



A general iterative algorithm for common solutions of quasi variational inclusion and fixed point problems

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Abstract

In this paper, quasi-variational inclusion and fixed point problems are investigated based on a general iterative process. Strong convergence theorems are established in the framework of Hilbert spaces. ©2016 All rights reserved.

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1. Introduction

Fixed point problems have been found with an explosive growth in theoretical advances, algorithmic development and applications across all the discipline of pure and applied sciences, see [4, 5, 7, 8, 10–17, 27] and the references therein. Analysis of these problems requires a blend of techniques from non-smooth analysis, convex analysis, functional analysis and numerical analysis. As a result of the interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving these problems and related convex optimization problems. Variational inclusions involving two operators are useful and important extension and generalizations

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of the variational inequalities with a wide range of applications in economics, decision sciences, network, mathematical, and engineering sciences, see [1, 2, 6, 18, 21, 25–27] and the references therein. It is well known that the projection method and its variant forms including the Wiener-Hopf equations cannot be extended and modified for solving the variational inclusions, which motivate us to use new techniques and methods. Resolvent techniques recently have been investigated by many authors in the framework of Hilbert spaces, see [3, 22–24, 27–32] and the references therein. The given operator is decomposed into the sum of two monotone operators whose resolvent is easier to evaluate than the resolvent of the original sum operator. Such type of methods are called the operator splitting methods and have proved to be very effective for solving inclusion problems involving two operators.

In this paper, we study a general iterative process for common solutions of quasi-variational inclusion and fixed point problems. Strong convergence theorems are established in the framework of Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, the strong convergence theorem is established in the framework of Hilbert spaces. Some sub-results and applications are provided to support our main results.

2. Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let B be a mapping on H . Recall that the following definitions.

B is said to be monotone iff

$$\langle Bx - By, x - y \rangle \geq 0 \quad \forall x, y \in H.$$

B is said to be r -strongly monotone iff there exists a constant $r > 0$ such that

$$\langle Bx - By, x - y \rangle \geq r\|x - y\|^2 \quad \forall x, y \in H.$$

B is said to be r -inverse-strongly monotone iff there exists a constant $r > 0$ such that

$$\langle Bx - By, x - y \rangle \geq r\|Bx - By\|^2 \quad \forall x, y \in H.$$

Recall that a set-valued mapping $M : H \rightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ implies $\langle x - y, f - g \rangle \geq 0$. The monotone mapping $M : H \rightarrow 2^H$ is maximal if the graph of $G(M)$ of M is not properly contained in the graph of any other monotone mapping.

Consider the following so-called quasi-variational inclusion problem: find an $u \in H$ for a given element $f \in H$ such that

$$f \in Bu + Mu, \tag{2.1}$$

where $B : H \rightarrow H$ and $M : H \rightarrow 2^H$ are two nonlinear mappings, see, for example, [9] and the references therein. A special case of problem (2.1) is to find an element $u \in H$ such that

$$0 \in Bu + Mu. \tag{2.2}$$

In this paper, we use $VI(H, B, M)$ to denote the solution of problem (2.2). A number of problems arising in structural analysis, mechanics, and economic can be studied in the framework of this class of variational inclusions.

Next, we consider two special cases of problem (2.2).

- (1) If $M = \partial\phi : H \rightarrow 2^H$, where $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the sub-differential of ϕ , then problem (2.2) is equivalent to find $u \in H$ such that

$$\langle Bu, v - u \rangle + \phi(v) - \phi(u) \geq 0 \quad \forall v \in H,$$

which is said to be the mixed quasi-variational inequality.

- (2) If ϕ is the indicator function of C , then problem (2.2) is equivalent to the classical variational inequality problem, denoted by $VI(C, B)$, which is to find $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0 \quad (2.3)$$

for all $v \in C$.

Let S be a nonlinear mapping on H . $F(S)$ stands for the fixed point set of S . Recall that S is said to be α -contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\| \quad \forall x, y \in H.$$

S is said to be non-expansive iff

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in H.$$

S is said to be k -strictly pseudo-contractive iff there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k \|x - y - Sx + Sy\|^2 \quad \forall x, y \in H.$$

The class of k -strictly pseudo-contractive mappings was introduced by Browder and Petryshn [5] in 1967.

