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Stationary distribution of stochastic nuclear spin generator systems

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Abstract

This paper discusses the stochastic nuclear spin generator systems under the influence of white noise. We prove the existence of a unique solution and a stationary distribution for stochastic nuclear spin generator systems. We analyze long-time behaviour of random attractor of the distributions of the solutions. Furthermore, we prove that the random attractor contains of only one point for particular parameters or can converge weakly to a stationary distribution. Numerical experiments illustrate the results.

Keywords: Existence of a unique solution, stationary distribution, random attractor, invariant measure, nuclear spin generator.
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1. Introduction

The nuclear spin generator chaotic systems was founded in 1963 by S. Sherman [14] for a third-order system generated by a autonomous differential equation which describes the behaviour of a typical nuclear

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reactor problem. A classical nuclear spin generator chaotic systems is written by

$$\begin{cases} \dot{x}(t) = -\beta x + y, \\ \dot{y}(t) = -x - \beta y + \beta kyz, \\ \dot{z}(t) = \alpha\beta - \alpha\beta z - \alpha ky^2, \end{cases} \tag{1.1}$$

with initial value $(x_0, y_0, z_0) = u_0$, where x denotes a neutron density; y denotes temperature which is associated with the fuel; z denotes temperature which is associated with the moderator or coolant; and parameters α, β and γ are nonnegative real numbers. This system exhibits the paradox of abundant nonlinear phenomenon for different parameter condition. So it is " a better archetypal system than the Lorenz system" [3]. Recently, there has been an increasing interest in investigating the nonlinear dynamics of nuclear spin generator (NSG) [10, 15–18, 20].

On the contrary, Vreeke [18] has pointed out that the parameters in the nuclear spin generator systems exhibit random fluctuation to a greater or lesser extent due to the local magnetic field of the nuclei in the sample. Scholars in general estimate them by average values plus some error terms. Usually, basing on the well-known central limit theorem, the distribution of residuals follows normal, that is, the corresponding Itô's-type of the stochastic NSG system is defined by

$$du = (-Au - B(u) + f)dt + G(u)dW_t, \qquad u(0) = u_0, \quad 0 \le t \le T < \infty \tag{1.2}$$

with the initial value u_0 independent of $\mathcal{F}_t^{\mathbb{P}}$ for all $t \ge 0$, where W_t is independent Brownian motions. The coefficients of the drift are given by

$$A = \begin{bmatrix} \beta & -1 & 0 \\ 1 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\beta kyz \\ \beta ky^2 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ 0 \\ \alpha\beta \end{bmatrix}.$$

The noise term $G(u) : \Re^3 \to \Re\{3 \times m\}$-matrices satisfies a linear growth condition and a Lipschitz.

We also interest in the asymptotic behavior of the stochastic nuclear spin generator systems. To investigate the stochastic ultimate bound, stationary distributions and random attractor for a stochastic dynamical system is important but quite challenging task in general [1, 2, 4–9, 12, 13, 21, 22]. Some results in recent literature in general have been obtained by the construction of Lyapunov functionals. Although a very useful method for proving the stationary distributions and random attractor, the construction of a Lyapunov functional is usually a very difficult task, and involves long computations. Moreover, a new Lyapunov functional is often required for each model under consideration. However, our approach does not require to make use of Lyapunov functional methods, and apply Krylov-Bogolyubov methods to a quite general framework.

To the best of the author's knowledge, comparably little progress has been made by now. Since the nonlinear part of the nuclear spin generator systems does not satisfy a linear growth condition, we cannot apply the existence and uniqueness standard theorems that prove the existence of a unique solution. In this paper, firstly, basing on truncation function methods, we prove the existence and uniqueness of the solution. Secondly, using Krylov-Bogolyubov methods, we prove the existence a stationary distribution and a random attractor. Finally, we prove that the random attractor contains of only one point for particular parameters or can converge weakly to a stationary distribution.

2. Preliminaries and notations

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a probability space. We define a flow θ of maps θ_t: $\Omega \to \Omega$ with $t \in R$, i.e.,

$$\theta_0 = id_\Omega \quad \theta_t \circ \theta_s = \theta_{t+s} \quad s, t \in R,$$

(for brevity we write $\theta_t \circ \theta_s = \theta_t\theta_s$) such that $(t, \omega) \to \theta_t\omega$ is $\mathcal{F} \otimes \mathcal{B}(R)$-measurable and $\theta_t\mathbb{P} = \mathbb{P}$ (measure preserving). In addition, $\mathbb{P}$ is assumed to be ergodic with respect to the flow θ. We call $\{\Omega, \mathcal{F}, \mathbb{P}, \theta_{tt\in R}\}$ or θ for short, a metric dynamical system.

Definition 2.1 ([19]). Let $t \in \Re^+$, ξ_t be a homogeneous Markov process on the measure space $(\Re^d, \mathcal{B}(\Re^d))$ with transition probability $P(t,x,A)$. If for any $f \in C_b(\mathcal{B}(\Re^d))$, where $C_b(\mathcal{B}(\Re^d))$ denotes the space of all continuous bounded function on $\Re^d$, the associated operators T_t are defined by

$$T_t f(x) = \int_{Re^d} f(y)P(t,x,dy) = E_x f(\xi_t), \qquad t \in \Re^+,$$

which are continuous at $x \in \Re^d$, i.e., $T_t : \mathcal{C}_b \to \mathcal{C}_b$, the Markov process $\xi(t)$ is said to satisfy the Feller property.

Definition 2.2 ([19]). For all $\mathcal{A} \in \mathcal{B}$ and $\nu \in \mathcal{M}_1(\Re^d)$, defining operators

$$\nu\mathfrak{T}_t(\mathcal{A}) = \int_{\Re^d} P(t,x,\mathcal{A})\nu(dx), \qquad t \in \Re^+,$$

then a measure $\mu \in \mathcal{M}_1(\Re^d)$ is called stationary distribution if $\mu = \mu\mathfrak{T}_t$ for all $t \geq 0$.

Definition 2.3 ([19]). For $f \in \mathcal{C}_b(\Re^d)$ and $\nu \in \mathcal{M}_1(\Re^d)$, defining the natural pairing

$$\langle f, \nu \rangle = \int_{\Re^d} f(x)\nu(dx),$$

then T_t is the dual $\mathfrak{T}_t$, i.e., $\langle T_t f, \nu \rangle = \langle f, \nu\mathfrak{T}_t \rangle$.

Definition 2.4 ([1]). Let $f, g : \Re^d \to \Re^d, t \in [t_0, T]$ and $\omega \in \Omega$. A function $\phi : t \to x$ is called solution in the sense of Stratonovich of the initial value problem

$$\frac{dx}{dt} = f(x) + g(x) \circ W_t, \qquad x(t_0) = x_0 \in \Re^d,$$

if there exists a neighborhood $\mathcal{N}(\omega)$ (we identify $\omega(t) = W_t(\omega)$) and a solution operators $\Phi : \mathcal{N}(\omega) \to \mathcal{C}^0(\Re, \Re^d)$ which is continuous with $\Phi(\omega) = \phi$ such that $\Phi(\varpi)$ is for all $\varpi \in \mathcal{N}(\omega) \cap \mathcal{C}^1(\Re, \Re)$ a solution of the ordinary differential equation

$$\frac{dx}{dt} = f(x) + g(x)\frac{d\varpi(t)}{dt}, \qquad x(t_0) = x_0 \in \Re^d.$$

Definition 2.5 ([2]). Let $\mathcal{D}$ be the set of all nonempty random sets $\{K(\omega)\}_{\omega\in\Omega}$, where $K(\omega)$ is compact, such that $K(\omega)$ is contained in a ball with center zero and measurable radius $r(\omega)$ such that for all $\omega \in \Omega$ and for all $\lambda > 0$

$$\lim_{t\to\infty} e^{-\lambda t} r(\theta_{-t}\omega) = 0.$$

Definition 2.6 ([2]). Let ϕ be a random dynamical system (RDS). A probability measure μ on $(\Omega \times \Re^d, \mathcal{F} \otimes \mathcal{B}(\Re^d))$ is called invariant measure w.r.t. ϕ if

(i) $\Theta(t)\mu = \mu$ for all $t \in \mathbb{T}$, where $\Theta(t)(\omega, x) := (\theta_t\omega, \phi(t,\omega)x)$. The $\{\Theta(t)\}_t$ is called a skew-product flow;

(ii) $\pi_\Omega\mu = \mathbb{P}$, where π_Ω is the projection of $\Omega \times \Re^d$ onto Ω.

Definition 2.7 ([2]). Let $\mathcal{D}$ be an inclusion closed system. A random compact set $A \in \mathcal{D}$ is called $\mathcal{D}$-attractor of a RDS ϕ, if

(i) A is invariant, i.e., $\phi(t, \omega, A(\omega)) = A(\theta_t\omega)$, for all $t > 0$, $\omega \in \Omega$;

(ii) A is $\mathcal{D}$-attracting, i.e., for all $\omega \in \Omega$ and $D \in \mathcal{D}$,

$$\lim_{t\to+\infty} dist(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), B(\omega)) = 0,$$

where $dist(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y)$ is the usual Hausdorff semi-metric.

