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On new traveling wave solutions of potential KdV and (3+1)-dimensional Burgers equations

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Abstract

This paper acquires soliton solutions of the potential KdV (PKdV) equation and the (3+1)-dimensional Burgers equation (BE) by the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$ expansion method (EM). Obtained soliton solutions are designated in terms of kink, bell-shaped solitary wave, periodic and singular periodic wave solutions. These solutions may be useful and desirable to explain some nonlinear physical phenomena.

Keywords: $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM, the PKdV equation, the (3+1)-dimensional BE, hyperbolic solution, periodic solution, rational solution.
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1. Introduction

In applied sciences, each physical event can be modeled mathematically. Nonlinear partial differential equations (NPDEs) have an important place for solution of problems in mechanic and geometry of the surface work. So, it is important to have information about general solutions of these type of problems.

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There is no general method to obtain general solutions of NPDEs. It is generally used many transformation methods. These transformation methods are used to convert NPDEs to ODEs. Most of these methods are based on finding balance term with balancing of the highest order linear and nonlinear term with the help of the homogeneous balance method. Hence, these type methods only use for NPDEs. Some of these methods are: tanh methods [3, 4, 11, 12, 19], Exp-function method [7], Jacobi elliptic function method [5], $\frac{G'}{G}$-EM [13] and the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM [8].

$\frac{G'}{G}$-expansion method firstly was presented by Wang et al. in 2008. This method contains an auxiliary equation as $G'' + \lambda G' + \mu G = 0$ and includes a solution series as $\sum_{i=0}^{M} a_i \left(\frac{G'}{G}\right)^i$. Later, Guo and Zhou introduced to extended $\frac{G'}{G}$-EM [6] inspired by the $\frac{G'}{G}$-EM in 2010. They used a solution series as $a_0 + \sum_{i=1}^{M} \left\{ a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\sigma\left(1+\frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} \right\}$ for same auxiliary equation. They obtained more different solutions by using the extended $\frac{G'}{G}$-EM. Finally, Lü et al. improved the generalized $\frac{G'}{G}$-EM [9]. They chose auxiliary equation as $f' = h_0 + h_1 f + h_2 f^2 + h_3 f^3$ and used solution series as $a_0 + \sum_{i=1}^{M} a_i \left(\frac{f'}{f}\right)^i$. Therefore, they found more solutions.

Recently, Li et al. [8] have presented the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM. They used auxiliary equation as $G''(\xi) + \lambda G(\xi) - \mu = 0$ and considered solutions series as $\sum_{i=0}^{M} a_i \phi^i + \sum_{i=1}^{M} b_i \phi^{i-1}\psi$ for $\phi = \frac{G'}{G}, \psi = \frac{1}{G}$. According to the studies in the literature, the solutions obtained by this method are more general from solutions which are obtained by using the $\frac{G'}{G}$- expansion method, the extended $\frac{G'}{G}$- and the generalized $\frac{G'}{G}$-EMs. To see these differences, we refer the reader to [16–18].

In this study, we obtain hyperbolic, periodic and rational solutions for the PKdV equation [14], the (3+1)-dimensional BE [2] by the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM.

2. An analysis of the method

We give a simple illustration of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM. Let us consider the ODE

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{2.1}$$

where $\phi = \frac{G'}{G}, \psi = \frac{1}{G}$, then we acquire

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{2.2}$$

Step 1. If $\lambda < 0$, we have the solution of Eq. (2.1)

$$G(\xi) = A_1 \sinh\left(\sqrt{-\lambda}\xi\right) + A_2 \cosh\left(\sqrt{-\lambda}\xi\right) + \frac{\mu}{\lambda},$$

where A_1 and A_2 are arbitrary constants. Thus, we acquire

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{2.3}$$

where $\sigma = A_1^2 - A_2^2$.

Step 2. If $\lambda > 0$, we write the solution of Eq. (2.1)

$$G(\xi) = A_1 \sin\left(\sqrt{\lambda}\xi\right) + A_2 \cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda},$$

and therefore,

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right).$$

Step 3. If $\lambda = 0$, we obtain the solution of Eq. (2.1)

