Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4148–4166
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Multi-step hybrid steepest-descent methods for split feasibility problems with hierarchical variational inequality problem constraints

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Communicated by Y. Yao

Abstract

In this paper, we introduce and analyze a multi-step hybrid steepest-descent algorithm by combining Korpelevich's extragradient method, viscosity approximation method, hybrid steepest-descent method, Mann's iteration method and gradient-projection method (GPM) with regularization in the setting of infinite-dimensional Hilbert spaces. Strong convergence was established.

Keywords: Hybrid steepest-descent method, split feasibility problem, generalized mixed equilibrium problem, variational inclusion, maximal monotone mapping, nonexpansive mapping.
2010 MSC: 49J30, 47H09, 47J20, 49M05.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous (or L-Lipschitzian) if there exists a constant $L \geq 0$ such that $\|Sx - Sy\| \leq L\|x - y\|, \forall x, y \in C$. In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in [0, 1)$ then S is called a contraction. Let $A : C \to H$

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Received 2016-06-23

be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, A)$.

In 1976, Korpelevich [26] proposed an iterative algorithm for solving the VIP (1.1) in Euclidean space $\mathbf{R}^n$, which is known as the extragradient method. Subsequently, many authors improved it in various ways; see e.g., [9–11, 15, 19, 27, 28] and references therein.

On the other hand, let C and Q be nonempty closed convex subsets of infinite-dimensional real Hilbert spaces H and $\mathcal{H}$, respectively. The split feasibility problem (SFP) is to find a point x^* with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.2}$$

where $A \in B(H, \mathcal{H})$ and $B(H, \mathcal{H})$ denotes the family of all bounded linear operators from H to $\mathcal{H}$. We denote by Γ the solution set of the SFP (1.2). In 1994, the SFP (1.2) was first introduced by Censor and Elfving [22], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

Assume that the SFP (1.2) is consistent, that is, the solution set Γ of the SFP (1.2) is nonempty. Let $f : H \to \mathbf{R}$ be a continuous differentiable function. The minimization problem $\min_{x\in C} f(x) := \|Ax - P_Q Ax\|^2/2$ is ill-posed. In 2010, Xu [35] considered the following Tikhonov regularization problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2,$$

where $\alpha > 0$ is the regularization parameter. Very recently, by combining the gradient-projection method with regularization and extragradient method due to Nadezhkina and Takahashi [27], Ceng, Ansari and Yao [12] proposed a Mann type extragradient-like algorithm, and proved that the sequences generated by the proposed algorithm converge weakly to a common solution of the SFP (1.2) and the fixed point problem of a nonexpansive mapping.

On the other hand, let S and T be two nonexpansive self-mappings on a nonempty closed convex subset C of a real Hilbert space H. In 2009, Yao, Liou and Marino [40, Theorem 3.2] considered the following hierarchical variational inequality problem (HVIP): find hierarchically a fixed point of T, which is a solution to the VIP for monotone mapping $I - S$; namely, find $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)\tilde{x}, p - \tilde{x}\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{1.3}$$

The solution set of the HVIP (1.3) is denoted by $\mho$. It is not hard to check that solving the HVIP (1.3) is equivalent to the fixed point problem of the composite mapping $P_{\mathrm{Fix}(T)}S$, that is, find $\tilde{x} \in C$ such that $\tilde{x} = P_{\mathrm{Fix}(T)}S\tilde{x}$. They introduced and analyzed an iterative algorithm for solving the HVIP (1.3).

Furthermore, let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [28] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

We denote the set of solutions of GMEP (1.4) by $\mathrm{GMEP}(\Theta, \varphi, A)$. The GMEP (1.4) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. The GMEP is further considered and studied; see e.g., [7, 9, 13–15, 19].