Theorem 2.8 ([2])**.** *Let ϕ be continuous RDS and let $\mathcal{D}$ be an IC-system. Moreover, let $B \in \mathcal{D}$ be a random compact set which is $\mathcal{D}$-absorbing. Then there exists a unique $\mathcal{D}$-attractor $A \in \mathcal{D}$ for the cocycle ϕ given by*

$$A(\omega) = \bigcap_{t\geq 0} \overline{\bigcup_{t\geq \tau} \phi(\tau, \theta_{-\tau}\omega), B(\theta_{-\tau}\omega)}.$$

If $B(\omega)$ is connected then so is $A(\omega)$.

Lemma 2.9. *Let $u_i = (x_i, y_i, z_i) \in \Re^3$ for $i = 1, 2, 3$.*

(1) *The matrix A is positive definite, i.e., $(Au, u) \geq \beta \min\{1, \alpha\}\|u\|^2$.*

(2) *The function $\tilde{B}(u_2, u_3) = (0, -y_2z_3, y_2y_3)$ has the following properties:*

(i) $B^T(u) = k\beta\widetilde{B}(u, u)$,

(ii) $\widetilde{B}(u_2, u_3)$ *is bilinear,*

(iii) $\left(\widetilde{B}(u_2, u_3), u_1\right) = -\left(\widetilde{B}(u_2, u_1), u_3\right)$, *in particular* $\left(\widetilde{B}(u_2, u_3), u_3\right) = 0$,

(iv) $\|\widetilde{B}(u_2, u_3)\| \leq \|u_2\|\|u_3\|$,

(v) $|\left(B^T(u_2) - B^T(u_3), u_2 - u_3\right)| \leq \frac{\|u_3\|^2\|u_2-u_3\|}{4L} + L|x_2 - x_3|^2$.

Proof.

(1) For $u = (x, y, x) \in \Re^3$, we have

$$(Au, u) = \beta x^2 + \beta y^2 + \alpha\beta z^2 \geq \gamma\|u\|^2,$$

where $\gamma = \beta \min\{1, \alpha\}$, $(Au, u) = 0$ if and only if $u = 0$. Next, we show that the assertion (2) is correct.

(i) By the definition of $\tilde{B}$, let $u = (x, y, z)$, we have

$$k\beta\tilde{B}(u, u) = (0, -yz, y^2) = (0, -k\beta yz, k\beta y^2) = B^T(u).$$

(ii) By the definition of $\tilde{B}$, for all $k_1, k_2 \in \Re$, we have

$$\begin{aligned}
\widetilde{B}(u_2, k_1u_3 + k_2\overline{u}_3) =&(0, -y_2(k_1z_3 + k_2\overline{z}_3), y_2(k_1y_3 + k_2\overline{y}_3))\\
=&k_1(0, -y_2z_3, y_2y_3) + k_2(0, -y_2\overline{z}_3, y_2\overline{y}_3)\\
=&k_1\widetilde{B}(u_2, u_3) + k_2\widetilde{B}(u_2, \overline{u}_3),\\
\widetilde{B}(k_1u_2 + k_2\overline{u}_2, u_3) =&(0, -(k_1y_2 + k_2\overline{y}_2)z_3, (k_1y_2 + k_2\overline{y}_2)y_3))\\
=&k_1(0, -y_2z_3, y_2y_3) + k_2(0, -y_2\overline{z}_3, y_2\overline{y}_3)\\
=&k_1\widetilde{B}(u_2, u_3) + k_2\widetilde{B}(u_2, \overline{u}_3).
\end{aligned}$$

By the definition of the bilinear, the assertion (ii) is correct.

(iii) By the definition of $\tilde{B}$ and the scalar product, we have

$$\begin{aligned}
&\left(\widetilde{B}(u_2, u_3), u_1\right) = ((0, -y_2z_3, y_2y_3), (x_1, y_1, z_{,1})) = -y_1y_2z_3 + z_1y_2y_3,\\
&\left(\widetilde{B}(u_2, u_1), u_3\right) = ((0, -y_2z_1, y_2y_1), (x_3, y_3, z_{,3})) = -z_1y_2y_3 + y_1y_2z_3,\\
&\left(\widetilde{B}(u_2, u_3), u_1\right) = -\left(\widetilde{B}(u_2, u_1), u_3\right),
\end{aligned}$$

in particular

$$\left(\widetilde{B}(u_2, u_3), u_3\right) = ((0, -y_2z_3, y_2y_3), (x_3, y_3, z_{,3})) = -y_2y_3z_3 + y_2y_3z_3 = 0.$$

Then, the assertion (iii) is correct.

(iv) By the definition of $\tilde{B}$, we have

$$\|\widetilde{B}(u_2,u_3)\| = \|(0,-y_2z_3,y_2y_3)\| = \sqrt{y_2^2z_3^2+y_2^2y_3^2} \le \sqrt{x_2^2+y_2^2+z_2^2}\sqrt{x_3^2+y_3^2+z_3^2} \le \|u_2\|\|u_3\|.$$

Then, the assertion (iv) is correct.

(v) By the bilinearity of $\tilde{B}$ and using the Schwarz inequality, we get

$$\begin{aligned}
|\left(B^T(u_2)-B^T(u_3),u_2-u_3\right)| &= |-\left(B^T(u_2),u_3\right)-\left(B^T(u_3),u_2\right)| \\
&= |\left(B^T(u_2,u_3),u_2-u_3\right)-\left(B^T(u_3,u_3),u_2-u_3\right)| \\
&= |\left(B^T(u_2-u_3,u_3),u_2-u_3\right)| \\
&\le \left(\sqrt{2\lambda}\right)^{-1}\|u_3\|\|u_2-u_3\|\sqrt{2\lambda}|x_2-x_3| \\
&\le (4\lambda)^{-1}\|u_3\|^2\|u_2-u_3\|^2+\lambda|x_2-x_3|,
\end{aligned}$$

where λ is a positive constant.

□

3. Stationary distribution

In this section, we will prove the existence of a unique solution and a stationary distribution for the stochastic nuclear spin generator systems.

Theorem 3.1. *Let $p \in \mathbb{N}$ be even and $\mathbb{E}\|u_0\|^p < \infty$. Then there exists a pathwise unique almost sure continuous solution in system* (1.2).

Proof. Let $H_N \in \mathcal{C}^1(\Re^3,\Re)$ with

$$H_N(u) = \begin{cases} 1, & \text{for } \|u\| \le N, \\ 0, & \text{for } \|u\| \ge N+1. \end{cases}$$

Setting $B_N(u) := H_N(u)B(u)$, then system (1.2) is modified by

$$du_N = (-A_N u - B_N(u_N) + f)dt + G(u_N)dW_t, \qquad u_N(0) = u_0, \quad 0 \le t \le T, \tag{3.1}$$

where u_0 is independent of $\mathcal{F}_t^{\mathbb{P}}$ for $t \ge 0$ such that $\mathbb{E}\|u_0\|^2 < \infty$.

Step 1: We show that system (1.2) has a continuous unique solution which is $\mathcal{F}_t^{\mathbb{P}}$-measurable. Due to the "truncation" function $H_N \in \mathcal{C}^1(\Re^3,\Re)$, the nonlinear part $B_N(u)$ of system (3.1) is also differentiable, and its derivative is a continuous compact support. Therefore, the nonlinear part $B_N(u)$ satisfies a linear growth condition and is continuous bounded. It is easy to see that all the other coefficients of system (3.1) satisfy Lipschitz condition and a linear growth condition. Then, by the existence and uniqueness standard theorem [1], it is easy to know that the assertions are directly proved.