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2,$$

and therefore,

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right).$$

Now, let us illustrate how this method works. Therefore, let us consider an NLPDE is given by

$$Q\left(u, u_t, u_x, u_{xx}, u_{tt}, ...\right) = 0,$$

where $u = u(x,t)$ is an unknown function. By using the transformation $u(x,t) = u(\xi)$, $\xi = x - Vt$ then we get a nonlinear ODE for $u(\xi)$

$$Q' = \left(u, u', u'', ...\right) = 0. \tag{2.4}$$

Assume that the solutions of Eq. (2.4) can be expressed by a polynomial ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{M} a_i\phi^i + \sum_{i=1}^{M} b_i\phi^{i-1}\psi, \tag{2.5}$$

where $a_i\,(i = 0, 1, ..., M)$ and $b_i\,(i = 1, ..., M)$ are constants to be determined later. M is a positive integer that can be determined by balancing the highest order derivative and with the highest nonlinear terms in Eq. (2.4). Substitute Eq. (2.5) into Eq. (2.4) along with Eq. (2.2) and Eq. (2.3), the Eq. (2.4) can be converted into a polynomial in ϕ and ψ. Equating the coefficients of each power of $\phi^i\psi^j$ to zero yields a system of algebraic equation for a_i, b_i, V, μ and λ. We solve this algebraic equation with the aid of Mathematica 7.0. Thus, we obtain the general solutions in terms of the hyperbolic functions for $\lambda < 0$. We acquire the general solutions in terms of the trigonometric functions for $\lambda > 0$ and we have the general solutions in terms of the rational function for $\lambda = 0$.

3. Applications

Example 3.1. The potential KdV equation has the form

$$u_t + 3u_x^2 + u_{xxx} = 0. \tag{3.1}$$

If we use $u(x,t) = u(\xi)$, $\xi = x - Vt$, Eq. (3.1) becomes

$$-Vu' + 3\left(u'\right)^2 + u''' = 0, \tag{3.2}$$

where V is velocity of soliton. We acquire balance $M = 1$. Thus, we choose a solution of Eq. (3.2) as

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi). \tag{3.3}$$

Case 1.1. For $\lambda < 0$, substituting Eq. (3.3) into Eq. (3.2) and by using Eq. (2.2) and Eq. (2.3), we acquire the system for $a_0, a_1, b_1, \mu, \sigma, \lambda$ and V. These systems are

$$a_1V\lambda - 2a_1\lambda^2 + 3a_1^2\lambda^2 + \frac{1}{\mu^2 + \lambda^2\sigma}3a_1\lambda^2\mu^2 - \frac{1}{\mu^2 + \lambda^2\sigma}3a_1^2\lambda^2\mu^2 = 0,$$

$$
\begin{aligned}
-a_1V\mu+5a_1\lambda\mu-6a_1^2\lambda\mu-\frac{1}{\mu^2+\lambda^2\sigma}6a_1\lambda\mu^3+\frac{1}{\mu^2+\lambda^2\sigma}6a_1^2\lambda\mu^3&=0,\\
b_1V-5b_1\lambda+6a_1b_1\lambda+\frac{1}{\mu^2+\lambda^2\sigma}12b_1\lambda\mu^2-\frac{1}{\mu^2+\lambda^2\sigma}12a_1b_1\lambda\mu^2&=0,\\
a_1V-8a_1\lambda+6a_1^2\lambda-\frac{1}{\mu^2+\lambda^2\sigma}3b_1^2\lambda^2+\frac{1}{\mu^2+\lambda^2\sigma}3a_1\lambda\mu^2-\frac{1}{\mu^2+\lambda^2\sigma}3a_1^2\lambda\mu^2&=0,\\
12a_1\mu-6a_1^2\mu+\frac{1}{\mu^2+\lambda^2\sigma}6b_1^2\lambda\mu&=0,\\
-6a_1+3a_1^2-\frac{1}{\mu^2+\lambda^2\sigma}3b_1^2\lambda&=0,\\
-\frac{1}{\mu^2+\lambda^2\sigma}6b_1\lambda\mu+\frac{1}{\mu^2+\lambda^2\sigma}6a_1b_1\lambda\mu&=0,\\
-\frac{1}{\mu^2+\lambda^2\sigma}6b_1\lambda^2\mu+\frac{1}{\mu^2+\lambda^2\sigma}6a_1b_1\lambda^2\mu&=0,\\
-6b_1+6a_1b_1&=0.
\end{aligned}
\tag{3.4}
$$

We obtain the roots of Eq. (3.4) with the aid of Mathematica as

$$
a_1=1,\quad \lambda\neq 0,\quad b_1=\pm\frac{1}{\sqrt{\lambda}}\sqrt{-\mu^2-\lambda^2\sigma},\quad V=-\lambda,\quad \mu^2+\lambda^2\sigma\neq 0. \tag{3.5}
$$

Substituting Eq. (3.5) into Eq. (3.3), we acquire the following solutions of Eq. (3.1):
Family 1.1.

$$
\begin{aligned}
u(x,t)=a_0&+\left(\frac{A_1\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)\\
&\pm\frac{\sqrt{-\mu^2-\lambda^2\sigma}}{\sqrt{\lambda}}\frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}},
\end{aligned}
\tag{3.6}
$$

where $\sigma=A_1^2-A_2^2$ and $\xi=x+\lambda t$.