A typical problem is to minimize a quadratic function over the set of the fixed points of a non-expansive mapping on a real Hilbert space H :

$$\min_{x \in F(S)} \left(\frac{1}{2} \langle Ax, x \rangle - h(x) \right), \quad (2.4)$$

where A is a linear bounded and strongly positive operator, $F(S)$ is the fixed point set of non-expansive mapping S and h is a potential function for γf , that is, $h'(x) = \gamma f(x)$ for $x \in H$.

Iterative methods for non-expansive mappings have recently been applied to solve convex minimization problems. Marino and Xu [20] studied the following iterative scheme

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0,$$

where f is a α -contractive mapping. They proved $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle x - x^*, (A - \gamma f)x^* \rangle \geq 0, \quad x \in F(S),$$

which is the optimality condition for minimization problem (2.4).

Recently, Zhang, Lee and Chan [32] considered problem (2.2). To be more precise, they proved the following theorem.

Theorem 2.1. *Let H be a real Hilbert space, $B : H \rightarrow H$ an α -inverse-strongly monotone mapping, $M : H \rightarrow 2^H$ a maximal monotone mapping, and $S : H \rightarrow H$ a nonexpansive mapping. Suppose that the set $F(S) \cap VI(H, B, M) \neq \emptyset$, where $VI(H, B, M)$ is the set of solutions of variational inclusion (2.2). Suppose $x_0 = x \in H$ and $\{x_n\}$ is the sequence defined by*

$$\begin{cases} x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S y_n, \\ y_n = J_{M, \lambda}(x_n - \lambda B x_n), \quad n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (b) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then, $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(H, B, M)} x_0$.

To prove our main results, we also need the following lemmas.

Lemma 2.2 ([32]). *Let $M : H \rightarrow 2^H$ be a multi-valued maximal monotone mapping. Then the single-valued mapping $J_{M,\lambda} : H \rightarrow H$ defined by $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u) \forall u \in H$ is called the resolvent operator associated with M , where λ is any positive number and I is the identity mapping. The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and non-expansive for all $\lambda > 0$. $u \in H$ is a solution of variational inclusion (2.2) if and only if $u = J_{M,\lambda}(u - \lambda Bu,) \forall \lambda > 0$, that is,*

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)) \quad \forall \lambda > 0.$$

Lemma 2.3 ([19]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (a) $\sum_{n=1}^\infty \gamma_n = \infty$;
- (b) $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^\infty |\delta_n| < \infty$.

Then, $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Lemma 2.4 ([4]). *Let H be a real Hilbert space H . Let T be a strictly pseudo-contractive mapping with fixed points. Then, $I - T$ is demiclosed at zero, that is, $x_n \rightarrow x$ and $x_n - Tx_n \rightarrow 0$, we have $x \in F(T)$.*

Lemma 2.5 ([12]). *Let H be a real Hilbert space and $M : H \rightarrow 2^H$ be a maximal monotone mapping and $P : H \rightarrow H$ be a hemi-continuous bounded monotone mapping with $D(M) = H$. Then, mapping $M + P : H \rightarrow 2^H$ is maximal monotone.*

3. Main results

Theorem 3.1. *Let H be a real Hilbert space and $M : H \rightarrow 2^H$ a maximal monotone mapping. Let $B : H \rightarrow H$ be a r -inverse-strongly monotone and let T be a k -strictly pseudo-contractive mapping on H . Let f be a contraction of H into itself with the contractive coefficient α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma} / \alpha$ and $\Omega = F(T) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in H$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \kappa T J_{M,\lambda}(x_n - \lambda Bx_n) + (1 - \kappa) J_{M,\lambda}(x_n - \lambda Bx_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n \quad \forall n \geq 1, \end{cases}$$

where $\kappa \in (0, 1 - k]$, $\lambda \in (0, 2r]$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solves the following variational inequality

$$\langle Az - \gamma f(z), z - w \rangle \leq 0 \quad \forall w \in \Omega. \tag{3.1}$$