Step 2: We will prove that there exists a constant $K_p := K(T,\mathbb{E}\|u_0\|^p,p)$ independent of N satisfying $\mathbb{E}\|u_N\|^p \le K_p$ for all $t \in [0,T]$. Define the Lyapunov function

$$V(u) = \|u\|^p = \left(x^2+y^2+z^2\right)^{\frac{p}{2}}$$

for $p \in \mathbb{N}$ even. Applying the chain rule to equation (3.1), we get

$$\begin{aligned}
d\|u_N\|^p =& p\|u_N\|^{p-2}\left[-(Au_N,u_N)-(H_N(u_N)B(u_N),u_N)+(f,u_N)\right]dt \\
&+\left(p\left(\frac{p}{2}-1\right)\|u_N\|^{p-4}\text{trace}\left(u_Nu_N^TG(u_N)G^T(u_N)\right)+\frac{p}{2}\|u_N\|^{p-2}\text{trace}\left(G(u_N)G^T(u_N)\right)\right) \\
&+p\|u_N\|^{p-2}u_N^TG(u_N)dW_t.
\end{aligned} \tag{3.2}$$

By the properties of Lemma 2.9, we get

$$(B(u_N), u_N) = H_N(u_N)(B(u_N), u_N) = 0.$$

All the other terms of equation (3.2) are bounded. Therefore,

(1) If $\alpha > 1$, we have

$$\begin{aligned}
-(Au, u) + (f, u) &= -\beta x^2 - \beta y^2 - \alpha\beta z^2 + \alpha\beta z \\
&= -\beta\|u\|^2 - (\alpha - 1)\beta\left(z - \frac{\alpha}{2(\alpha-1)}\right)^2 + \frac{\alpha^2\beta}{4(\alpha-1)} \\
&\leq -\beta\|u\|^2 + \frac{\alpha^2\beta}{4(\alpha-1)}.
\end{aligned}$$

(2) If $\alpha \leq 1$, we have

$$\begin{aligned}
-(Au, u) + (f, u) &= -\beta x^2 - \beta y^2 - \alpha\beta z^2 + \alpha\beta z \\
&= -l\|u\|^2 - (\alpha\beta - 1)\left(z - \frac{\alpha\beta}{2(\alpha\beta-1)}\right)^2 + \frac{\alpha^2\beta^2}{4(\alpha\beta-1)} \\
&\leq -\|u\|^2 + \frac{\alpha^2\beta^2}{4(\alpha\beta-1)},
\end{aligned} \tag{3.3}$$

where $l = \min\{1, \beta\}, \alpha\beta \geq 1$. Since the trace of $uu^T G(u)G^T(u)$ is no more than one eigenvalue, we conclude form

$$\left(uu^T G(u)G^T(u)\right)u = \left(u^T G(u)G^T(u)u\right)u,$$

that

$$\operatorname{trace}\left(u_N u_N^T G(u_N)G^T(u_N)\right) \leq \|u_N\|^2\|G(u_N)\|^2.$$

From (3.2), we get

$$\begin{aligned}
d\|u_N\|^p = &- pl\|u_N\|^p dt + p\|u_N\|^{p-2}\frac{\alpha^2\beta^2}{4(\alpha\beta-1)}dt \\
&+ \frac{p}{2}(p-1)\|u_N\|^{p-2}\|G(u_N)\|^2 dt + p\|u_N\|^{p-2}u_N^T G(u_N)dW_t + \xi(t)dt,
\end{aligned} \tag{3.4}$$

where $l = \min\{1, \beta\}, \alpha\beta \geq 1$ and $\xi(t)$ is an adapted process. For $M \in \mathbb{N}$, define the stopping time

$$\tau_M := \inf\{t \in [0, T] : \|u\| \geq M\}.$$

Note that

$$\int_0^{t\wedge\tau_M} g(s)ds \leq \int_0^t g(s \wedge \tau_M)ds$$

for all $f(t) \geq 0$. Since

$$\|u_N(s \wedge \tau_M)\|^q \leq M^q \quad \text{for all } q > 0$$

and using the linear growth condition

$$\|G(u)\|^2 \leq L^2(1 + \|u\|^2),$$

we obtain

$$\mathbb{E}\int_0^{t\wedge\tau_M} p\|u_N(s)\|^{p-2}u_N^T(s)G(u_N(s))dW_s = 0.$$

From (3.4), we get

$$\begin{aligned}\mathbb{E}\|u_N(t\wedge\tau_M)\|^p \leq&\mathbb{E}\|u_N(0)\|^p+\int_0^t\left(-pl+L^2\frac{p}{2}(p-1)\right)\mathbb{E}\|u_N(s\wedge\tau_M)\|^p ds\\&+\int_0^t\frac{p}{2}\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+L^2(p-1)\right)\mathbb{E}\|u_N(s\wedge\tau_M)\|^{p-2}ds.\end{aligned}\tag{3.5}$$

When $p=2$, (3.5) becomes

$$\begin{aligned}\mathbb{E}\|u_N(t\wedge\tau_M)\|^2 \leq&\mathbb{E}\|u_N(0)\|^2+\int_0^t\left(-pl+L^2\frac{p}{2}(p-1)\right)\mathbb{E}\|u_N(s\wedge\tau_M)\|^2 ds\\&+\int_0^t\frac{p}{2}\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+L^2(p-1)\right)ds.\end{aligned}$$

By Gronwall's inequality, we have

$$\sup_{t\in[0,T]}\mathbb{E}\|u_N(t\wedge\tau_M)\|^2\leq K_2,$$

where K_2 is a positive constant. By recursive computation, it is easy to know that there exists a constant K_p satisfying

$$\begin{aligned}\mathbb{E}\|u_N(t\wedge\tau_M)\|^p \leq&\mathbb{E}\|u_N(0)\|^p+\int_0^t\left(\left(-pl+L^2\frac{p}{2}(p-1)\right)\mathbb{E}\|u_N(s\wedge\tau_M)\|^p+K_2K_{p-2}\right)ds\\ \leq&K_p.\end{aligned}\tag{3.6}$$

It is easy to show that the stopping time satisfies $\tau_M\to T$ as $M\to\infty$. Since the solution u_N is continuous in t, the norm $\|u_N(t\wedge\tau_M)\|^p$ is bounded. Therefore, it converges ω-wise to $\|u_N(t)\|^p$ as $M\to\infty$. By the nonnegative bounded of the norm and Fatou's lemma, we obtain that for $t\leq T$

$$\mathbb{E}\|u_N(t)\|^p=\mathbb{E}\lim_{M\to\infty}\|u_N(t\wedge\tau_M)\|^p\leq\liminf_{M\to\infty}\mathbb{E}\|u_N(t\wedge\tau_M)\|^p\leq K_p.$$

Step 3: We will show that there exists a positive constant $\widetilde{K}_p:=\widetilde{K}(T,\mathbb{E}\|u_0\|^p,p)$ independent of N satisfying $\mathbb{E}\sup_{t\in[0,T]}\|u_N\|^p\leq\widetilde{K}_p$ for all $t\in[0,T]$. From (3.1) and (3.3), we have

$$\begin{aligned}du_N^2(t)=&2u_N(t)(-Au_N(t)+f-B(u))dt+G^T(u_N(t))G(u_N(t))dt+u_N^T(t)G(u_N(t))dW_t\\ \leq&\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+G^T(u_N(t))G(u_N(t))\right)dt+u_N^T(t)G(u_N(t))dW_t\end{aligned}$$

and

$$\begin{aligned}u_N^2(t)\leq&u_N^2(0)+\int_0^t\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+G^T(u_N(s))G(u_N(s))\right)ds\\&+\int_0^t u_N^T(s)G(u_N(s))dW_s.\end{aligned}$$

If $p>2$, using the inequality

$$\left|\sum_{i=1}^N a_i\right|^m\leq N^{m-1}\sum_{i=1}^N|a_i|^m\quad\text{for } m\geq 1,$$

we obtain

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|u_N(t\wedge\tau_M)\|^p\leq&3^{\frac{p-2}{2}}\mathbb{E}u_N^p(0)+3^{\frac{p-2}{2}}\mathbb{E}\left|\int_0^T\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+\|G(u_N(s))\|^2\right)ds\right|^{\frac{p}{2}}\\&+3^{\frac{p-2}{2}}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t u_N^T(s)G(u_N(s))dW_s\right|^{\frac{p}{2}}.\end{aligned}$$

By Fubini's theorem and using Step 2, we get

$$\begin{aligned}\mathbb{E}\left|\int_0^T \left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+\|G(u_N(s))\|^2\right)ds\right|^{\frac{p}{2}} \leq & \mathbb{E}\left|\int_0^T \left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}+\|G(u_N(s))\|^2\right)ds\right|^{\frac{p}{2}} \\ \leq & 3^{\frac{p-2}{2}}T^{\frac{p-2}{p}}\int_0^T\left(\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}\right)^{\frac{p}{2}}+L^p+L^pK_p\right)ds \\ = & 3^{\frac{p-2}{2}}T^{\frac{p-2}{p}}\left(\left(\frac{\alpha^2\beta^2}{2(\alpha\beta-1)}\right)^{\frac{p}{2}}+L^p+L^pK_p\right)T=\widetilde{K}_p^1,\end{aligned}$$

where $\widetilde{K}_p^1$ is bounded constant. Using the Burkholder-Davis-Gundt inequality [8], we can estimate the stochastic integral by