Setting $A_1=0$, $A_2>0$ and $\mu=0$ in Eq. (3.6), we have the hyperbolic solution

$$
u(x,t)=a_0\pm\frac{1}{A_2}\sqrt{-\lambda\sigma}\ \mathrm{sech}\left(\sqrt{-\lambda}(x+\lambda t)\right)+\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}(x+\lambda t)\right). \tag{3.7}
$$

Setting $A_2=0$, $A_1>0$ and $\mu=0$ in Eq. (3.6), we obtain the hyperbolic solution

$$
u(x,t)=a_0+\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}(x+\lambda t)\right)\pm\frac{1}{A_1}\sqrt{-\lambda\sigma}\mathrm{csch}\left(\sqrt{-\lambda}(x+\lambda t)\right).
$$

Case 1.2. For $\lambda>0$, substituting Eq. (3.3) into Eq. (3.2) and by using Eq. (2.2) and Eq. (2.3), we acquire the system for $a_0,a_1,b_1,\mu,\sigma,\lambda$ and V. These systems are

$$
\begin{aligned}
a_1V-8a_1\lambda+6a_1^2\lambda+\frac{1}{-\mu^2+\lambda^2\sigma}3b_1^2\lambda^2-\frac{1}{-\mu^2+\lambda^2\sigma}3a_1\lambda\mu^2+\frac{1}{-\mu^2+\lambda^2\sigma}3a_1^2\lambda\mu^2&=0,\\
a_1V\lambda-2a_1\lambda^2+3a_1^2\lambda^2-\frac{1}{-\mu^2+\lambda^2\sigma}3a_1\lambda^2\mu^2+\frac{1}{-\mu^2+\lambda^2\sigma}3a_1^2\lambda^2\mu^2&=0,\\
-a_1V\mu+5a_1\lambda\mu-6a_1^2\lambda\mu+\frac{1}{-\mu^2+\lambda^2\sigma}6a_1\lambda\mu^3-\frac{1}{-\mu^2+\lambda^2\sigma}6a_1^2\lambda\mu^3&=0,
\end{aligned}
$$

$$
\begin{aligned}
&b_1V-5b_1\lambda+6a_1b_1\lambda-\frac{1}{-\mu^2+\lambda^2\sigma}12b_1\lambda\mu^2+\frac{1}{-\mu^2+\lambda^2\sigma}12a_1b_1\lambda\mu^2=0,\\
&\frac{1}{-\mu^2+\lambda^2\sigma}6b_1\lambda^2\mu-\frac{1}{-\mu^2+\lambda^2\sigma}6a_1b_1\lambda^2\mu=0,\\
&12a_1\mu-6a_1^2\mu-\frac{1}{-\mu^2+\lambda^2\sigma}6b_1^2\lambda\mu=0,\\
&\frac{1}{-\mu^2+\lambda^2\sigma}6b_1\lambda\mu-\frac{1}{-\mu^2+\lambda^2\sigma}6a_1b_1\lambda\mu=0\\
&-6a_1+3a_1^2+\frac{1}{-\mu^2+\lambda^2\sigma}3b_1^2\lambda=0,\\
&-6b_1+6a_1b_1=0.
\end{aligned}
\tag{3.8}
$$

We have the roots of Eq. (3.8) with the aid of Mathematica as

$$
a_1=1,\quad \lambda\neq 0,\quad b_1=\pm\frac{1}{\sqrt{\lambda}}\sqrt{-\mu^2+\lambda^2\sigma},\quad V=-\lambda,\quad \mu^2+\lambda^2\sigma\neq 0.
$$

Thus, we acquire the following solutions of Eq. (3.1).

Family 1.2.

$$
\begin{aligned}
u(x,t)=a_0&+\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right)-A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right)}{A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)\\
&\pm\frac{\sqrt{-\mu^2+\lambda^2\sigma}}{\sqrt{\lambda}}\frac{1}{A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right)+\frac{\mu}{\lambda}},
\end{aligned}
\tag{3.9}
$$

where $\sigma=A_1^2+A_2^2$ and $\xi=x+\lambda t$.