Proof. Putting $S = \kappa T + (1 - \kappa)I$, we see that S is nonexpansive with $F(S) = F(T)$. Indeed, we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \kappa \|Tx - Ty\|^2 + (1 - \kappa) \|x - y\|^2 - \kappa(1 - \kappa) \|(Tx - Ty) - (x - y)\|^2 \\ &\leq \kappa \|x - y\|^2 + \kappa k \|(Tx - Ty) - (x - y)\|^2 + (1 - \kappa) \|x - y\|^2 \\ &\quad - \kappa(1 - \kappa) \|(Tx - Ty) - (x - y)\|^2 \\ &\leq \|x - y\|^2 - \kappa[(1 - \kappa) - k] \|(Tx - Ty) - (x - y)\|^2 \\ &\leq \|x - y\|^2 \quad \forall x, y \in H. \end{aligned}$$

From the strong monotonicity of $A - \gamma f$, we get the uniqueness of the solution of the variational inequality (3.1). Suppose $z_1 \in \Omega$ and $z_2 \in \Omega$ both are solutions to (3.1). It follows that

$$\langle Az_2 - \gamma f(z_2), z_2 - z_1 \rangle \leq 0$$

and

$$\langle Az_1 - \gamma f(z_1), z_1 - z_2 \rangle \leq 0.$$

Adding up the two inequalities, we see that

$$\langle z_2 - z_1, (A - \gamma f)z_1 - (A - \gamma f)z_2 \rangle \geq 0.$$

The strong monotonicity of $A - \gamma f$ implies that $z_1 = z_2$ and the uniqueness is proved. Below we use z to denote the unique solution of (3.1). From the condition on λ , we have

$$\begin{aligned} \|(I - \lambda B)x - (I - \lambda B)y\|^2 &= \lambda^2 \|Bx - By\|^2 + \|x - y\|^2 - 2\lambda \langle x - y, Bx - By \rangle \\ &\leq \|x - y\|^2 - \lambda(2r - \lambda) \|Bx - By\|^2 \\ &\leq \|x - y\|^2, \end{aligned}$$

which implies mapping $I - \lambda B$ is nonexpansive. Taking $x^* \in \Omega$, we find from Lemma 2.2 that $x^* = J_{M,\lambda}(x^* - \lambda Bx^*)$. It follows that

$$\|y_n - x^*\| \leq \|J_{M,\lambda}(x_n - \lambda Bx_n) - J_{M,\lambda}(x^* - \lambda Bx^*)\| \leq \|x_n - x^*\|.$$

Note that from the conditions, we may assume, without loss of generality, that $\alpha_n \leq \|A\|^{-1}$ for all $n \geq 1$. Since A is a strongly positive linear bounded self-adjoint operator, we have $\|A\| = \sup\{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$. Now, for $x \in C$ with $\|x\| = 1$, we see that

$$0 \leq 1 - \alpha_n \|A\| \leq 1 - \alpha_n \langle Ax, x \rangle = \langle (I - \alpha_n A)x, x \rangle,$$

that is, $I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|I - \alpha_n A\| &= \sup\{\langle (I - \alpha_n A)x, x \rangle : x \in C, \|x\| = 1\} \\ &= \sup\{1 - \alpha_n \langle Ax, x \rangle : x \in C, \|x\| = 1\} \\ &\leq 1 - \alpha_n \bar{\gamma}. \end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n \|\gamma f(x_n) - Ax^*\| + \|I - \alpha_n A\| \|S J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ax^*\| + (1 - \alpha_n \bar{\gamma}) \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ax^*\| + (1 - \alpha_n \bar{\gamma}) \|x_n - x^*\| \\ &\leq \alpha_n \|\gamma f(x_n) - \gamma f(x^*)\| + \alpha_n \|\gamma f(x^*) - Ax^*\| \\ &\quad + (1 - \alpha_n \bar{\gamma}) \|x_n - x^*\| \\ &\leq \alpha \alpha_n \gamma \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Ax^*\| + (1 - \alpha_n \bar{\gamma}) \|x_n - x^*\| \\ &= [1 - \alpha_n (\bar{\gamma} - \alpha \gamma)] \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Ax^*\|. \end{aligned}$$