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t u_N^T(s)G(u_N(s))dW_s\right|^{\frac{p}{2}} \leq C_p\mathbb{E}\left|\int_0^T \|u_N(s)G(u_N(s))\|^2 ds\right|^{\frac{p}{4}},$$

where $C_p=\left(\frac{34}{p}\right)^{\frac{p}{4}}$ is positive constant for $0<p<4$ and $C_p=\left(\frac{p^{\frac{p}{2}+1}}{2^{\frac{p}{2}+2}\left(\frac{p}{2}-1\right)^{\frac{p}{2}-1}}\right)$ for $p\geq 4$. By similar way, we handle the Lebesgue integral, hence there exists a constant $\widetilde{K}_p^2$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u_N(t\wedge\tau_M)\|^p\right)\leq 3^{\frac{p-2}{2}}\left(\mathbb{E}u_N^p(0)+\widetilde{K}_p^1+\widetilde{K}_p^2\right)\leq\widetilde{K}_p.$$

Since the solution $u_N(t)$ is continuous at t, thus $\sup_{t\in[0,T]}\|u_N(t\wedge\tau_M)\|^p$ is bounded and converges ω-wise to $\sup_{t\in[0,T]}\|u_N(t)\|^p$ as $M\to\infty$. Hence, using Fatou's Lemma, we have

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|u_N(t)\|^p &= \mathbb{E}\lim_{M\to\infty}\sup_{t\in[0,T]}\|u_N(t\wedge\tau_M)\|^p \\ &\leq \liminf_{M\to\infty}\mathbb{E}\sup_{t\in[0,T]}\|u_N(t\wedge\tau_M)\|^p\leq\widetilde{K}_p.\end{aligned}\tag{3.7}$$

Step 4: By Step 1, it is easy to see that system (3.1) has the solution $u_N(t)$. Using Chebyshev's inequality and Step 3, we obtain

$$\begin{aligned}\mathbb{P}\{\tau_M<T\}\leq&\mathbb{P}\left\{\sup_{t\in[0,T]}\|u_N(t)\|\geq N\right\}\leq\frac{\mathbb{E}\sup_{t\in[0,T]}\|u_N(t)\|^2}{N^2} \\ \leq&\frac{\widetilde{K}_p\left(T,\mathbb{E}\|u_0\|^2,2\right)}{N^2}\xrightarrow{N\to\infty}0.\end{aligned}$$

Note that we can find an $N_0(\omega)$ satisfying $\tau_{N_0(\omega)}=T$ for almost every $\omega\in\Omega$. Moreover, we have

$$B_{N'}(u)=B_N(u)=B(u),\qquad N'\geq N>0$$

for all $\|u\|\leq N$. Hence, by Theorem 2 in Gihman and Skorokhog [[4], p.44], it implies $\tau_{N'}\geq\tau_N$ and $u_{N'}^{u_0}(\cdot,\omega)=u_N^{u_0}(\cdot,\omega)$ on $[0,\tau_N]$ for all $N'\geq N>0$. Thus if $\tau_N=T$, it is easy to see that $\tau_{N'}=T$ for all $N'\geq N>0$. Therefore, the set $\{\omega:\tau_N=T\}$ is monotonically increasing and converges to Ω as $N\to\infty$. Note that $u_N^{u_0}$ is only a version of $u(\cdot)$ on $[0,\tau_N]$, that is, there is an exceptional $\mathbb{P}$-null set $\mathcal{N}(N)$. In fact, there exist countable many such sets, and the union over all these $\mathbb{P}$-null sets is also a null set. Furthermore, since $u_N(t)$ is continuous as well as converges uniformly in t to $u(t)$, hence, $u(t)$ is continuous at t.

To complete the proof, we must show that the limit function $u(t)$ actually solves the nuclear spin equation. For $t = 0$, this is true, because $u(0) = u_0$ for all $N \in \mathbb{N}$. Since $B_N(u_N(t \wedge \tau_N)) = B(u(t \wedge \tau_N))$ and $u_N(t \wedge \tau_N) = u(t \wedge \tau_N)$ for all $t \leq T$, then almost sure convergence of τ_N to T implies

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]}\left\|\int_{t_0}^{t}(A(u_N(s)-u(s))+(f-f))+B_N(u(s)-B(u(s)))\,ds\right.\right.\\
&\qquad\left.\left.+\int_{t_0}^{t}(G(u_N(s))-G(u(s)))\,dW_S\right\|>0\right\}\\
&\leq \mathbb{P}\{\tau_M<t\}\xrightarrow{N\to\infty}0.
\end{aligned}$$

Hence $u(\cdot)$ is a solution of the stochastic nuclear spin generator system (1.2) on $[0, T]$. □

Corollary 3.2. *The solution $u(t)$ of stochastic nuclear spin generator system* (1.2) *with $\mathbb{E}\|u_0\|^2 < \infty$ possesses the following properties:*

(i) *$u(t)$ is $\mathcal{F}_t^{\mathbb{P}} \otimes \mathcal{B}([0, T])$-measurable and is a Markov process.*

(ii) *In addition, if $\mathbb{E}\|u_0\|^p < \infty$ for $p \in \mathbb{N}$, then there is a positive constant $\widetilde{K}(T, \mathbb{E}\|u_0\|^p, p) > 0$ satisfying*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u_t\|^p\right)\leq \mathbb{E}\|u_t\|^p\leq \widetilde{K}(T,\mathbb{E}\|u_0\|^p,p),\qquad \forall t\in[0,T].$$

Furthermore, for every deterministic as well as bounded set $\mathcal{B} \subset \Re^3$, the positive constant $\sup_{u_0\in\mathcal{B}} \widetilde{K}(T, \mathbb{E}\|u_0\|^p, p)$ is finite, where u_0 is deterministic.

Proof.

(i) Denote by $\mathfrak{F}_t$ the minimal ϱ-algebra of events relative to which $u(0)$ and W_s for $s \leq t$ are measurable, and H^t the ϱ-algebra generated by $W(s) - W(t)$ for $s \geq t$. It is obvious that the events of the ϱ-algebra H^t are independent of those $\mathfrak{F}_t$. The value of $u_{u_0,t}(s)$ is completely determined by the increments $W(v) - W(t)$ for $v \geq t$ and is measurable w.r.t. H^t. We note that $u(s) = u_{u(t),t}(s)$ since for $s > t, u(s)$ and $u_{u(t),t}(s)$ satisfy

$$u(s)=u_t+\int_t^s(-Av-B(v)+f)dv+\int_t^s G(v)dW_v,$$

whose solution is unique. Therefore, $u(s) = h(u(t), \omega)$, where $h(u(t), \omega)$ is a random function independent of the event of $\mathfrak{F}_t$. Assume $h(u(t), \omega) = \sum_{i=1}^{N} \psi_i(u)\lambda_i(\omega)$, where ψ_i is a nonrandom function. Then for any random variable ζ and ξ, measurable w.r.t. $\mathfrak{F}_t$, we have

$$\mathbb{E}\left(h(\xi,\omega)\zeta\right)=\mathbb{E}\left(\sum_{i=1}^{N}\psi_i(\xi)\lambda_i(\omega)\zeta\right)=\mathbb{E}\left(\sum_{i=1}^{N}\psi_i(\xi)\zeta\mathbb{E}\lambda_i(\omega)\zeta\right).$$

Since $\sum_{i=1}^{N} \psi_i(\xi)\mathbb{E}\lambda_i(\omega)$ can be approximated by an arbitrary measurable bounded function. Then

$$\mathbb{E}h(\xi,\omega)\zeta=\mathbb{E}\left(\sum_{i=1}^{N}\psi_i(\xi)\zeta\mathbb{E}\lambda_i(\omega)\zeta\right)=\mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{N}\psi_i(\xi)\zeta\lambda_i(\omega)\right)\zeta\right).$$

By passage to the limit, we get

$$\mathbb{E}\left(h(\xi,\omega)|\mathfrak{F}_t\right)=\mathbb{E}h(\xi,\omega).$$

Therefore, we find

$$\mathbb{E}(\chi_{\mathcal{A}}(u(s))|\mathfrak{F}_t) = \mathbb{E}\chi_{\mathcal{A}}(u_{u_t}(s)) = P(t, u_t, \mathcal{A}) = \mathbb{P}((u_{u_t}(s)) \in \mathcal{A}).$$

By the definition of Markov process, we prove that $u(t)$ is $\mathcal{F}_t^{\mathbb{P}} \otimes \mathcal{B}([0,T])$-measurable and is homogeneous Markov process.