Setting $A_1=0$, $A_2>0$ and $\mu=0$ in Eq. (3.9), we find the trigonometric solution

$$
u(x,t)=a_0\pm\frac{1}{A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}(x+\lambda t)\right)-\sqrt{\lambda}\tan\left(\sqrt{\lambda}(x+\lambda t)\right).
$$

Setting $A_2=0$, $A_1>0$ and $\mu=0$ in Eq. (3.9), we have the trigonometric solution

$$
u(x,t)=a_0+\sqrt{\lambda}\cot\left(\sqrt{\lambda}(x+\lambda t)\right)\pm\frac{1}{A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}(x+\lambda t)\right).
$$

Case 1.3. For $\lambda=0$, we acquire the following system:

$$
\begin{aligned}
&b_1V-\frac{1}{A_1^2-2A_2\mu}12b_1\mu^2+\frac{1}{A_1^2-2A_2\mu}12a_1b_1\mu^2=0,\\
&a_1V-\frac{1}{A_1^2-2A_2\mu}3a_1\mu^2+\frac{1}{A_1^2-2A_2\mu}3a_1^2\mu^2=0,\\
&-a_1V\mu+\frac{1}{A_1^2-2A_2\mu}6a_1\mu^3-\frac{1}{A_1^2-2A_2\mu}6a_1^2\mu^3=0,\\
&\frac{1}{A_1^2-2A_2\mu}6b_1\mu-\frac{1}{A_1^2-2A_2\mu}6a_1b_1\mu=0,\\
&12a_1\mu-6a_1^2\mu-\frac{1}{A_1^2-2A_2\mu}6b_1^2\mu=0,
\end{aligned}
\tag{3.10}
$$

$$-6a_1 + 3a_1^2 + \frac{1}{A_1^2 - 2A_2\mu} 3b_1^2 = 0,$$
$$-6b_1 + 6a_1 b_1 = 0.$$

We have the roots of Eq. (3.10) with the aid of Mathematica as

$$a_1 = 1, \quad b_1 = \pm\sqrt{A_1^2 - 2A_2\mu}, \quad V = 0, \quad A_1^2 - 2A_2\mu \neq 0.$$

Family 1.3 It is impossible to write solution of Eq. (3.1) because $V = 0$.

Remark 3.2. Wang et al. [14] obtained the singular 1-soliton solution by using the ansatz method for Eq. (3.1). Also, they found hyperbolic and trigonometric solutions by the aid of $\frac{G'}{G}$-expansion method. Our solutions (3.6) and (3.9) are different from their solutions (17) and (18), and more general.

Example 3.3. Let us consider the (3+1)-dimensional BE of the form,

$$\begin{aligned} u_t - 2uu_y - 2vu_x - 2wu_z - u_{xx} - u_{yy} - u_{zz} = 0, \\ u_x - v_y = 0, \\ u_z - w_y = 0. \end{aligned} \tag{3.11}$$

If we use $u(x,t) = u(\xi)$, $\xi = x - Vt$, Eq. (3.11) becomes

$$\begin{aligned} -Vu' - 2uu' - 2vu' - 2wu' - 3u'' = 0, \\ u' - v' = 0, \\ u' - w' = 0. \end{aligned} \tag{3.12}$$

From Eq. (3.12), we have

$$\begin{aligned} u(\xi) &= a_0 + a_1\phi(\xi) + b_1\psi(\xi), \\ w(\xi) &= c_0 + c_1\phi(\xi) + d_1\psi(\xi), \\ v(\xi) &= e_0 + e_1\phi(\xi) + f_1\psi(\xi). \end{aligned} \tag{3.13}$$

Case 2.1. For $\lambda < 0$, we acquire the following another systems:

$$\begin{aligned} &2a_0a_1\lambda + 2a_1c_0\lambda + 2a_1e_0\lambda + a_1V\lambda - \frac{1}{\mu^2+\lambda^2\sigma}3b_1\lambda^2\mu + \frac{1}{\mu^2+\lambda^2\sigma}2a_1b_1\lambda^2\mu \\ &\quad + \frac{1}{\mu^2+\lambda^2\sigma}2a_1d_1\lambda^2\mu + \frac{1}{\mu^2+\lambda^2\sigma}2a_1f_1\lambda^2\mu = 0, \\ \\ &-3b_1\lambda + 2a_1b_1\lambda + 2a_1d_1\lambda + 2a_1f_1\lambda - 2a_0a_1\mu - 2a_1c_0\mu - 2a_1e_0\mu - a_1V\mu + \frac{1}{\mu^2+\lambda^2\sigma}6b_1\lambda\mu^2 \\ &\quad - \frac{1}{\mu^2+\lambda^2\sigma}4a_1b_1\lambda\mu^2 - \frac{1}{\mu^2+\lambda^2\sigma}4a_1d_1\lambda\mu^2 - \frac{1}{\mu^2+\lambda^2\sigma}4a_1f_1\lambda\mu^2 = 0, \\ \\ &6a_1\lambda + 2a_1^2\lambda + 2a_1c_1\lambda + +2a_1e_1\lambda - \frac{1}{\mu^2+\lambda^2\sigma}2b_1^2\lambda^2 - \frac{1}{\mu^2+\lambda^2\sigma}2b_1d_1\lambda^2 - \frac{1}{\mu^2+\lambda^2\sigma}2b_1f_1\lambda^2 = 0, \\ \\ &2a_0b_1 + 2b_1c_0 + 2b_1e_0 + b_1V + 9a_1\mu - 2a_1^2\mu - 2a_1c_1\mu - 2a_1e_1\mu + \frac{1}{\mu^2+\lambda^2\sigma}4b_1^2\lambda\mu \\ &\quad + \frac{1}{\mu^2+\lambda^2\sigma}4b_1d_1\lambda\mu + \frac{1}{\mu^2+\lambda^2\sigma}4b_1f_1\lambda\mu = 0, \end{aligned} \tag{3.14}$$

