if and only if the function $f$ is $T$-orbitally lower semi-continuous at $z$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point with $x_0 \notin Tx_0$. It follows from (a1) that there exists $x_1 \in Tx_0 - Tx_1$ satisfying

$$F(d(x_0, x_1)) \leq F(f(x_0)) + \tau, \quad F(f(x_1)) + \tau + \eta(f(x_0)) \leq F(d(x_0, x_1)). \tag{2.1}$$

In light of (2.1) and $\eta(f(x_0)) > 0$, we deduce that

$$\begin{aligned} F(f(x_1)) &\leq F(d(x_0, x_1)) - \tau - \eta(f(x_0)) \\ &\leq F(f(x_0)) + \tau - \tau - \eta(f(x_0)) \\ &= F(f(x_0)) - \eta(f(x_0)) \\ &< F(f(x_0)). \end{aligned}$$

In terms of (a1) there exists $x_2 \in Tx_1 - Tx_2$ with

$$F(d(x_1, x_2)) \leq F(f(x_1)) + \tau, \quad F(f(x_2)) + \tau + \eta(f(x_1)) \leq F(d(x_1, x_2)),$$

which together with (2.1), $\eta(f(x_0)) > 0$ and $\eta(f(x_1)) > 0$ mean that

$$\begin{aligned} F(f(x_2)) &\leq F(d(x_1, x_2)) - \tau - \eta(f(x_1)) \\ &\leq F(f(x_1)) + \tau - \tau - \eta(f(x_1)) \\ &= F(f(x_1)) - \eta(f(x_1)) \\ &< F(f(x_1)), \end{aligned}$$

$$\begin{aligned}F(d(x_1,x_2)) &\le F(f(x_1))+\tau\\&\le F(d(x_0,x_1))-\tau-\eta(f(x_0))+\tau\\&= F(d(x_0,x_1))-\eta(f(x_0))\\&< F(d(x_0,x_1)).\end{aligned}$$

Repeating this process, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}_0}\subset X$ of $T$ satisfying

$$\begin{aligned}&F(d(x_n,x_{n+1}))\le F(f(x_n))+\tau,\\&F(f(x_{n+1}))+\tau+\eta(f(x_n))\le F(d(x_n,x_{n+1})),\qquad x_{n+1}\in Tx_n-Tx_{n+1},\quad \forall n\in\mathbb{N}_0.\end{aligned}\tag{2.2}$$

In view of (2.2) and $\eta(f(x_{n-1}))>0$ for each $n\in\mathbb{N}$, we have

$$\begin{aligned}F(f(x_n)) &\le F(d(x_{n-1},x_n))-\tau-\eta(f(x_{n-1}))\\&\le F(f(x_{n-1}))+\tau-\tau-\eta(f(x_{n-1}))\\&= F(f(x_{n-1}))-\eta(f(x_{n-1}))\\&< F(f(x_{n-1})),\quad \forall n\in\mathbb{N}.\end{aligned}\tag{2.3}$$

It follows from (2.3) and $(F1)$ that

$$0<f(x_n)<f(x_{n-1}),\quad \forall n\in\mathbb{N}.\tag{2.4}$$

Note that (2.4) implies that there exists a constant $a\in\mathbb{R}^+$ with

$$\lim_{n\to\infty} f(x_n)=a.\tag{2.5}$$

By virtue of (a2) there exists a constant $b>0$ satisfying

$$\liminf_{s\to a^+}\eta(s)=2b,$$

which means that for $\varepsilon=b$, there exists $\delta>0$ satisfying

$$\eta(s)-2b>-\varepsilon,\quad \forall s\in(a,a+\delta),$$

that is,

$$\eta(s)>b,\quad \forall s\in(a,a+\delta).\tag{2.6}$$

Clearly, (2.4)-(2.6) ensure that there exists $n_0\in\mathbb{N}$ satisfying

$$a<f(x_n)<a+\delta,\quad \eta(f(x_n))>b,\quad \forall n\ge n_0.\tag{2.7}$$

Making use of (2.3) and (2.7), we arrive at

$$\begin{aligned}F(f(x_n)) &\le F(f(x_{n-1}))-\eta(f(x_{n-1}))\\&\le F(f(x_{n-2}))-\eta(f(x_{n-2}))-\eta(f(x_{n-1}))\\&\vdots\\&\le F(f(x_{n_0}))-\eta(f(x_{n_0}))-\eta(f(x_{n_0+1}))-\cdots-\eta(f(x_{n-1}))\\&\le F(f(x_{n_0}))-(n-n_0)b,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}F(f(x_n))=-\infty.\tag{2.8}$$

By means of $(F2)$, (2.5) and (2.8), we conclude immediately that

$$a=\lim_{n\to\infty}f(x_n)=0.\tag{2.9}$$

Using (2.2) and (2.7), we infer that

$$\begin{aligned} F(d(x_n, x_{n+1})) &\le F(f(x_n)) + \tau \\ &\le F(d(x_{n-1}, x_n)) - \tau - \eta(f(x_{n-1})) + \tau \\ &= F(d(x_{n-1}, x_n)) - \eta(f(x_{n-1})) \\ &\le F(d(x_{n-2}, x_{n-1})) - \eta(f(x_{n-2})) - \eta(f(x_{n-1})) \\ &\vdots \\ &\le F(d(x_{n_0}, x_{n_0+1})) - \eta(f(x_{n_0})) - \eta(f(x_{n_0+1})) - \cdots - \eta(f(x_{n-1})) \\ &\le F(d(x_{n_0}, x_{n_0+1})) - (n - n_0)b \\ &\to -\infty \quad \text{as } n \to \infty. \end{aligned} \tag{2.10}$$