(ii) By Step 3 in proof of Theorem 3.1, we can prove that there exist the asserted constant. By Step 2 in proof of Theorem 3.1, we can prove that $\mathbb{E}\|u\|$ is bounded. Ever bounded set $\mathcal{B}$ is contained in a ball of appropriate radius R and center zero. Let $\|u_0\| = R$, the assertion holds to dependent on a bounded constant by equation (3.6) and (3.7), respectively. □

Corollary 3.3. *Let $\mathbb{E}\|u_0\|^2 < \infty$. If either $L^2 < 2l$ or $G(u) \leq L^2$, then there is a positive constant $K\left(\mathbb{E}\|u_0\|^2, 2\right)$ satisfying*

$$\sup_{t\in[0,\infty]} \mathbb{E}\|u(t)\|^2 \leq K\left(\mathbb{E}\|u_0\|^2, 2\right).$$

Proof. By the proof of Step 2 given in proof of Theorem 3.1, it is easy to show that the bounded constant $-pl + \frac{pL^2}{2}$ is negative which is only dependent on the initial value but independent of t. □

Remark 3.4. The $G(u)$ of intensities of random noises influence on the bound for the nuclear spin generator system (1.1). However if $G(u) = 0,\ p = 2$, then the deterministic nuclear spin generator system (1.1) implies that

$$\limsup_{t\to\infty} \|u(t)\|^2 \leq \frac{\alpha^2\beta^2}{4(\alpha\beta - 1)}.$$

Proof. One can present a proof for the deterministic case that there exists an attractor in to which every solution enters in finite time. Under conditions of the Theorem 3.1, if $G(u) \neq 0,\ p = 2$, we have

$$\limsup_{t\to\infty} \mathbb{E}\|u(t)\|^2 \leq \frac{\alpha^2\beta^2 + 2L^2(\alpha\beta - 1)}{2(\alpha\beta - 1)(2 - L^2)}. \tag{3.8}$$

That is, the positively invariant set for nuclear spin generator system (1.1) has changed. The results show that the white noise can make the solution bounds to undergo change under some conditions. It pointed out that the parameters in the nuclear spin generator system (1.1) exhibit random fluctuation. □

Lemma 3.5. *Let $g \in \mathcal{C}_b(\Re^3)$. Then operators T_t (respectively $\mathfrak{T}_t$) of the stochastic nuclear spin generator system* (1.2) *are*

(i) *continuous w.r.t. to t, i.e., $T_{t_n}g(x) \xrightarrow{t_n\to t_0} T_{t_0}g(x)$, and weakly continuous at t, i.e.,*

$$\int_{\Re^3} g(x)d(\mu\mathfrak{T}_{t_n})(x) \xrightarrow{t_n\to t_0} \int_{\Re^3} g(x)d(\mu\mathfrak{T}_{t_0})(x);$$

(ii) *continuous w.r.t. to x (Feller), i.e., $T_tg(x_n) \xrightarrow{t_n\to t_0} T_tg(x_0)$, and weakly continuous at x, i.e.,*

$$\int_{\Re^3} g(x)d(\delta_{x_n}\mathfrak{T}_t)(x) \xrightarrow{t_n\to t_0} \int_{\Re^3} g(x)d(\delta_{x_0}\mathfrak{T}_t)(x).$$

Proof.

(i) By Theorem 3.1, it is easy to known that the solution of system (1.2) is continuous at t, and f is continuous bounded for all $t \geq 0$. By Lebesgue's theorem, we obtain

$$T_f f(x) = \mathbb{E}f(u^x(t)) = \mathbb{E}\lim_{n\to\infty} f(u^x(t_n)) = \lim_{n\to\infty} \mathbb{E}f(u^x(t_n)) = \lim_{n\to\infty} T_{t_n}f(x)$$

for any sequence $t_n \to t_0$ and all $x \in \Re^d$. Therefore, the operators $T_tf(x)$ are continuous at t.

Using the natural pairing $\langle T_tf, \mu\rangle = \langle f, \mu\mathfrak{T}_t\rangle$ and Definition 2.3, then operators $\mathfrak{T}_tf(x)$ are weakly continuous at t.

(ii) For every ε_1 and $\varepsilon_2 > 0$, there exists an $n_0(x_0, t) > 0$ satisfying

$$\mathbb{P}\{\|u^x(t) - u^{x_0}(t)\| > \varepsilon_1\} < \varepsilon_2$$

for all $n > n_0(x_0, t)$ and every sequence $x_n \to x_0$. By Chebyshev's inequality and Corollary 3.2, there exists an $N(T, \varepsilon_2)$ satisfying

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \|u^{x_i}(t)\| \geq N\right\} \leq \frac{\widetilde{K}_{\max}}{N^2} < \frac{\varepsilon_2}{3} \tag{3.9}$$

for a given sequence $x_n \to x_0$ and for all $N > N(T, \varepsilon_2)$ and $x_i (i \in \mathbb{N})$.

Since Step 1 in proof of Theorem 3.1, it is easy to know that the solution u_N is continuous w.r.t. the initial value. Therefore, there exists an $n_0(t, x_0) > 0$ satisfying

$$\mathbb{P}\{\|u_N^x(t) - u_N^{x_0}(t)\| > \varepsilon_1\} < \frac{\varepsilon_2}{3} \tag{3.10}$$

for all $n > n_0(t, x_0)$.

Using equations (3.9) and (3.10), there exists a positive constant $n_0(t, x_0)$ satisfying

$$\begin{aligned}\mathbb{P}\{\|u^{x_n}(t) - u^{x_0}(t)\| > \varepsilon_1\} =&\mathbb{P}\left\{\{\|u^{x_n}(t)\| < N\} \cap \{\|u^{x_0}(t)\| < N\} \cap \{\|u^{x_n}(t) - u^{x_0}(t)\| > \varepsilon_1\}\right\}\\ &+ \mathbb{P}\left\{\{\|u^{x_n}(t)\| \geq N\} \cup \{\|u^{x_0}(t)\| \geq N\} \cap \{\|u^{x_n}(t) - u^{x_0}(t)\| > \varepsilon_1\}\right\}\\ \leq&\mathbb{P}\{\|u^{x_n}(t) - u^{x_0}(t)\| > \varepsilon_1\} + \mathbb{P}\{\|u^{x_n}(t)\| \geq N\} + \mathbb{P}\{\|u^{x_0}(t)\| \geq N\}\\ <&\frac{\varepsilon_2}{3} + \frac{\varepsilon_2}{3} + \frac{\varepsilon_2}{3} = \varepsilon_2\end{aligned}$$

for all $n > n_0(t, x_0)$. Therefore, it is easy to know that the solution is continuous. By the theorem of dominate convergence, we know that $\mathbb{E}|f(u^{x_n}(t)) - f(u^{x_0}(t))|$ converges to zero as $n \to \infty$. Hence the operators $T_t f(x)$ are continuous at x.

Using again the natural pairing $\langle T_t f, \mu\rangle = \langle f, \mu\mathfrak{T}_t\rangle$ and Definition 2.3, then operators $\mathfrak{T}_t f(x)$ are weakly continuous at x. □

Theorem 3.6. *If $L^2 < 2$ and $\mathbb{E}\|u_0\| < \infty$, then, there exists a stationary distribution for the stochastic nuclear spin generator systems.*

Proof. Denoting the operators $\mathfrak{T}_t$ generated by the solution of the stochastic nuclear spin system and δ_{u_0} is a Dirac-measure. Let $0 = t_0 \leq \ldots \leq t_n = t$ be partition of the internal $[0, t]$ and set $\Delta_n = \max_{1\leq i\leq n-1}(t_i - t_{i-1})$. By the linear combinations of measure, we obtain

$$\begin{aligned}\int_{\Re^3} H_N(u)\|u\|^2 d\left(\frac{1}{t}\int_0^t \delta_{u_0}\mathfrak{T}_\tau(u)d\tau\right) &= \int_{\Re^3} H_N(u)\|u\|^2 d\left(\frac{1}{t}\sum_{i=1}^n \delta_{u_0}\mathfrak{T}_{t_i}(u)(t_i - t_{i-1})\right)\\ &= \frac{1}{t}\sum_{i=1}^n \int_{\Re^3} H_N(u)\|u\|^2 d\left(\delta_{u_0}\mathfrak{T}_{t_i}(u)\right)(t_i - t_{i-1}).\end{aligned}$$

By Lemma 3.5, it is easy to see that $\delta_{u_0}\mathfrak{T}_{t_i}$ is weakly continuous at t. Therefore, the limit of the right hand side as $\Delta_n \to 0$ is a well-defined Riemann integral. We define

$$\int_0^t \delta_{u_0}\mathfrak{T}_\tau(u)d\tau = \operatorname*{w-lim}_{\Delta_n\longrightarrow 0}\sum_{i=1}^n \delta_{u_0}\mathfrak{T}_{t_i}(u)(t_i - t_{i-1}).$$

It is easy to see that the limit of the left hand side also exists as $\Delta_n \longrightarrow 0$. Hence, we obtain

$$\int_{\Re^3} H_N(u)\|u\|^2 d\left(\frac{1}{t}\int_0^t \delta_{u_0}\mathfrak{T}_\tau(u)d\tau\right) = \frac{1}{t}\int_0^t \int_{\Re^3} H_N(u)\|u\|^2 d\left(\delta_{u_0}\mathfrak{T}_\tau(u)\right) d\tau.$$

Note that the truncation function $H_N(u)$ defined in Theorem 3.1 and $H_N(u)\|u\|^2 \le \|u\|^2$.