$$2a_0a_1+2a_1c_0+2a_1e_0+a_1V-\frac{1}{\mu^2+\lambda^2\sigma}3b_1\lambda\mu+\frac{1}{\mu^2+\lambda^2\sigma}2a_1b_1\lambda\mu$$
$$+\frac{1}{\mu^2+\lambda^2\sigma}2a_1d_1\lambda\mu+\frac{1}{\mu^2+\lambda^2\sigma}2a_1f_1\lambda\mu=0,$$

$$-6a_1+2a_1^2+2a_1c_1+2a_1e_1-\frac{1}{\mu^2+\lambda^2\sigma}2b_1^2\lambda-\frac{1}{\mu^2+\lambda^2\sigma}2b_1d_1\lambda-\frac{1}{\mu^2+\lambda^2\sigma}2b_1f_1\lambda=0,$$
$$-6b_1+4a_1b_1+2b_1c_1+2a_1d_1+2b_1e_1+2a_1f_1=0.$$

We find the roots of Eq. (3.14) with the aid of Mathematica as

$$\begin{gathered}a_0=a_0,\quad a_1=\frac{1}{2},b_1=\pm\frac{1}{2\sqrt{\lambda}}\sqrt{-\mu^2-\lambda^2\sigma},\\ c_0=c_0,\quad c_1=\frac{1}{2},d_1=\pm\frac{1}{2\sqrt{\lambda}}\sqrt{-\mu^2-\lambda^2\sigma},\\ e_0=e_0,\quad e_1=\frac{1}{2},f_1=\pm\frac{1}{2\sqrt{\lambda}}\sqrt{-\mu^2-\lambda^2\sigma},\\ \lambda\neq 0,\quad V=-2\left(a_0+c_0+e_0\right),\quad \mu^3+\lambda^2\mu\sigma\neq 0.\end{gathered}\tag{3.15}$$

Substituting Eq. (3.15) into Eq. (3.13), we obtain the following solutions of Eq. (3.11).
Family 2.1.

$$\begin{aligned}u\left(x,y,z,t\right)&=a_0+\frac{1}{2}\left(\frac{A_1\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)\\&\pm\frac{\sqrt{-\mu^2-\lambda^2\sigma}}{2\sqrt{\lambda}}\frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}},\\ v\left(x,y,z,t\right)&=e_0+\frac{1}{2}\left(\frac{A_1\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)\\&\pm\frac{\sqrt{-\mu^2-\lambda^2\sigma}}{2\sqrt{\lambda}}\frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}},\\ w\left(x,y,z,t\right)&=c_0+\frac{1}{2}\left(\frac{A_1\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)\\&\pm\frac{\sqrt{-\mu^2-\lambda^2\sigma}}{2\sqrt{\lambda}}\frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}},\end{aligned}\tag{3.16}$$

where $\sigma=A_1^2-A_2^2$ and $\xi=x+y+z+2\left(a_0+c_0+e_0\right)t$.

Setting $A_1=0$, $A_2>0$ and $\mu=0$ in Eq. (3.16), we have the hyperbolic solution

$$\begin{aligned}u\left(x,y,z,t\right)&=a_0\pm\frac{1}{A_2}\sqrt{-\lambda\sigma}\operatorname{sech}\left(\sqrt{-\lambda}\xi\right)+\frac{1}{2}\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right),\\ v\left(x,y,z,t\right)&=e_0\pm\frac{1}{A_2}\sqrt{-\lambda\sigma}\operatorname{sech}\left(\sqrt{-\lambda}\xi\right)+\frac{1}{2}\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right),\\ w\left(x,y,z,t\right)&=c_0\pm\frac{1}{A_2}\sqrt{-\lambda\sigma}\operatorname{sech}\left(\sqrt{-\lambda}\xi\right)+\frac{1}{2}\sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\xi\right).\end{aligned}$$