That is,

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty.$$

It follows from (2.10) and $(F2)$ that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

It is clear that $(F3)$ and (2.11) ensure that there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} [d^k(x_n, x_{n+1})F(d(x_n, x_{n+1}))] = 0. \tag{2.12}$$

Using (2.10)-(2.12), we derive that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} [(n - n_0)bd^k(x_n, x_{n+1})] \\ &\le \limsup_{n\to\infty} \{(F(d(x_{n_0}, x_{n_0+1})) - F(d(x_n, x_{n+1})))d^k(x_n, x_{n+1})\} \\ &= 0, \end{aligned}$$

which yields that

$$\lim_{n\to\infty} (n - n_0)bd^k(x_n, x_{n+1}) = 0,$$

that is,

$$\lim_{n\to\infty} nd^k(x_n, x_{n+1}) = 0. \tag{2.13}$$

It follows from (2.13) that there exists $n_1 \ge n_0$ satisfying

$$nd^k(x_n, x_{n+1}) \le 1, \quad \forall n \ge n_1,$$

that is,

$$d(x_n, x_{n+1}) \le \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \ge n_1,$$

which gives that

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\le \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}, \quad \forall m > n \ge n_1, \end{aligned}$$

which together with the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ means that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since

$(X, d)$ is a complete metric space, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.14}$$

Suppose that $f$ is $T$-orbitally lower semi-continuous at $z$. It follows from (2.9) and (2.14) that

$$d(z, Tz) = f(z) \le \liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n) = 0,$$

that is, $z \in X$ is a fixed point of $T$.

Conversely, suppose that $z \in X$ is a fixed point of $T$. For each orbit $\{y_n\}_{n\in\mathbb{N}_0}$ of $T$ with $\lim_{n\to\infty} y_n = z$, we deduce that

$$f(z) = d(z, Tz) = 0 \le \liminf_{n\to\infty} f(y_n),$$

which implies that $f$ is $T$-orbitally lower semi-continuous in $z$. This completes the proof. □

**Theorem 2.2.** *Let $(X, d)$ be a complete metric space, $T : X \to CL(X)$ be a multi-valued mapping such that*

(a3) *for any $x \in X - Tx$ there is $y \in Tx - Ty$ with*

$$F(d(x, y)) \le F(f(x)) + \tau, \quad F(f(y)) + \tau + \eta(d(x, y)) \le F(d(x, y)),$$

*where $F \in \mathcal{F}$, $\tau > 0$ and $\eta : (0, +\infty) \to (0, +\infty)$ satisfies* (a2).

*Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of $T$ and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z$ is a fixed point of $T$ in $X$ if and only if the function $f$ is $T$-orbitally lower semi-continuous at $z$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point with $x_0 \notin Tx_0$. It follows from (a2) that there exists $x_1 \in Tx_0 - Tx_1$ satisfying

$$F(d(x_0, x_1)) \le F(f(x_0)) + \tau, \quad F(f(x_1)) + \tau + \eta(d(x_0, x_1)) \le F(d(x_0, x_1)). \tag{2.15}$$

In view of (a3), there exists $x_2 \in Tx_1 - Tx_2$ with

$$F(d(x_1, x_2)) \le F(f(x_1)) + \tau, \quad F(f(x_2)) + \tau + \eta(d(x_1, x_2)) \le F(d(x_1, x_2)),$$

which together with (2.15) and $\eta(d(x_0, x_1)) > 0$ we have

$$\begin{aligned} F(d(x_1, x_2)) &\le F(f(x_1)) + \tau \\ &\le F(d(x_0, x_1)) - \tau - \eta(d(x_0, x_1)) + \tau \\ &= F(d(x_0, x_1)) - \eta(d(x_0, x_1)) \\ &< F(d(x_0, x_1)). \end{aligned}$$

Repeating this process, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ of $T$ satisfying

$$\begin{aligned} &F(d(x_n, x_{n+1})) \le F(f(x_n)) + \tau, \\ &F(f(x_{n+1})) + \tau + \eta(d(x_n, x_{n+1})) \le F(d(x_n, x_{n+1})), \qquad x_{n+1} \in Tx_n - Tx_{n+1}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{2.16}$$

In light of (2.16) and $\eta(d(x_{n-1}, x_n)) > 0$ for each $n \in \mathbb{N}$, we deduce that

$$\begin{aligned} F(d(x_n, x_{n+1})) &\le F(f(x_n)) + \tau \\ &\le F(d(x_{n-1}, x_n)) - \tau - \eta(d(x_{n-1}, x_n)) + \tau \\ &= F(d(x_{n-1}, x_n)) - \eta(d(x_{n-1}, x_n)) \\ &< F(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{2.17}$$

which together with $(F1)$ implies that

$$0 < d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{2.18}$$

Consequently, (2.18) means that the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ converges to a constant $a \in \mathbb{R}^+$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = a. \tag{2.19}$$

As in the proof of Theorem 2.1, we conclude that (2.6) holds. It follows from (2.6), (2.18) and (2.19) that there exists $n_0 \in \mathbb{N}$ satisfying

$$a < d(x_n, x_{n+1}) < a + \delta, \quad \eta(d(x_n, x_{n+1})) > b, \quad \forall n \geq n_0. \tag{2.20}$$