By Corollary 3.3, it is easy to show that there exists a constant $K(\mathbb{E}\|u_0\|^2, 2)$ independent of t satisfying

$$\int_{\Re^3} T_t\|u\|^2 d\delta_{u_0}(u) = \mathbb{E}\|u^{u_0}(t)\|^2 \le K\left(\mathbb{E}\|u_0\|^2, 2\right),$$

where the conjugated operators T_t are defined in Lemma 3.5. Using Chebyshev's inequality and Levi's theorem, we get

$$\begin{aligned}
\left(\frac{1}{t}\int_0^t \delta_{u_0}\mathfrak{T}_\tau d\tau\right)\{\|u\| > r\} \le& \frac{1}{r^2}\lim_{N\to\infty}\int_{\Re^3} H_N(u)\|u\|^2 d\left(\frac{1}{t}\int_0^t \delta_{u_0}\mathfrak{T}_\tau(u)d\tau\right)\\
=&\frac{1}{r^2}\lim_{N\to\infty}\frac{1}{t}\int_0^t\int_{\Re^3} H_N(u)\|u\|^2 d\left(\delta_{u_0}\mathfrak{T}_\tau(u)\right)d\tau\\
=&\frac{1}{r^2}\lim_{N\to\infty}\frac{1}{t}\int_0^t\int_{\Re^3} T_t H_N(u)\|u\|^2 d\left(\delta_{u_0}\right)d\tau\\
\le&\frac{1}{r^2}\lim_{N\to\infty}\frac{1}{t}\int_0^t K\left(\mathbb{E}\|u_0\|^2, 2\right)d\tau\\
=&\frac{K\left(\mathbb{E}\|u_0\|^2, 2\right)}{r^2}\xrightarrow{r\to\infty} 0.
\end{aligned}$$

Therefore, for every $\varepsilon > 0$, there is a compact set $\Lambda_\varepsilon \subset \Re^3$ satisfying $\mu(\Lambda_\varepsilon) \ge 1-\varepsilon$ for all $\mu \in \Gamma$ defined by

$$\Gamma = \left\{\frac{1}{t}\int_0^t \delta_{u_0}\mathfrak{T}_\tau d\tau\right\}_{t>0}.$$

By Theorem 6.7 in Parthasarathy [11, p.47], it is easy to know that this is sufficient for the relative compactness of Γ. Since Γ is relatively compact, there exists a sequence $t_n \to \infty$ satisfying

$$\overline{\Gamma} \ni \mu = \operatorname*{w-lim}_{n\longrightarrow\infty}\frac{1}{t_n}\int_0^{t_n}\delta_{u_0}\mathfrak{T}_\tau d\tau.$$

Since the operators $\mathfrak{T}_t$ are weakly continuous at t, we can change the order of $\mathfrak{T}_t$ and weak limit. As the mapping $t \to \delta_{u_0}\mathfrak{T}_t$ is weakly continuous at t, using the dual operators T_t and Feller property, we have

$$\begin{aligned}
\int_{\Re^3} H_N(u)\|u\|^2 d\left(\int_0^t \delta_{u_0}\mathfrak{T}_\tau(u)d\tau\mathfrak{T}_s(u)\right) &= \int_0^t\left(\int_{\Re^3} T_s\left(H_N(u)\|u\|^2\right)d\left(\delta_{u_0}\mathfrak{T}_\tau(u)\right)\right)d\tau\\
&= \int_{\Re^3} H_N(u)\|u\|^2 d\left(\int_0^t \delta_{u_0}\mathfrak{T}_{\tau+s}(u)d\tau\right)\\
&\overset{h=\tau+s}{=\!=\!=} \int_{\Re^3} H_N(u)\|u\|^2 d\left(\int_s^{t+s}\delta_{u_0}\mathfrak{T}_h(u)dh\right).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mu\mathfrak{T}_s =& \operatorname*{w-lim}_{n\longrightarrow\infty}\left(\frac{1}{t_n}\int_s^{t_n+s}\delta_{u_0}\mathfrak{T}_h dh\right)\\
=& \operatorname*{w-lim}_{n\longrightarrow\infty}\left(\frac{1}{t_n}\int_0^{t_n}\delta_{u_0}\mathfrak{T}_h dh + \frac{1}{t_n}\int_{t_n}^{t_n+s}\delta_{u_0}\mathfrak{T}_h dh - \frac{1}{t_n}\int_0^s \delta_{u_0}\mathfrak{T}_h dh\right).
\end{aligned}$$

Since

$$\frac{1}{t_n}\int_{t_n}^{t_n+s}\delta_{u_0}\mathfrak{T}_h dh \xrightarrow{t_n\to\infty} 0, \qquad \frac{1}{t_n}\int_0^s \delta_{u_0}\mathfrak{T}_h dh \xrightarrow{t_n\to\infty} 0,$$

then

$$\mu\mathfrak{T}_s = \underset{n\longrightarrow\infty}{\text{w-lim}} \left(\frac{1}{t_n} \int_s^{t_n+s} \delta_{u_0} \mathfrak{T}_h dh \right) = \mu.$$

Therefore, by Definition 2.2, μ is an invariant measure. That is, the stochastic nuclear spin generator systems (1.2) possesses a stationary distribution. $\square$

4. Random attractor

In this section we will prove the existence of random attractors for the stochastic nuclear spin generator system.

Theorem 4.1. *Let the noise coefficient $G(u) = \sqrt{\gamma}u$ and the initial value $u_0 \in \Re^3$, then there exists a solution in the sense of Stratonovich for stochastic nuclear spin generator system* (1.2)*. Furthermore, a continuous RDS $\phi : \Re^+ \times \Omega \times \Re^3 \to \Re^3$ is generated by the solution operators $u^{u_0}(t,\omega)$ of stochastic nuclear spin generator system* (1.2) *via the relation*

$$\phi(t, \omega, u_0) := u^{u_0}(t, \omega) \quad \textit{over } (\Omega, \mathcal{F}, \mathbb{P}).$$

Proof. First, we consider the following time varying equation

$$\frac{dv(t)}{dt} = -Av(t) - e^{-\sqrt{\gamma}\omega(t)} B(e^{\sqrt{\gamma}\omega(t)} v(t)) + e^{-\sqrt{\gamma}\omega(t)} f, \qquad v(0) = e^{-\sqrt{\gamma}\omega(0)} u_0$$

with $u_0 \in \Re^3$. By the proof of Theorem 3.1, we obtain

$$\frac{d\|v(t)\|^2}{dt} \leq -2l\|v(t)\|^2 + \frac{\alpha^2\beta^2}{2(\alpha\beta - 1)} e^{-2\sqrt{\gamma}\omega(t)}, \qquad \|v(0)\|^2 = \|u_0\|^2.$$

As for every closed time interval, $\omega \in \Omega$ is bounded, then $\|v(t)\|^2$ is also bounded but depending on u_0 and ω for every fixed $t \geq 0$. Since the coefficient satisfies the local Lipschitz condition, there exists a solution for all $t \geq 0$. Moreover, the solution is continuous at (t, ω, u_0). The function defined by $\phi(t, \omega, u_0) = v(t, \omega, u_0)e^{\sqrt{\gamma}\omega(t)}$ is also continuous at (t, ω, u_0). Furthermore, $\phi(t, \omega, u_0)$ solves the following equation

$$\frac{du(t)}{dt} = -Au(t) - B(u(t)) + f + \sqrt{\gamma}u(t)\frac{d\omega(t)}{dt}, \ \omega \in \mathcal{C}^\infty(\Re, \Re), \quad u(0) = u_0.$$

Therefore, $\phi(t, \omega, u_0)$ is a solution in the sense of Stratonovich.