This implies that

$$\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \alpha \gamma}\},$$

which gives that sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Next, we show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Note that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|J_{M,\lambda}(x_n - \lambda Bx_n) - J_{M,\lambda}(x_{n+1} - \lambda Bx_{n+1})\| \\ &\leq \|(x_n - \lambda Bx_n) - (x_{n+1} - \lambda Bx_{n+1})\| \\ &\leq \|x_n - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_{n+1} - x_n\| = \|\alpha_{n+1} \gamma f(x_{n+1}) - \alpha_n \gamma f(x_n) + (I - \alpha_{n+1} A)y_{n+1} - (I - \alpha_n A)y_n\|$$

$$\begin{aligned} &\leq \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + \gamma|\alpha_{n+1} - \alpha_n|\|f(x_n)\| + \|I - \alpha_{n+1}A\|\|y_{n+1} - y_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|Ay_n\| \\ &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + \gamma|\alpha_{n+1} - \alpha_n|\|f(x_n)\| + (1 - \alpha_{n+1}\bar{\gamma})\|y_{n+1} - y_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|Ay_n\| \\ &\leq [1 - \alpha_n(\bar{\gamma} - \alpha\gamma)]\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|(\gamma\|f(x_n)\| + \|Ay_n\|). \end{aligned}$$

In view of Lemma 2.3, one has

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.2}$$

Since

$$x_{n+1} - y_n = \alpha_n(\gamma f(x_n) - Ay_n),$$

which implies from the restriction imposed on $\{\alpha_n\}$ that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0. \tag{3.3}$$

Combing (3.2) with (3.3), one finds that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{3.4}$$

On the other hand, one has

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (\alpha_n\|\gamma f(x_n) - Ax^*\| + (1 - \alpha_n\bar{\gamma})\|SJ_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|)^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Ax^*\|^2 + (1 - \alpha_n\bar{\gamma})\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - Ax^*\|\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n\|\gamma f(x_n) - Ax^*\|^2 + (1 - \alpha_n\bar{\gamma})\|(x_n - \lambda Bx_n) - (x^* - \lambda Bx^*)\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - Ax^*\|\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n\|\gamma f(x_n) - Ax^*\|^2 + (1 - \alpha_n\bar{\gamma})\|x_n - x^*\|^2 - (1 - \alpha_n\bar{\gamma})\lambda(2r - \lambda)\|Bx_n - Bx^*\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - Ax^*\|\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n\|\gamma f(x_n) - Ax^*\|^2 + \|x_n - x^*\|^2 - (1 - \alpha_n\bar{\gamma})\lambda(2r - \lambda)\|Bx_n - Bx^*\|^2 \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} (1 - \alpha_n\bar{\gamma})\lambda(2r - \lambda)\|Bx_n - Bx^*\|^2 &\leq \alpha_n\|\gamma f(x_n) - Ax^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n\|\gamma f(x_n) - Ax^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Ax^*\|\|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|. \end{aligned}$$

From (3.2), one has

$$\lim_{n \rightarrow \infty} \|Bx_n - Bx^*\| = 0. \tag{3.5}$$

Since $J_{M,\lambda}$ is firmly nonexpansive, one has

$$\begin{aligned} \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|^2 &\leq \langle (x_n - \lambda Bx_n) - (x^* - \lambda Bx^*), J_{M,\lambda}(x_n - \lambda Bx_n) - x^* \rangle \\ &\leq \frac{1}{2} \left(\|(x_n - \lambda Bx_n) - (x^* - \lambda Bx^*)\|^2 + \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|^2 \right. \\ &\quad \left. - \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n) - \lambda(Bx_n - Bx^*)\|^2 \right). \end{aligned}$$