Setting $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ in Eq. (3.16), we obtain the hyperbolic solutions

$$\begin{aligned}
u(x,y,z,t) &= a_0 + \frac{1}{2}\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) \pm \frac{1}{2A_1}\sqrt{-\lambda\sigma}\operatorname{csch}\left(\sqrt{-\lambda}\xi\right),\\
v(x,y,z,t) &= e_0 + \frac{1}{2}\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) \pm \frac{1}{2A_1}\sqrt{-\lambda\sigma}\operatorname{csch}\left(\sqrt{-\lambda}\xi\right),\\
w(x,y,z,t) &= c_0 + \frac{1}{2}\sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\xi\right) \pm \frac{1}{2A_1}\sqrt{-\lambda\sigma}\operatorname{csch}\left(\sqrt{-\lambda}\xi\right).
\end{aligned} \tag{3.17}$$

Case 2.2. For $\lambda > 0$, substituting Eq. (3.13) into Eq. (3.12) and by using Eq. (2.2) and Eq. (2.3), it yields a set of algebraic equations for $a_0, a_1, b_1, c_0, c_1, d_1, e_0, e_1, f_1, \mu, \sigma, \lambda$ and V. These systems are

$$\begin{aligned}
&2a_0a_1\lambda + 2a_1c_0\lambda + 2a_1e_0\lambda + a_1V\lambda + \frac{1}{-\mu^2+\lambda^2\sigma}3b_1\lambda^2\mu - \frac{1}{-\mu^2+\lambda^2\sigma}2a_1b_1\lambda^2\mu\\
&\quad - \frac{1}{-\mu^2+\lambda^2\sigma}2a_1d_1\lambda^2\mu - \frac{1}{-\mu^2+\lambda^2\sigma}2a_1f_1\lambda^2\mu = 0,\\
&-3b_1\lambda + 2a_1b_1\lambda + 2a_1d_1\lambda + 2a_1f_1\lambda - 2a_0a_1\mu - 2a_1c_0\mu - 2a_1e_0\mu - a_1V\mu - \frac{1}{-\mu^2+\lambda^2\sigma}6b_1\lambda\mu^2\\
&\quad + \frac{1}{-\mu^2+\lambda^2\sigma}4a_1b_1\lambda\mu^2 + \frac{1}{-\mu^2+\lambda^2\sigma}4a_1d_1\lambda\mu^2 + \frac{1}{-\mu^2+\lambda^2\sigma}4a_1f_1\lambda\mu^2 = 0,\\
&-6a_1\lambda + 2a_1^2\lambda + 2a_1c_1\lambda + +2a_1e_1\lambda + \frac{1}{-\mu^2+\lambda^2\sigma}2b_1^2\lambda^2\\
&\quad + \frac{1}{-\mu^2+\lambda^2\sigma}2b_1d_1\lambda^2 + \frac{1}{-\mu^2+\lambda^2\sigma}2b_1f_1\lambda^2 = 0,\\
&2a_0a_1 + 2a_1c_0 + 2a_1e_0 + a_1V + \frac{1}{-\mu^2+\lambda^2\sigma}3b_1\lambda\mu - \frac{1}{-\mu^2+\lambda^2\sigma}4a_1b_1\lambda\mu\\
&\quad - \frac{1}{-\mu^2+\lambda^2\sigma}2a_1d_1\lambda\mu - \frac{1}{-\mu^2+\lambda^2\sigma}2a_1f_1\lambda\mu = 0,\\
&-6a_1 + 2a_1^2 + 2a_1c_1 + 2a_1e_1 + \frac{1}{-\mu^2+\lambda^2\sigma}2b_1^2\lambda + \frac{1}{-\mu^2+\lambda^2\sigma}2b_1d_1\lambda + \frac{1}{-\mu^2+\lambda^2\sigma}2b_1f_1\lambda = 0,\\
&-6b_1 + 4a_1b_1 + 2b_1c_1 + 2a_1d_1 + 2b_1e_1 + 2a_1f_1 - a_1\lambda + e_1\lambda = 0,\\
&a_1\mu - e_1\mu = 0,\\
&-b_1 + f_1 = 0,\\
&-a_1 + e_1 = 0,\\
&-a_1\lambda + c_1\lambda = 0,\\
&a_1\mu - c_1\mu = 0,\\
&-b_1 + d_1 = 0,\\
&-a_1 + c_1 = 0.
\end{aligned} \tag{3.18}$$

We find the roots of Eq. (3.18) with the aid of Mathematica as

$$a_0 = a_0, \quad a_1 = \frac{1}{2}, \quad b_1 = \pm\frac{1}{2\sqrt{\lambda}}\sqrt{-\mu^2+\lambda^2\sigma},$$

$$
\begin{gathered}
c_0 = c_0, \quad c_1 = \frac{1}{2}, \quad d_1 = \pm\frac{1}{2\sqrt{\lambda}}\sqrt{-\mu^2+\lambda^2\sigma}, \\
e_0 = e_0, \quad e_1 = \frac{1}{2}, \quad f_1 = \pm\frac{1}{2\sqrt{\lambda}}\sqrt{-\mu^2+\lambda^2\sigma}, \\
\lambda \neq 0, \quad V = -2\left(a_0+c_0+e_0\right), \quad \mu^3+\lambda^2\mu\sigma \neq 0.
\end{gathered} \tag{3.19}
$$

Substituting Eq. (3.19) into Eq. (3.13), we acquire the following solutions of Eq. (3.11).