Using (2.17) and (2.20), we obtain that

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \eta(d(x_{n-1}, x_n)) \\ &\leq F(d(x_{n-2}, x_{n-1})) - \eta(d(x_{n-2}, x_{n-1})) - \eta(d(x_{n-1}, x_n)) \\ &\vdots \\ &\leq F(d(x_{n_0}, x_{n_0+1})) - \eta(d(x_{n_0}, x_{n_0+1})) - \eta(d(x_{n_0+1}, x_{n_0+2})) - \cdots - \eta(d(x_{n-1}, x_n)) \\ &\leq F(d(x_{n_0}, x_{n_0+1})) - (n - n_0)b \\ &\to -\infty \quad \text{as } n \to \infty, \end{aligned}$$

which implies (2.11). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

**Theorem 2.3.** *Let $(X, d)$ be a complete metric space, $T : X \to CL(X)$ be a multi-valued mapping such that*

(a4) *for any $x \in X - Tx$ there is $y \in Tx - Ty$ with*

$$F(d(x, y)) \leq F(f(x)) + \frac{1}{2}\eta(f(x)), \quad F(f(y)) + \eta(f(x)) \leq F(d(x, y)),$$

*where $F \in \mathcal{F}$, $\eta : (0, +\infty) \to (0, +\infty)$ satisfies* (a2) *and*

(a5) $\limsup_{s\to 0^+} \eta(s) < +\infty$.

*Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of $T$ and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z$ is a fixed point of $T$ in $X$ if and only if the function $f$ is $T$-orbitally lower semi-continuous at $z$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point with $x_0 \notin Tx_0$. It follows from (a4) that there exists $x_1 \in Tx_0 - Tx_1$ satisfying

$$F(d(x_0, x_1)) \leq F(f(x_0)) + \frac{1}{2}\eta(f(x_0)), \quad F(f(x_1)) + \eta(f(x_0)) \leq F(d(x_0, x_1)). \tag{2.21}$$

It follows from (2.21) and $\eta(f(x_0)) > 0$ that

$$\begin{aligned} F(f(x_1)) &\leq F(d(x_0, x_1)) - \eta(f(x_0)) \\ &\leq F(f(x_0)) + \frac{1}{2}\eta(f(x_0)) - \eta(f(x_0)) \\ &= F(f(x_0)) - \frac{1}{2}\eta(f(x_0)) \\ &< F(f(x_0)). \end{aligned}$$

(a4) implies that there exists $x_2 \in Tx_1 - Tx_2$ with

$$F(d(x_1,x_2)) \le F(f(x_1)) + \frac{1}{2}\eta(f(x_1)), \quad F(f(x_2)) + \eta(f(x_1)) \le F(d(x_1,x_2)),$$

which together with (2.21) and $\eta(f(x_1)) > 0$ give that

$$\begin{aligned} F(f(x_2)) &\le F(d(x_1,x_2)) - \eta(f(x_1)) \\ &\le F(f(x_1)) + \frac{1}{2}\eta(f(x_1)) - \eta(f(x_1)) \\ &= F(f(x_1)) - \frac{1}{2}\eta(f(x_1)) \\ &< F(f(x_1)), \end{aligned}$$

$$\begin{aligned} F(d(x_1,x_2)) &\le F(f(x_1)) + \frac{1}{2}\eta(f(x_1)) \\ &\le F(d(x_0,x_1)) - \eta(f(x_0)) + \frac{1}{2}\eta(f(x_1)). \end{aligned}$$

Repeating this process, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}_0} \in X$ of $T$ satisfying

$$\begin{aligned} &F(d(x_n,x_{n+1})) \le F(f(x_n)) + \frac{1}{2}\eta(f(x_n)), \\ &F(f(x_{n+1})) + \eta(f(x_n)) \le F(d(x_n,x_{n+1})), \qquad x_{n+1} \in Tx_n - Tx_{n+1}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{2.22}$$

In view of (2.22) and $\eta(f(x_{n-1})) > 0$ for each $n \in \mathbb{N}$, we deduce that

$$\begin{aligned} F(f(x_n)) &\le F(d(x_{n-1},x_n)) - \eta(f(x_{n-1})) \\ &\le F(f(x_{n-1})) + \frac{1}{2}\eta(f(x_{n-1})) - \eta(f(x_{n-1})) \\ &\le F(f(x_{n-1})) - \frac{1}{2}\eta(f(x_{n-1})) \\ &< F(f(x_{n-1})), \quad \forall n \in \mathbb{N} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} F(d(x_n,x_{n+1})) &\le F(f(x_n)) + \frac{1}{2}\eta(f(x_n)) \\ &\le F(d(x_{n-1},x_n)) - \eta(f(x_{n-1})) + \frac{1}{2}\eta(f(x_n)), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.24}$$

Similar to the arguments of Theorem 2.1, we conclude that (2.4)-(2.7) hold. In terms of (2.23) and (2.7), we arrive at

$$\begin{aligned} F(f(x_n)) &\le F(f(x_{n-1})) - \frac{1}{2}\eta(f(x_{n-1})) \\ &\le F(f(x_{n-2})) - \frac{1}{2}\eta(f(x_{n-2})) - \frac{1}{2}\eta(f(x_{n-1})) \\ &\vdots \\ &\le F(f(x_{n_0})) - \frac{1}{2}\eta(f(x_{n_0})) - \frac{1}{2}\eta(f(x_{n_0+1})) - \cdots - \frac{1}{2}\eta(f(x_{n-1})) \\ &\le F(f(x_{n_0})) - \frac{1}{2}(n-n_0)b \\ &\to -\infty \quad \text{as } n \to \infty, \end{aligned}$$

which together with (2.5) and $(F2)$, we derive that (2.8) and (2.9) hold.