Therefore, we arrive at

$$\begin{aligned} \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\|^2 \\ &\quad - \lambda \|Bx_n - Bx^*\|^2 + 2\lambda \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\| \|Bx_n - Bx^*\|. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (\alpha_n \|\gamma f(x_n) - Ax^*\| + (1 - \alpha_n \bar{\gamma}) \|SJ_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|)^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + (1 - \alpha_n \bar{\gamma}) \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|^2 \\ &\quad + 2\alpha_n (1 - \alpha_n \bar{\gamma}) \|\gamma f(x_n) - Ax^*\| \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + (1 - \alpha_n \bar{\gamma}) \|x_n - x^*\|^2 - (1 - \alpha_n \bar{\gamma}) \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\|^2 \\ &\quad - \lambda (1 - \alpha_n \bar{\gamma}) \|Bx_n - Bx^*\|^2 + 2\lambda (1 - \alpha_n \bar{\gamma}) \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\| \|Bx_n - Bx^*\| \\ &\quad + 2\alpha_n (1 - \alpha_n \bar{\gamma}) \|\gamma f(x_n) - Ax^*\| \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \|x_n - x^*\|^2 - (1 - \alpha_n \bar{\gamma}) \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\|^2 \\ &\quad + 2\lambda \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\| \|Bx_n - Bx^*\| \\ &\quad + 2\alpha_n \|\gamma f(x_n) - Ax^*\| \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} (1 - \alpha_n \bar{\gamma}) \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\|^2 &\leq \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + 2\lambda \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\| \|Bx_n - Bx^*\| \\ &\quad + 2\alpha_n \|\gamma f(x_n) - Ax^*\| \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\| \\ &\leq \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_n - x_{n+1}\| \\ &\quad + 2\lambda \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\| \|Bx_n - Bx^*\| \\ &\quad + 2\alpha_n \|\gamma f(x_n) - Ax^*\| \|J_{M,\lambda}(x_n - \lambda Bx_n) - x^*\|. \end{aligned}$$

Using (3.2) and (3.5), one gets that

$$\lim_{n \rightarrow \infty} \|x_n - J_{M,\lambda}(x_n - \lambda Bx_n)\| = 0. \tag{3.6}$$

Now, we are in a position to prove that

$$\limsup_{n \rightarrow \infty} \langle x_n - z, (\gamma f - A)z \rangle \leq 0,$$

where $z = P_\Omega[I - (A - \gamma f)]z$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle x_n - z, (\gamma f - A)z \rangle = \lim_{i \rightarrow \infty} \langle x_{n_i} - z, (\gamma f - A)z \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w . Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. Next, we show that $w \in F(S) \cap VI(H, M, B)$. Note that

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - SJ_{M,\lambda}(x_n - \lambda Bx_n)\| + \|SJ_{M,\lambda}(x_n - \lambda Bx_n) - Sx_n\| \\ &\leq \|x_n - y_n\| + \|J_{M,\lambda}(x_n - \lambda Bx_n) - x_n\|. \end{aligned}$$

Using (3.4) and (3.6), one has $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$. From Lemma 2.4, one gets $w \in F(S) = F(T)$.

Next, we prove $x \in VI(H, M, B)$. In fact, since B is r -inverse-strongly monotone, it follows from B is Lipschitz continuous. It follows from Lemma 2.5 that $M + B$ is a maximal monotone operator. Let $(u, v) \in G(M + A)$, that is, $v - Bu \in M(u)$. Setting $t_n = J_{M,\lambda}(x_n - \lambda Bx_n)$, we have $x_n - \lambda Bx_n \in t_n + \lambda Mt_n$, that is,

$$\frac{x_n - t_n}{\lambda} - Bx_n \in Mt_n.$$

By virtue of the maximal monotonicity of $M + B$, we have

$$\langle u - t_n, v - Bu - \frac{x_n - t_n}{\lambda} + Bx_n \rangle \geq 0.$$

Hence, we have

$$\begin{aligned} \langle u - t_n, v \rangle &\geq \langle u - t_n, Bu + \frac{x_n - t_n}{\lambda} - Bx_n \rangle \\ &= \langle u - t_n, Bu - Bt_n + Bt_n - Bx_n + \frac{x_n - t_n}{\lambda} \rangle \\ &\geq \langle u - t_n, Bt_n - Bx_n \rangle + \langle u - t_n, \frac{x_n - t_n}{\lambda} \rangle. \end{aligned}$$

From (3.6), we have $\langle u - w, v \rangle \geq 0$. Since $B + M$ is maximal monotone, this implies that $0 \in (M + B)(w)$, that is, $w \in VI(H, M, B)$, and so $w \in F(T) \cap VI(H, M, B)$.