Family 2.2

$$
\begin{aligned}
u\left(x,y,z,t\right) = a_0 &+ \frac{1}{2}\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right) - A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right)}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}}\right) \\
&\pm \frac{\sqrt{-\mu^2+\lambda^2\sigma}}{2\sqrt{\lambda}}\frac{1}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}}, \\
v\left(x,y,z,t\right) = e_0 &+ \frac{1}{2}\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right) - A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right)}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}}\right) \\
&\pm \frac{\sqrt{-\mu^2+\lambda^2\sigma}}{2\sqrt{\lambda}}\frac{1}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}}, \\
w\left(x,y,z,t\right) = c_0 &+ \frac{1}{2}\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\xi\right) - A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\xi\right)}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}}\right) \\
&\pm \frac{\sqrt{-\mu^2+\lambda^2\sigma}}{2\sqrt{\lambda}}\frac{1}{A_1\sin\left(\sqrt{\lambda}\xi\right) + A_2\cos\left(\sqrt{\lambda}\xi\right) + \frac{\mu}{\lambda}},
\end{aligned} \tag{3.20}
$$

where $\sigma = A_1^2 + A_2^2$ and $\xi = x + y + z + 2\left(a_0 + c_0 + e_0\right)t$.

Setting $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ in Eq. (3.20), we have the trigonometric solutions

$$
\begin{aligned}
u\left(x,y,z,t\right) &= a_0 \pm \frac{1}{2A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}\xi\right) - \frac{1}{2}\sqrt{\lambda}\tan\left(\sqrt{\lambda}\xi\right), \\
v\left(x,y,z,t\right) &= e_0 \pm \frac{1}{2A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}\xi\right) - \frac{1}{2}\sqrt{\lambda}\tan\left(\sqrt{\lambda}\xi\right), \\
w\left(x,y,z,t\right) &= c_0 \pm \frac{1}{2A_2}\sqrt{\lambda\sigma}\sec\left(\sqrt{\lambda}\xi\right) - \frac{1}{2}\sqrt{\lambda}\tan\left(\sqrt{\lambda}\xi\right).
\end{aligned}
$$

Setting $A_2 = 0$, $A_1 > 0$ and $\mu = 0$ in Eq. (3.20), we have the trigonometric solutions

$$
\begin{aligned}
u\left(x,y,z,t\right) &= a_0 + \frac{1}{2}\sqrt{\lambda}\cot\left(\sqrt{\lambda}\xi\right) \pm \frac{1}{2A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}\xi\right), \\
v\left(x,y,z,t\right) &= e_0 + \frac{1}{2}\sqrt{\lambda}\cot\left(\sqrt{\lambda}\xi\right) \pm \frac{1}{2A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}\xi\right), \\
w\left(x,y,z,t\right) &= c_0 + \frac{1}{2}\sqrt{\lambda}\cot\left(\sqrt{\lambda}\xi\right) \pm \frac{1}{2A_1}\sqrt{\lambda\sigma}\csc\left(\sqrt{\lambda}\xi\right).
\end{aligned} \tag{3.21}
$$

Case 2.3. For $\lambda = 0$, we acquire the algebraic system

$$
\begin{aligned}
&-2a_0a_1\mu - 2a_1c_0\mu - 2a_1e_0\mu - a_1V\mu - \frac{1}{A_1^2 - 2A_2\mu}6b_1\mu^2 + \frac{1}{A_1^2 - 2A_2\mu}4a_1b_1\mu^2 \\
&\quad + \frac{1}{A_1^2 - 2A_2\mu}4a_1d_1\mu^2 + \frac{1}{A_1^2 - 2A_2\mu}4a_1f_1\mu^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2a_0b_1 + 2b_1c_0 + 2b_1e_0 + b_1V + 9a_1\mu - 2a_1^2\mu - 2a_1c_1\mu - 2a_1e_1\mu - \frac{1}{A_1^2 - 2A_2\mu}4b_1^2\mu \\
&\quad - \frac{1}{A_1^2 - 2A_2\mu}4b_1d_1\lambda\mu - \frac{1}{A_1^2 - 2A_2\mu}4b_1f_1\lambda\mu = 0,
\end{aligned}
$$