In light of (2.7) and (2.24), we get that

$$\begin{aligned}F(d(x_n,x_{n+1})) &\le F(d(x_{n-1},x_n)) - \eta(f(x_{n-1})) + \frac{1}{2}\eta(f(x_n))\\ &\le F(d(x_{n-2},x_{n-1})) - \eta(f(x_{n-2})) - \frac{1}{2}\eta(f(x_{n-1})) + \frac{1}{2}\eta(f(x_n))\\ &\ \vdots \\ &\le F(d(x_{n_0},x_{n_0+1})) - \eta(f(x_{n_0})) - \frac{1}{2}\eta(f(x_{n_0+1})) - \cdots - \frac{1}{2}\eta(f(x_{n-1})) + \frac{1}{2}\eta(f(x_n))\\ &\le F(d(x_{n_0},x_{n_0+1})) - \frac{1}{2}(n-n_0-1)b + \frac{1}{2}\eta(f(x_n)), \quad \forall n \ge n_0.\end{aligned} \tag{2.25}$$

Taking upper limit in (2.25) and using (2.7), (2.9) and (a5), we get that

$$\begin{aligned}\limsup_{n\to\infty} F(d(x_n,x_{n+1})) &\le \limsup_{n\to\infty}\left[F(d(x_{n_0},x_{n_0+1})) - \frac{1}{2}(n-n_0-1)b + \frac{1}{2}\eta(f(x_n))\right]\\ &\le \limsup_{n\to\infty}\left[F(d(x_{n_0},x_{n_0+1})) - \frac{1}{2}(n-n_0-1)b\right] + \frac{1}{2}\limsup_{n\to\infty}\eta(f(x_n))\\ &= -\infty,\end{aligned}$$

that is, (2.11) holds. Similarly, we know that (2.12) holds.

It follows from (a5), (2.11), (2.12), and (2.25) that

$$\begin{aligned}0 &\le \limsup_{n\to\infty}\left[\frac{1}{2}(n-n_0-1)bd^k(x_n,x_{n+1})\right]\\ &\le \limsup_{n\to\infty}\left\{\left(F(d(x_{n_0},x_{n_0+1})) - F(d(x_n,x_{n+1})) + \frac{1}{2}\eta(f(x_n))\right)d^k(x_n,x_{n+1})\right\}\\ &\le \limsup_{n\to\infty}\{(F(d(x_{n_0},x_{n_0+1})) - F(d(x_n,x_{n+1})))d^k(x_n,x_{n+1})\}\\ &\quad + \frac{1}{2}\limsup_{n\to\infty}[\eta(f(x_n))d^k(x_n,x_{n+1})]\\ &\le 0 + \frac{1}{2}\limsup_{n\to\infty}\eta(f(x_n))\cdot\limsup_{n\to\infty}d^k(x_n,x_{n+1})\\ &= 0,\end{aligned}$$

which means that

$$\limsup_{n\to\infty}[(n-n_0-1)bd^k(x_n,x_{n+1})] = 0,$$

which yields (2.13). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

**Theorem 2.4.** *Let $(X,d)$ be a complete metric space, $T : X \to CL(X)$ be a multi-valued mapping such that*

(a6) *for any $x \in X - Tx$ there is $y \in Tx - Ty$ with*

$$F(d(x,y)) \le F(f(x)) + \frac{1}{2}\eta(d(x,y)), \quad F(f(y)) + \eta(d(x,y)) \le F(d(x,y)),$$

*where $F \in \mathcal{F}$, $\eta : (0,+\infty) \to (0,+\infty)$ satisfies*

(a7) *$\eta$ is decreasing,*

(a8) $\lim_{s\to 0^+}\eta(s) > 0$.

*Then, for each $x_0 \in X$ there exists an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of $T$ and $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Furthermore, $z$ is a fixed point of $T$ in $X$ if and only if the function $f$ is $T$-orbitally lower semi-continuous at $z$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point with $x_0 \notin Tx_0$. It follows from (a6) that there exists $x_1 \in Tx_0 - Tx_1$ satisfying

$$F(d(x_0,x_1)) \le F(f(x_0)) + \frac{1}{2}\eta(d(x_0,x_1)), \quad F(f(x_1)) + \eta(d(x_0,x_1)) \le F(d(x_0,x_1)). \tag{2.26}$$

In view of (2.26) and $\eta(d(x_0,x_1)) > 0$, we arrive at

$$\begin{aligned} F(f(x_1)) &\le F(d(x_0,x_1)) - \eta(d(x_0,x_1)) \\ &\le F(f(x_0)) + \frac{1}{2}\eta(d(x_0,x_1)) - \eta(d(x_0,x_1)) \\ &= F(f(x_0)) - \frac{1}{2}\eta(d(x_0,x_1)) \\ &< F(f(x_0)). \end{aligned}$$