Finally, we show that $x_n \rightarrow z$, as $n \rightarrow \infty$. Indeed,

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(I - \alpha_n A)(Sy_n - z) + \alpha_n(\gamma f(x_n) - Az)\|^2 \\ &\leq \|(I - \alpha_n A)(Sy_n - z)\|^2 + 2\alpha_n \langle \gamma f(x_n) - Az, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma}) \|y_n - z\|^2 + 2\alpha_n \langle \gamma f(x_n) - Az, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma}) \|x_n - z\|^2 + 2\alpha_n \gamma \langle f(x_n) - f(z), x_{n+1} - z \rangle + 2\alpha_n \langle \gamma f(z) - Az, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma}) \|x_n - z\|^2 + 2\alpha_n \gamma \|f(x_n) - f(z)\| \|x_{n+1} - z\| + 2\alpha_n \langle \gamma f(z) - Az, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma}) \|x_n - z\|^2 + 2\alpha_n \gamma \alpha \|x_n - z\| \|x_{n+1} - z\| + 2\alpha_n \langle \gamma f(z) - Az, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n \bar{\gamma}) \|x_n - z\|^2 + \alpha_n \gamma \alpha (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + 2\alpha_n \langle \gamma f(z) - Az, x_{n+1} - z \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \frac{(1 - \alpha_n \bar{\gamma}^2) + \alpha_n \gamma \alpha}{1 - \alpha_n \gamma \alpha} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(z) - Az, x_{n+1} - z \rangle \\ &\leq \frac{(1 - 2\alpha_n \bar{\gamma} + \alpha_n \gamma \alpha)}{1 - \alpha_n \gamma \alpha} \|x_n - z\|^2 + \frac{\alpha_n^2 \bar{\gamma}^2}{1 - \alpha_n \gamma \alpha} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma \alpha} \langle \gamma f(z) - Az, x_{n+1} - z \rangle \\ &\leq \left(1 - \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha}\right) \|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n \gamma \alpha} \left(\frac{1}{\bar{\gamma} - \alpha\gamma} \langle \gamma f(z) - Az, x_{n+1} - z \rangle + \frac{1}{2(\bar{\gamma} - \alpha\gamma)} M\right), \end{aligned}$$

where M is an appropriate constant such that $M \geq \sup_{n \geq 1} \{\|x_n - z\|\}$. Using Lemma 2.3, we find the desired conclusion immediately. This completes the proof. \square

From Theorem 3.1, the following results are not hard to derive.

Corollary 3.2. *Let H be a real Hilbert space and $M : H \rightarrow 2^H$ a maximal monotone mapping. Let $B : H \rightarrow H$ be a r -inverse-strongly monotone and let T be a nonexpansive mapping on H . Let f be a contraction of H into itself with the contractive coefficient α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\Omega = F(T) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in H$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = TJ_{M,\lambda}(x_n - \lambda Bx_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n \quad \forall n \geq 1, \end{cases}$$

where $\lambda \in (0, 2r]$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solves the following variational inequality

$$\langle Az - \gamma f(z), z - w \rangle \leq 0 \quad \forall w \in \Omega.$$

Corollary 3.3. *Let H be a real Hilbert space and $M : H \rightarrow 2^H$ a maximal monotone mapping. Let $B : H \rightarrow H$ be a r -inverse-strongly monotone. Let f be a contraction of H into itself with the contractive coefficient α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $VI(H, B, M) \neq \emptyset$. Let $x_1 \in H$ and $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) J_{M, \lambda}(x_n - \lambda B x_n) \quad \forall n \geq 1,$$

where $\lambda \in (0, 2r]$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solves the following variational inequality

$$\langle Az - \gamma f(z), z - w \rangle \leq 0 \quad \forall w \in VI(H, B, M).$$

4. Applications

If T is k -strictly pseudocontractive, then $I - T$ is $\frac{1-k}{2}$ -inverse-strongly monotone. We are in a position to give a result on common fixed points of a pair of strictly pseudocontractive mappings.