$$
\begin{gathered}
2a_0a_1 + 2a_1c_0 + 2a_1e_0 + a_1V + \frac{1}{A_1^2 - 2A_2\mu}3b_1\mu - \frac{1}{A_1^2 - 2A_2\mu}2a_1b_1\mu \\
- \frac{1}{A_1^2 - 2A_2\mu}2a_1d_1\mu - \frac{1}{A_1^2 - 2A_2\mu}2a_1f_1\mu = 0, \\
-6a_1 + 2a_1^2 + 2a_1c_1 + 2a_1e_1 + \frac{1}{A_1^2 - 2A_2\mu}2b_1f_1 + \frac{1}{A_1^2 - 2A_2\mu}2b_1^2 + \frac{1}{A_1^2 - 2A_2\mu}2b_1d_1 = 0, \\
-6b_1 + 4a_1b_1 + 2b_1c_1 + 2a_1d_1 + 2b_1e_1 + 2a_1f_1 = 0, \\
a_1\mu - e_1\mu = 0, \\
-b_1 + d_1 = 0, \\
-a_1 + c_1 = 0, \\
a_1\mu - c_1\mu = 0, \\
-b_1 + f_1 = 0, \\
-a_1 + e_1 = 0.
\end{gathered}
\tag{3.22}
$$

We find the roots of Eq. (3.22) with the aid of Mathematica as

$$
\begin{gathered}
a_0 = a_0, \quad a_1 = \frac{1}{2}, \quad b_1 = \pm\frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}, \\
c_0 = c_0, \quad c_1 = \frac{1}{2}, \quad d_1 = \pm\frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}, \\
e_0 = e_0, \quad e_1 = \frac{1}{2}, \quad f_1 = \pm\frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}, \\
\lambda \neq 0, \quad V = -2\left(a_0 + c_0 + e_0\right), \quad A_1^2\mu - 2A_2\mu^2 \neq 0.
\end{gathered}
\tag{3.23}
$$

Substituting Eq. (3.23) into Eq. (3.13), we obtain the rational solutions

$$
u\left(x, y, z, t\right) = a_0 + \frac{1}{2\left(\frac{\mu}{2}\xi^2 + A_1\xi + A_2\right)}\left(\mu\xi + A_1\right) + \frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}\left(\frac{1}{\left(\frac{\mu}{2}\xi^2 + A_1\xi + A_2\right)}\right),
$$

$$
v\left(x, y, z, t\right) = e_0 + \frac{1}{2\left(\frac{\mu}{2}\xi^2 + A_1\xi + A_2\right)}\left(\mu\xi + A_1\right) + \frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}\left(\frac{1}{\left(\frac{\mu}{2}\xi^2 + A_1\xi + A_2\right)}\right),
$$

$$
w\left(x, y, z, t\right) = c_0 + \frac{1}{2\left(\frac{\mu}{2}\xi^2 + A_1\xi + A_2\right)}\left(\mu\xi + A_1\right) + \frac{1}{2}\sqrt{A_1^2 - 2A_2\mu}\left(\frac{1}{\left(\frac{\mu}{2}\xi^2 + A_1\xi + A_2\right)}\right),
$$

where $\xi = x + y + z + 2\left(a_0 + c_0 + e_0\right)t$.

Remark 3.4. Lu et al. [10] obtained the exact solutions of Eq. (3.11) by using the first integral method. Their solutions (56) and (61) only contain type of tan and tanh solutions. Also, we have type of coth, csch, sech, cot, sec and rational solutions.

Remark 3.5. Dai and Wang [1] found the exact solutions of Eq. (3.11) by using the Exp-function method. Their solutions (43) and (44) are similar to our solutions (3.17) and (3.21), respectively. Also, we obtained rational solutions of Eq. (3.11).

Remark 3.6. Wazwaz [15] obtained the single soliton solutions, multiple regular and singular kink solutions of Eq. (3.11) by using the tanh-coth method and Hirota's bilinear method. We acquired Wazwaz's solutions similar and rational solutions of Eq. (3.11).

Remark 3.7. All solutions obtained in this paper have been satisfied with Mathematica 7.0 by putting them back into the original Eq. (3.1) and (3.11).

4. Conclusions

In this study, the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM is successfully applied to KdV equation and the (3+1)-dimensional BE. We obtain traveling wave solutions of these equations by using the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EM. This method can be used to search for the traveling wave solutions of the other NPDEs. As one of the different sides of this method, algebraic equations system is simpler. Furthermore, this method gives more different solutions compared to the other analytical methods and can be applied to the variable coefficients or fractional differential equations. Therefore, many mathematicians work on this method.

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