Basing on the uniqueness of the solution and $(\theta_t\omega(\cdot))' = \omega'(t + \cdot)$, it is easy to prove that the solution of system (1.2) has the cocycle property for $\omega \in \mathcal{C}^1(\Re, \Re)$. By the continuity of the solution in $\omega \in \mathcal{C}^0(\Re, \Re)$ and t, it is easy to know that the perfect cocycle property of the solution $u(t)$ is continuous at $\omega \in \mathcal{C}^0(\Re, \Re)$ and t. Note that the exceptional $\mathbb{P}$-null set of the solution is independent on the initial value. Therefore, the solution operators $u^{u_0}(t, \omega)$ of stochastic nuclear spin generator system (1.2) generate a continuous RDS $\phi : \Re^+ \times \Omega \times \Re^3 \to \Re^3$. $\square$

Theorem 4.2. *Let $\alpha \leq 1, \alpha\beta > 1$ and the noise coefficient $G(u) = \sqrt{\gamma}u$, then the stochastic nuclear spin generator system* (1.2) *possesses a $\mathcal{D}$-absorbing set defined by*

$$D(\omega) = \left\{ u \in \Re^3 : \|u\|^2 \leq \widetilde{r}(\omega) = (1 + \sigma)\frac{\alpha^2\beta^2}{2(\alpha\beta - 1)} \int_{-\infty}^0 e^{2t - 2\sqrt{\gamma}\omega(t)} dt \right\},$$

with $\sigma > 0$ and for all

$$\omega \in \Omega_1 := \{\omega \in \Omega : \lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0\}.$$

Proof. By the equation (3.3), we have

$$d\|u\|^2 = (-2\|u\|^2 + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)} + \xi(t))dt + 2\sqrt{\gamma}\|u\|^2 \circ dW_t \tag{4.1}$$

with an adapted process $\xi(t) \leq 0$ and initial value $\|u(0)\| = \|u_0\|$. Note that $\phi(t,\omega,u_0)$ solves the equation (4.1). We will compute the following equation

$$\psi(t,\omega,x) = x^2 e^{-2t+2\sqrt{\gamma}\omega(t)} + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}\int_0^t e^{2(s-t)+2\sqrt{\gamma}\omega(t)-2\sqrt{\gamma}\omega(s)}ds. \tag{4.2}$$

It is easy to see that $\phi(t,\omega,u_0) \leq \psi(t,\omega,u_0)$. Replacing $\omega(t)$ by $\theta_{-t}\omega(t)$ in equation (4.2), we have

$$\lim_{t\to\infty}\psi(t,\theta_{-t}\omega(t),x) := r^2(\omega) = \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}\int_{-\infty}^0 e^{2s-2\sqrt{\gamma}\omega(s)}ds.$$

Since, for all initial values u_0, $\psi(t,\omega,u_0)$ converges to the solution $r^2(\omega)$ which is the stationary solution of equation (4.2). It is easy to see that $D(\omega)$ with $\widetilde{r}^2(\omega) := (1+\rho)r^2(\omega)$ for a fixed $\rho > 0$ defines an absorbing set for the cocycle ϕ.

Let $g_{\varepsilon,p}(t) = \varepsilon t + p\omega(t)$ with $\varepsilon > 0, p \in \Re$ and $t \leq 0$. As the paths of the Wiener process satisfy the law of the iterated logarithm, hence we get

$$\sup_{t\in(-\infty,0]} g_{\varepsilon,p}(t) =: \kappa_{\varepsilon,p}(\omega) < \infty.$$

Now let $\varepsilon > 0$ satisfying $(\sigma - 2\varepsilon) > 0$ and $(2-\varepsilon) > 0$. Then we have

$$e^{(\sigma-2\varepsilon)t}\int_{-t}^0 e^{(2-\varepsilon)\tau}e^{g_{\varepsilon,-2\sqrt{\gamma}}(t+\tau)+g_{\varepsilon,2\sqrt{\gamma}}(t)}d\tau \leq \frac{e^{(\sigma-2\varepsilon)t} + \kappa_{\varepsilon,-2\sqrt{\gamma}} + \kappa_{\varepsilon,2\sqrt{\gamma}}}{2-\varepsilon} \xrightarrow{t\to-\infty} 0,$$

therefore, we obtain

$$\lim_{t\to-\infty} e^{\sigma t}\int_t^0 e^{2\tau-2\sqrt{\gamma}\theta_t\omega(t)}d\tau = 0.$$

Thus, we get

$$\lim_{t\to+\infty} e^{-\sigma}\widetilde{r}(\theta_{-t}\omega) = 0.$$

Hence, the random compact set $D(\omega)$ belongs to $\mathcal{D}$. Moreover, given an $\mathcal{A} \in \mathcal{D}$, then $\mathcal{A}(\omega)$ is defined by including in a ball of radius $\hat{r}(\omega)$. Inserting $\hat{r}(\omega)$ in equation (4.2), it is easy to know that $\mathcal{A}$ is absorbed by $D(\omega)$ when $\hat{r}(\omega)e^{-2+2\sqrt{\gamma}\omega(t)}$ converges to zero. □

Theorem 4.3. *Let $\alpha \leq 1, \alpha\beta > 1, \mathbb{E}\|u_0\|^2 < \infty$ and the noise coefficient $G(u) = \sqrt{\gamma}u$. Moreover, let*

$$0 \leq \gamma < \frac{16(\alpha\beta-1)(\beta-\alpha) - \alpha\beta^2}{16(\alpha\beta-1)(\beta-\alpha)}.$$

Then the stochastic nuclear spin generator system (1.2) possesses a one point random attractor, i.e., $\mathcal{B}(\omega) = \{a(\omega)\}$, where $\{a(\omega)\}$ is a random fixed point.

Proof. By the equation (4.2) and stochastic Itô integral, we obtain

$$\begin{aligned}
d\psi(t,\omega,x) =& x^2 e^{-2t+2\sqrt{\gamma}\omega(t)}(-2+2\gamma)dt + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}dt \\
&+ \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}(-2+2\gamma)dt\int_0^t e^{2(s-t)+2\sqrt{\gamma}\omega(t)-2\sqrt{\gamma}\omega(s)}ds \\
=&(-2+2\gamma)\left(x^2 e^{-2t+2\sqrt{\gamma}\omega(t)} + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}\int_0^t e^{2(s-t)+2\sqrt{\gamma}\omega(t)-2\sqrt{\gamma}\omega(s)}ds\right) + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}dt \\
=&(-2+2\gamma)\psi(t,\omega,x)dt + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}dt,
\end{aligned}$$

thus,

$$\frac{d}{dt}\mathbb{E}\psi(t,\omega,x) = (-2+2\gamma)\mathbb{E}\psi(t,\omega,x) + \frac{\alpha^2\beta^2}{2(\alpha\beta-1)}, \quad \psi(0,\omega,x) = x. \tag{4.3}$$

Then the solution corresponding to the equation (4.3) is

$$\mathbb{E}\psi(t,\omega,x) = xe^{-2(1-\gamma)t} + \frac{\alpha^2\beta^2}{4(1-\gamma)(\alpha\beta-1)}\left(1-e^{-2(1-\gamma)t}\right).$$

Since $\gamma < 1$, we have

$$\mathbb{E}\widetilde{r}^2(\omega) = (1+\sigma)\frac{\alpha^2\beta^2}{4(1-\gamma)(\alpha\beta-1)}.$$

Consider $u_1(t)$ and $u_2(t)$ as two different solutions contained in the random attractor. By the (v) of Lemma 2.9 with $L = \beta - \alpha$, we have

$$\begin{aligned}
\|u_1(t)-u_2(t)\|^2 =& \|u_1(0)-u_2(0)\|^2 \\
& - 2\int_0^t \left(\beta|x_1(s)-x_2(s)|^2 + \beta|y_1(s)-y_2(s)|^2 + \alpha\beta|z_1(s)-z_2(s)|^2\right) ds \\
& + 2\int_0^t (B(u_1(s)) - B(u_2(s)), u_1(s)-u_2(s))\, ds + 2\sqrt{\gamma}\int_0^t \|u_1(s)-u_2(s)\|^2 \circ dW_s \\
\leq& \|u_1(0)-u_2(0)\|^2 - 2\int_0^t \left(\beta|x_1-x_2|^2 + \beta|y_1-y_2|^2 + \alpha\beta|z_1-z_2|^2\right) ds \\
& + 2(4L)^{-1}\int_0^t \|u_2(s)\|\|u_1(s)-u_2(s)\|^2 ds + 2L\int_0^t |x_1(s)-x_2(s)|^2 ds \\
& + 2\sqrt{\gamma}\int_0^t \|u_1(s)-u_2(s)\|^2 \circ dW_s \\
=& \|u_1(0)-u_2(0)\|^2 - 2\alpha\int_0^t \|u_1(s)-u_2(s)\|^2 ds + 2\sqrt{\gamma}\int_0^t \|u_1(s)-u_2(s)\|^2 \circ dW_s \\
& + \frac{1}{2(\beta-\alpha)}\int_0^t \|u_2(s)\|\|u_1(s)-u_2(s)\|^2 ds.
\end{aligned}$$

Hence, we obtain the following inequality

$$\|u_1(t)-u_2(t)\|^2 \leq \|u_1(0)-u_2(0)\|^2 \exp\left(t\left(-2\alpha + \frac{1}{2(\beta-\alpha)}\frac{1}{t}\int_0^t \|u_2(s)\|ds + 2\sqrt{\gamma}\frac{\omega(t)}{t}\right)\right).$$