(a6) implies that there exists $x_2 \in Tx_1 - Tx_2$ with

$$F(d(x_1,x_2)) \le F(f(x_1)) + \frac{1}{2}\eta(d(x_1,x_2)), \quad F(f(x_2)) + \eta(d(x_1,x_2)) \le F(d(x_1,x_2)),$$

which together with (2.26) and $\eta(d(x_1,x_2)) > 0$ show that

$$\begin{aligned} F(f(x_2)) &\le F(d(x_1,x_2)) - \eta(d(x_1,x_2)) \\ &\le F(f(x_1)) + \frac{1}{2}\eta(d(x_1,x_2)) - \eta(d(x_1,x_2)) \\ &= F(f(x_1)) - \frac{1}{2}\eta(d(x_1,x_2)) \\ &< F(f(x_1)), \end{aligned}$$

$$\begin{aligned} F(d(x_1,x_2)) &\le F(f(x_1)) + \frac{1}{2}\eta(d(x_1,x_2)) \\ &\le F(d(x_0,x_1)) - \eta(d(x_0,x_1)) + \frac{1}{2}\eta(d(x_1,x_2)). \end{aligned}$$

Repeating this process, we obtain an orbit $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ of $T$ satisfying

$$\begin{aligned} &F(d(x_n,x_{n+1})) \le F(f(x_n)) + \frac{1}{2}\eta(d(x_n,x_{n+1})), \\ &F(f(x_{n+1})) + \eta(d(x_n,x_{n+1})) \le F(d(x_n,x_{n+1})), \quad x_{n+1} \in Tx_n - Tx_{n+1}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{2.27}$$

Suppose that there exists some $n_0 \in \mathbb{N}$ satisfying

$$d(x_{n_0},x_{n_0+1}) \ge d(x_{n_0-1},x_{n_0}), \tag{2.28}$$

which together with (a7) gives that

$$\eta(d(x_{n_0},x_{n_0+1})) \le \eta(d(x_{n_0-1},x_{n_0})). \tag{2.29}$$

In terms of (2.27)-(2.29) and $\eta(d(x_{n_0},x_{n_0+1})) > 0$, we deduce that

$$\begin{aligned} F(d(x_{n_0-1},x_{n_0})) &\le F(d(x_{n_0},x_{n_0+1})) \\ &\le F(f(x_{n_0})) + \frac{1}{2}\eta(d(x_{n_0},x_{n_0+1})) \end{aligned}$$

$$\begin{aligned}
&\le F(d(x_{n_0-1}, x_{n_0})) - \eta(d(x_{n_0-1}, x_{n_0})) + \frac{1}{2}\eta(d(x_{n_0}, x_{n_0+1}))\\
&\le F(d(x_{n_0-1}, x_{n_0})) - \eta(d(x_{n_0}, x_{n_0+1})) + \frac{1}{2}\eta(d(x_{n_0}, x_{n_0+1}))\\
&= F(d(x_{n_0-1}, x_{n_0})) - \frac{1}{2}\eta(d(x_{n_0}, x_{n_0+1}))\\
&< F(d(x_{n_0-1}, x_{n_0})),
\end{aligned}$$

which is contradiction. Therefore,

$$0 < d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{2.30}$$

It is clear that (2.30) implies (2.19) for some $a \in \mathbb{R}$. (a7), (a8), (2.19), and (2.30) imply that

$$\lim_{n\to\infty} \eta(d(x_n, x_{n+1})) = 2b \tag{2.31}$$

for some $b > 0$. It is easy to see that (2.19), (2.30), and (2.31) ensure that there exists $n_1 > n_0$ satisfying

$$a < d(x_n, x_{n+1}) < a + \delta, \quad \eta(d(x_n, x_{n+1})) > b, \quad \forall n \ge n_1. \tag{2.32}$$

It follows from (2.27), (2.30), and (2.32) that

$$\begin{aligned}
F(d(x_n, x_{n+1})) &\le F(f(x_n)) + \frac{1}{2}\eta(d(x_n, x_{n+1}))\\
&\le F(d(x_{n-1}, x_n)) - \eta(d(x_{n-1}, x_n)) + \frac{1}{2}\eta(d(x_n, x_{n+1}))\\
&\le F(d(x_{n-2}, x_{n-1})) - \eta(d(x_{n-2}, x_{n-1})) - \frac{1}{2}\eta(d(x_{n-1}, x_n)) + \frac{1}{2}\eta(d(x_n, x_{n+1}))\\
&\ \vdots\\
&\le F(d(x_{n_1}, x_{n_1+1})) - \eta(d(x_{n_1}, x_{n_1+1})) - \frac{1}{2}\eta(d(x_{n_1+1}, x_{n_1+2})) - \cdots\\
&\quad - \frac{1}{2}\eta(d(x_{n-1}, x_n)) + \frac{1}{2}\eta(d(x_n, x_{n+1}))\\
&\le F(d(x_{n_1}, x_{n_1+1})) - \frac{1}{2}(n - n_1 - 1)b + \frac{1}{2}\eta(d(x_n, x_{n+1})), \quad \forall n \ge n_1.
\end{aligned} \tag{2.33}$$

Using (2.33) and (a7), we arrive at

$$\begin{aligned}
\limsup_{n\to\infty} F(d(x_n, x_{n+1})) &\le \limsup_{n\to\infty} \left[F(d(x_{n_1}, x_{n_1+1})) - \frac{1}{2}(n - n_1 - 1)b + \frac{1}{2}\eta(d(x_n, x_{n+1}))\right]\\
&\le \limsup_{n\to\infty} \left[F(d(x_{n_1}, x_{n_1+1})) - \frac{1}{2}(n - n_1 - 1)b\right] + \frac{1}{2}\limsup_{n\to\infty} \eta(d(x_n, x_{n+1}))\\
&= -\infty,
\end{aligned}$$

that is,

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty.$$

In view of $(F2)$ and (2.19), we get that

$$a = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.34}$$

In view of $(F3)$ and (2.33), ensure that there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty}[d^k(x_n, x_{n+1})F(d(x_n, x_{n+1}))] = 0. \tag{2.35}$$