Theorem 4.1. *Let H be a real Hilbert space. Let T be a k -strictly pseudo-contractive mapping on H and let S be a \bar{k} -strictly pseudo-contractive mapping on H . Let f be a contraction of H into itself with the contractive coefficient α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\Omega = F(T) \cap F(S) \neq \emptyset$. Let $x_1 \in H$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \lambda S x_n + (1 - \lambda)x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)(\kappa T y_n + (1 - \kappa)y_n) \end{cases} \quad \forall n \geq 1,$$

where $\kappa \in (0, 1 - k]$, $\lambda \in (0, 1 - \bar{k}]$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solves the following variational inequality

$$\langle Az - \gamma f(z), z - w \rangle \leq 0 \quad \forall w \in \Omega.$$

Proof. Putting $B := I - S$, we find B is $\frac{1-\bar{k}}{2}$ -inverse-strongly monotone. We also have $VI(H, B) = F(S)$ and $\lambda S x_n + (1 - \lambda)x_n = J_{M, \lambda}(x_n - \lambda S x_n)$. From Theorem 3.1, we obtain the desired result immediately. \square

Let C be a nonempty closed and convex subset of H and $B : C \rightarrow H$ be a mapping. Recall that the classical variational inequality is to find an $x \in C$ such that $\langle Bx, y - x \rangle \geq 0 \quad \forall y \in C$. The solution set of the variational inequality is denoted by $VI(C, A)$. It is known that x is a solution to the variational inequality iff x is a fixed point of the mapping $P_C(I - \lambda A)$, where I denotes the identity on H . Let i_C be a function defined by $i_C(x) = 0, x \in C, i_C(x) = \infty, x \notin C$. It is easy to see that i_C is a proper lower and semicontinuous convex function on H , and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent $J_{i_C, \lambda} := (I + \lambda \partial i_C)^{-1}$ of the subdifferential operator ∂i_C . Letting $x = J_{i_C, \lambda} y$, we find that

$$y \in x + \lambda \partial i_C x \iff y \in x + \lambda N_C x \iff x = P_C y,$$

where $N_C x := \{e \in H : \langle e, v - x \rangle \leq 0 \quad \forall v \in C\}$. Putting $M = \partial i_C$ in Theorems 3.1, we find the following results immediately.

From the above and Theorem 3.1, we immediately find the results.

Theorem 4.2. *Let C be a nonempty closed and convex subset of a real Hilbert space. Let $B : C \rightarrow H$ be a r -inverse-strongly monotone and let T be a k -strictly pseudo-contractive mapping on C . Let f be a contraction of C into itself with the contractive coefficient α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded*

self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\Omega = F(T) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \kappa TP_C(x_n - \lambda Bx_n) + (1 - \kappa)P_C(x_n - \lambda Bx_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n \quad \forall n \geq 1, \end{cases}$$

where $\kappa \in (0, 1 - k]$, $\lambda \in (0, 2r]$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solves the following variational inequality

$$\langle Az - \gamma f(z), z - w \rangle \leq 0 \quad \forall w \in \Omega.$$

For the class of nonexpansive mappings, we have the following results.

Corollary 4.3. Let C be a nonempty closed and convex subset of a real Hilbert space. Let $B : C \rightarrow H$ be a r -inverse-strongly monotone and let T be a nonexpansive mapping on C . Let f be a contraction of C into itself with the contractive coefficient α ($0 < \alpha < 1$) and let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\Omega = F(T) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)TP_C(x_n - \lambda Bx_n) \quad \forall n \geq 1,$$

where $\lambda \in (0, 2r]$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then, $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solves the following variational inequality

$$\langle Az - \gamma f(z), z - w \rangle \leq 0 \quad \forall w \in \Omega.$$

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