As the random attractor is a only subset of $\mathcal{D}$-absorbing set $D(\omega)$, taking two initial value in $D(\omega)$, we also estimate

$$\|u^x_{\theta_{-t}\omega}(s)\|^2 \leq \widetilde{r}^2(\theta_{-t+s}\omega) \quad \text{for all} \quad s,t \geq 0$$

and compute

$$\begin{aligned}
\lim_{t_n\to\infty} \sup_{x\in A(\theta_{-t}\omega)} \frac{1}{t_n}\int_0^{t_n} \|u^x_{\theta_{-t}\omega}(s)\|^2 ds &\leq (1+\rho)\frac{\alpha^2\beta^2}{4(1-\gamma)(\alpha\beta-1)} \lim_{t_n\to\infty}\frac{1}{t_n}\int_{-t_n}^0\int_{-\infty}^0 e^{2\tau-2\sqrt{\gamma}\theta_s\omega(\tau)} d\tau ds \\
&= (1+\rho)\frac{\alpha^2\beta^2}{4(1-\gamma)(\alpha\beta-1)}
\end{aligned}$$

for a sequence $t_n \to \infty$. By the two-side ergodic theorem [2], there is a sequence $t_n \to \infty$ so that the last transformation is true for a set Ω' of full measure. It is obvious that the supremum of the difference of two solution converges to zero if

$$1 > (1+\rho)\frac{\alpha\beta^2}{16(1-\gamma)(\alpha\beta-1)(\beta-\alpha)}.$$

Therefore, the random attractor consists in no more than one point $\{a(\omega)\}$ which is a random fixed point by the definition of random attractors. □

Corollary 4.4. *Let ϕ be an RDS generated by the solution of the stochastic nuclear spin generator system on $\Re^d$, if ϕ possesses a one point attractor $a(\omega)$ of ϕ w.r.t. an inclusion closed system (IC-system) $\mathcal{D}$ so that contains all deterministic compact sets, then*

$$w\text{-}\lim_{t\longrightarrow\infty} \mu\mathfrak{T}_t = \mathbb{E}\delta_{a(\cdot)} = \mathbb{P}\{a \in \cdot\}$$

for every $\mu \in \mathcal{P}(\Re^3)$.

Proof. Let $f \in \mathcal{C}_b(\Re^3)$, note that $\mathbb{E}X(\cdot) = \mathbb{E}X(\theta_{-t}\cdot)$ for any random variable X with finite expectation. By the definition of natural pairing and using Lebesgue's theorem, we have

$$\begin{aligned}
\lim_{t\to\infty}\int_{\Re^3} f(x)d(\mu\mathfrak{T}_t(x)) &= \lim_{t\to\infty}\int_{\Re^3} \mathbb{E}f(\phi(t,\cdot,x))d(\mu(x)) \\
&= \lim_{t\to\infty}\int_{\Re^3} \mathbb{E}f(\phi(t,\theta_{-t}\cdot,x))d(\mu(x)) \\
&= \int_{\Re^3} \mathbb{E}\lim_{t\to\infty} f(\phi(t,\theta_{-t}\cdot,x))d(\mu(x)) \\
&= \int_{\Re^3} \mathbb{E}f(a(\cdot))d(\mu(x)) \\
&= \mathbb{E}f(a(\cdot)) \\
&= \int_{\Re^3} f(x)d(\mathbb{E}\delta_{a(\cdot)})(x).
\end{aligned}$$

Then, we get

$$\text{w-}\lim_{t\longrightarrow\infty} \mu\mathfrak{T}_t = \mathbb{E}\delta_{a(\cdot)} = \mathbb{P}\{a \in \cdot\}.$$

Then the assertion is true. □

Corollary 4.5. *If the random $\mathcal{D}$-attractor possesses only a one point, $\mathcal{B}(\omega) = \{a(\omega)\}$, then the stochastic nuclear spin generator system possesses a unique stationary distribution $\varrho = \mathbb{E}\delta_{a(\cdot)}$.*

Proof. Let ϱ_1 and ϱ_2 denote two stationary distributions. By Corollary 4.4, we have

$$\varrho_1 = \text{w-}\lim_{t\longrightarrow\infty} \varrho_1\mathfrak{T}_t = \mathbb{E}\delta_{a(\cdot)} = \text{w-}\lim_{t\longrightarrow\infty} \varrho_2\mathfrak{T}_t = \varrho_2.$$

Then there exists a unique stationary distribution of the stochastic nuclear spin generator system. □

5. Numerical simulation results

According to our analytical results, the stochastic nuclear spin systems is bounded and possesses a one point random attractor under conditions specified in Corollary 3.3 and Theorem 4.3. We now try and support our analytical results by simulations (Fig.1–4).

For the stochastic nuclear spin system, formula (3.8) given in Remark 3.4 can provide estimations for the boundaries of the attractive sets formula (4.1) given in Theorem 4.2. System (1.2) can possesses a one point random attractor. The numerical results indicate that except for the case γ_i, all other cases exhibit either generalized Lorenz attractor or generalized Chen attractor. To illustrate the stochastic effects clearly, we performed simulations first for the deterministic case and a corresponding stochastic simulation (Fig. 1–4). The parameter values used in simulations have all been taken from published papers [3, 14, 16, 18]. The typical values for α, β and k are chosen to satisfy formula (3.8) or (4.1) and the values of γ_i are varied from 0 to 1.

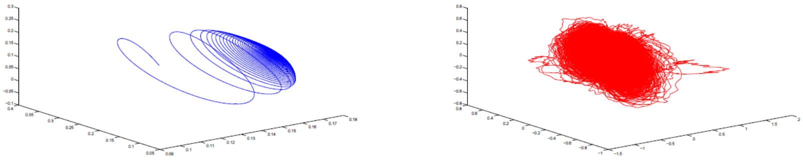

Figure 1: Condition of Theorem 4.3 or Corollary 3.3: $0 < \alpha \leq 1$ and $\alpha\beta > 1$. Simulated phase portraits of the stability of the bifurcation periodic solution of the nuclear spin systems (1.2) with the initial conditions $\alpha = 0.9, \beta = 1.12, k = 21.5$ and $\gamma_1 = 0.8, \gamma_1 = 0.5, \gamma_1 = 0.2$. Blue represent the simulation of the x-y-z of deterministic nuclear spin systems. Red represent the simulation of the x-y-z of stochastic nuclear spin systems.

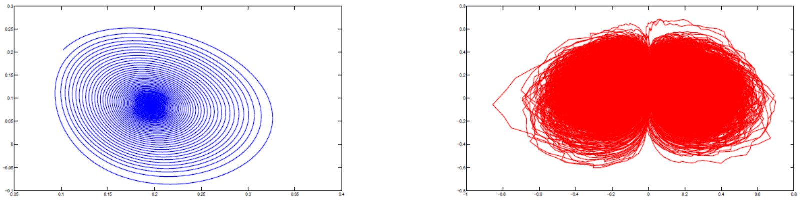

Figure 2: Condition of Theorem 4.3 or Corollary 3.3: $0 < \alpha \leq 1$ and $\alpha\beta > 1$. Simulated phase portraits of the stochastic nuclear spin systems (1.2) with the initial conditions $\alpha = 0.9, \beta = 1.12, k = 21.5$ and $\gamma_1 = 0.8, \gamma_1 = 0.5, \gamma_1 = 0.2$. Blue represent the simulation of the y-z of deterministic nuclear spin systems. Red represent the simulation of the y-z of stochastic nuclear spin systems.

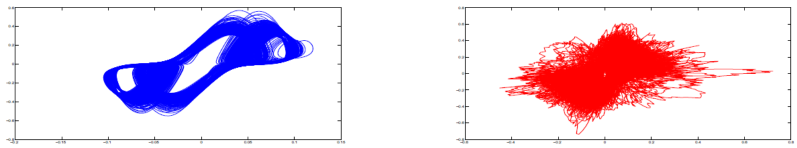

Figure 3: Condition: $0 < \alpha < 1$ and $\alpha\beta < 1$. Simulated phase portraits of the stochastic nuclear spin systems (1.2) with the initial conditions $\alpha = 0.495, \beta = 1.4, k = 21.5$ and $\gamma_1 = 0.8, \gamma_1 = 0.5, \gamma_1 = 0.2$. Blue represent the simulation of the x-y of deterministic nuclear spin systems. Red represent the simulation of the x-y of stochastic nuclear spin systems.

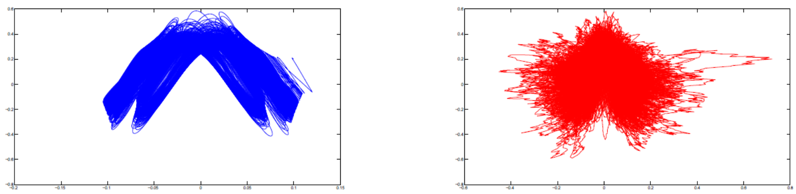

Figure 4: Condition: $0 < \alpha < 1$ and $\alpha\beta < 1$. Simulated phase portraits of the stochastic nuclear spin systems (1.2) with the initial conditions $\alpha = 0.495, \beta = 1.4, k = 21.5$ and $\gamma_1 = 0.8, \gamma_1 = 0.5, \gamma_1 = 0.2$. Blue represent the simulation of the x-z of deterministic nuclear spin systems. Red represent the simulation of the x-z of stochastic nuclear spin systems.

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