In light of (a7) and (2.33)-(2.35), we deduce that

$$\begin{aligned}
0 &\le \limsup_{n\to\infty}\left[\frac{1}{2}(n-n_0-1)bd^k(x_n,x_{n+1})\right]\\
&\le \limsup_{n\to\infty}\left[\left(F(d(x_{n_1},x_{n_1+1}))-F(d(x_n,x_{n+1}))+\frac{1}{2}\eta(d(x_n,x_{n+1}))\right)d^k(x_n,x_{n+1})\right]\\
&\le \limsup_{n\to\infty}[(F(d(x_{n_1},x_{n_1+1}))-F(d(x_n,x_{n+1})))d^k(x_n,x_{n+1})]\\
&\quad+\limsup_{n\to\infty}\left[\frac{1}{2}\eta(d(x_n,x_{n+1})d^k(x_n,x_{n+1})\right]\\
&\le 0+\limsup_{n\to\infty}\frac{1}{2}\eta(d(x_n,x_{n+1})\cdot\limsup_{n\to\infty}d^k(x_n,x_{n+1})\\
&=0,
\end{aligned}$$

which connotes (2.13). The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

## 3. Remarks and examples

*Remark* 3.1. The following examples show that Theorems 2.1-2.4 differ from Theorems 1.1-1.3.

**Example 3.2.** Let $X=\mathbb{R}$ be endowed with the Euclidean metric $d=|\cdot|$. Let $\tau=\ln\frac{4}{3}$, $T:X\to CL(X)$, $F:(0,+\infty)\to\mathbb{R}$ and $\eta:(0,+\infty)\to(0,+\infty)$ be defined by

$$Tx=\begin{cases}(-\infty,2x]\cup\left[\frac{x}{2},0\right), & x\in(-\infty,0),\\ \left[0,\frac{x}{3}\right]\cup[3x,+\infty), & x\in[0,+\infty),\end{cases}$$

$$F(t)=\ln t,\quad \eta(t)=\ln\frac{6}{5},\quad \forall t\in(0,+\infty).$$

It is easy to see that

$$f(x)=d(x,Tx)=\begin{cases}-\frac{x}{2}, & x\in(-\infty,0),\\ \frac{2x}{3}, & x\in[0,+\infty),\end{cases}$$

is continuous in $X$,

$$\liminf_{s\to t^+}\eta(s)=\liminf_{s\to t^+}\ln\frac{6}{5}>0,\quad \forall t\in\mathbb{R}^+.$$

Put $x\in X-Tx$. In order to verify (a1) and (a3), we consider the following two possible cases:

**Case 1.** Let $x\in(-\infty,0)-Tx$. It follows that $x\in\left(2x,\frac{x}{2}\right)$. Put

$$y=\frac{x}{2}\in(-\infty,2x]\cup\left[\frac{x}{2},0\right)-(-\infty,x]\cup\left[\frac{x}{4},0\right)=Tx-Ty.$$

It follows that

$$F(d(x,y))=\ln\left|\frac{x}{2}\right|\le\ln\left|\frac{x}{2}\right|+\ln\frac{4}{3}=F(f(x))+\tau,$$

and

$$\begin{aligned}
F(f(y))+\tau+\eta(f(x))&=F(f(y))+\tau+\eta(d(x,y))\\
&=\ln\left|\frac{x}{4}\right|+\ln\frac{4}{3}+\ln\frac{6}{5}\\
&=\ln\left|\frac{2x}{5}\right|\le\ln\left|\frac{x}{2}\right|\\
&=F(d(x,y)).
\end{aligned}$$

**Case 2.** Let $x \in [0,+\infty) - Tx$. It follows that $x \in \left(\frac{x}{3}, 3x\right)$. Put

$$y = \frac{x}{3} \in \left[0, \frac{x}{3}\right] \cup [3x,+\infty) - \left[0, \frac{x}{9}\right] \cup [x,+\infty) = Tx - Ty.$$

It is clear that

$$F(d(x,y)) = \ln\frac{2x}{3} \leq \ln\frac{2x}{3} + \ln\frac{4}{3} = F(f(x)) + \tau,$$

and

$$\begin{aligned} F(f(y)) + \tau + \eta(f(x)) &= F(f(y)) + \tau + \eta(d(x,y)) \\ &= \ln\frac{2x}{9} + \ln\frac{4}{3} + \ln\frac{6}{5} \\ &= \ln\frac{16x}{45} \leq \ln\frac{2x}{3} \\ &= F(d(x,y)). \end{aligned}$$

That is, (a1) and (a3) hold. It follows from both of Theorems 2.1 and 2.2 that $T$ has a fixed point in $X$. However, the mapping $T$ does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3, respectively. In fact, put $x_0 = -1$ and $y_0 = 1$. It is clear that

$$\begin{aligned} H(Tx_0, Ty_0) &= H\left((-\infty,-2] \cup \left[-\frac{1}{2}, 0\right), \left[0, \frac{1}{3}\right] \cup [3,+\infty)\right) \\ &= +\infty \nleq 2r = rd(x_0,y_0), \quad \forall r \in [0,1), \end{aligned}$$

$$H(Tx_0, Ty_0) = +\infty \nleq 2\varphi(d(x_0,y_0)) = \varphi(d(x_0,y_0))d(x_0,y_0)$$

for any mapping $\varphi : (0,+\infty) \to [0,1)$ with each of (1.3) and (1.5).

**Example 3.3.** Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$, $F : (0,+\infty) \to \mathbb{R}$, $\eta : (0,+\infty) \to (0,+\infty)$ be defined by

$$Tx = \begin{cases} \left[0, \frac{x^2}{2}\right], & x \in [0,1], \\ \left[0, \frac{1}{4}\right], & x \in (1,+\infty), \end{cases}$$

$$F(t) = \ln t, \quad \eta(t) = \ln\frac{4}{3}, \quad \forall t \in (0,+\infty).$$

It is easy to see that

$$f(x) = d(x,Tx) = \begin{cases} x - \frac{x^2}{2}, & x \in [0,1], \\ x - \frac{1}{4}, & x \in (1,+\infty), \end{cases}$$

is lower semi-continuous in $X$,

$$\limsup_{s\to 0^+} \eta(s) = \ln\frac{4}{3} < +\infty, \quad \liminf_{s\to t^+} \eta(s) = \ln\frac{4}{3} > 0, \quad \forall t \in \mathbb{R}^+.$$

In order to verify (a4), we consider the following two possible cases:

**Case 1.** Let $x \in [0,1] \cap (X - Tx)$. It follows that $x \in \left(\frac{x^2}{2}, 1\right]$. Put $y = \frac{x^2}{2} \in \left[0, \frac{x^2}{2}\right] - \left[0, \frac{x^4}{8}\right] = Tx - Ty$. It follows that

$$F(d(x,y)) = \ln\left(x - \frac{x^2}{2}\right) \leq \ln\left(x - \frac{x^2}{2}\right) + \frac{1}{2}\ln\frac{4}{3} = F(f(x)) + \frac{1}{2}\eta(f(x)),$$

and

$$F(f(y)) + \eta(f(x)) = \ln\left(\frac{x^2}{2} - \frac{x^4}{8}\right) + \ln\frac{4}{3}$$

$$
\begin{aligned}
&= \ln\left(\frac{1}{2}\left(x+\frac{x^2}{2}\right)\right) + \ln\left(x-\frac{x^2}{2}\right) + \ln\frac{4}{3} \\
&\leq \ln\frac{3}{4} + \ln\left(x-\frac{x^2}{2}\right) + \ln\frac{4}{3} \\
&= F(d(x,y)).
\end{aligned}
$$

**Case 2.** Let $x \in (1,+\infty) \cap (X - Tx)$. It follows that $x \in (1,+\infty)$. Put $y = \frac{1}{4} \in \left[0,\frac{1}{4}\right] - \left[0,\frac{1}{32}\right] = Tx - Ty$. It is clear that

$$
F(d(x,y)) = \ln\left(x-\frac{1}{4}\right) \leq \ln\left(x-\frac{1}{4}\right) + \frac{1}{2}\ln\frac{4}{3} = F(f(x)) + \frac{1}{2}\eta(f(x)),
$$

and

$$
F(f(y)) + \eta(f(x)) = \ln\frac{7}{32} + \ln\frac{4}{3} = \ln\frac{7}{24} < \ln\frac{3}{4} < \ln\left(x-\frac{1}{4}\right) = F(d(x,y)).
$$

That is, (a4) holds. It follows from Theorem 2.3 that $T$ has a fixed point in $X$. However, the mappings $T$ does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3, respectively. In fact, put $x_0 = 1$ and $y_0 = \frac{9}{8}$. It is clear that

$$
\begin{aligned}
&H(Tx_0,Ty_0) = H\left(\left[0,\frac{1}{2}\right],\left[0,\frac{1}{4}\right]\right) = \frac{1}{4} \not\leq \frac{1}{8}c = cd(x_0,y_0), \quad \forall c \in [0,1), \\
&H(Tx_0,Ty_0) = \frac{1}{4} \not\leq \frac{1}{8}\varphi(d(x_0,y_0)) = \varphi(d(x_0,y_0))d(x_0,y_0)
\end{aligned}
$$

for any mapping $\varphi : (0,+\infty) \to [0,1)$ with each of (1.3) and (1.5).

**Example 3.4.** Let $X = [0,1]$ be endowed with the Euclidean metric $d = |\cdot|$. Let $T : X \to CL(X)$, $F : (0,+\infty) \to \mathbb{R}$, $\eta : (0,+\infty) \to (0,+\infty)$ be defined by

$$
\begin{aligned}
Tx &= \begin{cases} \left\{\frac{x^2}{3}\right\}, & x \in \left[0,\frac{17}{36}\right) \cup \left(\frac{17}{36},1\right], \\ \left[\frac{1}{8},\frac{5}{48}\right], & x = \frac{17}{36}, \end{cases} \\
F(t) &= \ln t, \quad \forall t \in (0,+\infty), \\
\eta(t) &= \begin{cases} \ln 10, & t \in \left[0,\frac{1}{10}\right), \\ \ln\frac{1}{t}, & t \in \left[\frac{1}{10},\frac{1}{5}\right), \\ \ln\frac{9}{4}, & t \in \left[\frac{1}{5},+\infty\right). \end{cases}
\end{aligned}
$$

It is easy to see that

$$
f(x) = d(x,Tx) = \begin{cases} x - \frac{x^2}{3}, & x \in \left[0,\frac{17}{36}\right) \cup \left(\frac{17}{36},1\right], \\ \frac{25}{72}, & x = \frac{17}{36} \end{cases}
$$

is lower semi-continuous in $X$ and

$$
\lim_{s\to 0^+} \eta(s) = \ln 10 > 0.
$$

Put $x \in X - Tx$. In order to verify (a6), we consider the following two possible cases:

**Case 1.** Let $x \in \left(0,\frac{17}{36}\right) \cup \left(\frac{17}{36},1\right] - \left\{\frac{x^2}{3}\right\}$. Put $y = \frac{x^2}{3} \in \left\{\frac{x^2}{3}\right\} - \left\{\frac{x^4}{27}\right\} = Tx - Ty$. Note that $x - \frac{x^2}{3} \in \left(0,\frac{2}{3}\right]$. Assume that $x - \frac{x^2}{3} \in \left(0,\frac{1}{10}\right)$. It follows that

$$
\frac{1}{3}\left(x+\frac{x^2}{3}\right) < x - \frac{x^2}{3} < \frac{1}{10},
$$

which yields that

$$\ln \frac{1}{3}\left(x+\frac{x^2}{3}\right)+\ln 10<0.$$

Consequently, we have

$$F(d(x,y))=\ln\left(x-\frac{x^2}{3}\right)\le \ln\left(x-\frac{x^2}{3}\right)+\frac{1}{2}\ln 10=F(f(x))+\frac{1}{2}\eta(d(x,y)),$$

and

$$\begin{aligned}F(f(y))+\eta(d(x,y))&=\ln\left(\frac{x^2}{3}-\frac{x^4}{27}\right)+\ln 10\\&=\ln\frac{1}{3}\left(x+\frac{x^2}{3}\right)+\ln\left(x-\frac{x^2}{3}\right)+\ln 10\\&<\ln\left(x-\frac{x^2}{3}\right)\\&=F(d(x,y)).\end{aligned}$$

Assume that $x-\frac{x^2}{3}\in\left[\frac{1}{10},\frac{1}{5}\right)$. It follows that

$$F(d(x,y))=\ln\left(x-\frac{x^2}{3}\right)\le \ln\left(x-\frac{x^2}{3}\right)+\frac{1}{2}\ln\frac{1}{(x-\frac{x^2}{3})}=F(f(x))+\frac{1}{2}\eta(d(x,y)),$$

and

$$\begin{aligned}F(f(y))+\eta(d(x,y))&=\ln\left(\frac{x^2}{3}-\frac{x^4}{27}\right)+\ln\frac{1}{(x-\frac{x^2}{3})}\\&=\ln\frac{1}{3}\left(x+\frac{x^2}{3}\right)+\ln\left(x-\frac{x^2}{3}\right)+\ln\frac{1}{(x-\frac{x^2}{3})}\\&=\ln\frac{1}{3}\left(x+\frac{x^2}{3}\right)<\ln\left(x-\frac{x^2}{3}\right)\\&=F(d(x,y)).\end{aligned}$$

Assume that $x-\frac{x^2}{3}\in\left[\frac{1}{5},+\infty\right)$. It follows that

$$F(d(x,y))=\ln\left(x-\frac{x^2}{3}\right)\le \ln\left(x-\frac{x^2}{3}\right)+\frac{1}{2}\ln\frac{9}{4}=F(f(x))+\frac{1}{2}\eta(d(x,y)),$$

and

$$\begin{aligned}F(f(y))+\eta(d(x,y))&=\ln\left(\frac{x^2}{3}-\frac{x^4}{27}\right)+\ln\frac{9}{4}\\&=\ln\frac{1}{3}\left(x+\frac{x^2}{3}\right)+\ln\left(x-\frac{x^2}{3}\right)+\ln\frac{9}{4}\\&\le\ln\frac{4}{9}+\ln\left(x-\frac{x^2}{3}\right)+\ln\frac{9}{4}=\ln\left(x-\frac{x^2}{3}\right)\\&=F(d(x,y)).\end{aligned}$$

**Case 2.** Let $x=\frac{17}{36}$. Put $y=\frac{1}{8}\in\left\{\frac{1}{8},\frac{5}{48}\right\}-\left\{\frac{1}{192}\right\}=Tx-Ty$. It follows that

$$F(d(x,y))=\ln\frac{25}{72}\le\ln\frac{25}{72}+\frac{1}{2}\ln\frac{9}{4}=F(f(x))+\frac{1}{2}\eta(d(x,y)),$$

and

$$F(f(y))+\eta(d(x,y))=\ln\frac{23}{192}+\ln\frac{9}{4}<-1.31<-1.06<\ln\frac{25}{72}=F(d(x,y)).$$

That is, (a6) holds. It follows from Theorem 2.4 that $T$ has a fixed point in $X$. However, the mappings $T$ does not satisfy (1.1), (1.2) and (1.4) in Theorems 1.1-1.3, respectively. In fact, put $x_0 = \frac{1}{2}$ and $y_0 = \frac{17}{36}$. It is clear that

$$H(Tx_0, Ty_0) = H\left(\frac{1}{12}, \left\{\frac{1}{8}, \frac{5}{48}\right\}\right) = \frac{1}{24} = \frac{1}{36} \cdot \frac{3}{2} \nleq \frac{1}{36}c = cd(x_0, y_0), \quad \forall c \in [0, 1),$$
$$H(Tx_0, Ty_0) = \frac{1}{24} \nleq \frac{1}{36}\varphi(d(x_0, y_0)) = \varphi(d(x_0, y_0))d(x_0, y_0)$$

for any mapping $\varphi : (0, +\infty) \to [0, 1)$ with each of (1.3) and (1.5).

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