It was assumed in [28] that $\Theta : C\times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2):

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x,y)+\Theta(y,x)\le 0$ for any $x,y\in C$;

(A3) Θ is upper-hemicontinuous, that is, for each $x,y,z\in C$,

$$\limsup_{t\to 0^+}\Theta(tz+(1-t)x,y)\le \Theta(x,y);$$

(A4) $\Theta(x,\cdot)$ is convex and lower semicontinuous for each $x\in C$;

(B1) for each $x\in H$ and $r>0$, there exists a bounded subset $D_x\subset C$ and $y_x\in C$ such that for any $z\in C\setminus D_x$, $\Theta(z,y_x)+\varphi(y_x)-\varphi(z)+\frac{1}{r}\langle y_x-z,z-x\rangle<0$;

(B2) C is a bounded set.

In addition, let B be a single-valued mapping of C into H and R be a multivalued mapping with $D(R)=C$. Consider the following variational inclusion: find a point $x\in C$ such that

$$0\in Bx+Rx. \tag{1.5}$$

We denote by $\mathrm{I}(B,R)$ the solution set of the variational inclusion (1.5). Let a set-valued mapping $R: D(R)\subset H\to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda}:H\to\overline{D(R)}$ associated with R and λ by $J_{R,\lambda}x=(I+\lambda R)^{-1}x,\quad \forall x\in H$, where λ is a positive number. In 1998, Huang [25] studied problem (1.5) in the case where R is maximal monotone and B is strongly monotone and Lipschitz continuous with $D(R)=C=H$. Subsequently, Zeng, Guu and Yao [45] further studied problem (1.5) in the case which is more general than Huang's one [25].

In this paper, we introduce and analyze a multi-step hybrid steepest-descent algorithm by combining Korpelevich's extragradient method, viscosity approximation method, hybrid steepest-descent method, Mann's iteration method and gradient-projection method (GPM) with regularization in the setting of infinite-dimensional Hilbert spaces. It is proven that under appropriate assumptions the proposed algorithm converges strongly to a solution of the SFP (1.2) with constraints of several problems: finitely many GMEPs, finitely many variational inclusions and the fixed point problem of an infinite family of nonexpansive mappings. Our results improve, extend and develop the corresponding results in the literature; see e.g., [40, Theorems 3.1 and 3.2] and [12, Theorem 3.2]. Recent results in this directions can be also found, e.g., in [1, 2, 4, 5, 8, 16, 17, 20, 21, 32, 33, 37–39, 42–44].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n\rightharpoonup x$ (resp. $x_n\to x$) to indicate that the sequence $\{x_n\}$ converges weakly (resp. strongly) to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of $\{x_n\}$, that is, $\omega_w(x_n):=\{x\in H: x_{n_i}\rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}$. The metric projection from H onto C is the mapping $P_C:H\to C$ which assigns to each point $x\in H$, the unique point $P_Cx\in C$ such that $\|x-P_Cx\|=\inf_{y\in C}\|x-y\|$.

Definition 2.1. Let T be a mapping with domain $D(T)\subset H$ and range $R(T)\subset H$. Then T is said to be

(i) monotone if $\langle Tx-Ty,x-y\rangle\ge 0,\forall x,y\in D(T)$;

(ii) β-strongly monotone if $\langle Tx-Ty,x-y\rangle\ge\eta\|x-y\|^2,\forall x,y\in D(T)$, for some $\beta>0$;

(iii) ν-inverse-strongly monotone if $\langle Tx-Ty,x-y\rangle\ge\nu\|Tx-Ty\|^2,\forall x,y\in D(T)$, for some $\nu>0$.

It is clear that if T is ν-inverse-strongly monotone, then T is monotone and $\frac{1}{\nu}$-Lipschitz continuous. Moreover, one also has that, for all $u,v\in D(T)$ and $\lambda>0$,

$$\|(I-\lambda T)u-(I-\lambda T)v\|^2\le\|u-v\|^2+\lambda(\lambda-2\nu)\|Tu-Tv\|^2. \tag{2.1}$$

So, if $\lambda\le 2\nu$, then $I-\lambda T$ is a nonexpansive mapping. Next, some important properties of projections are gathered in the following proposition.

Proposition 2.2 ([24]). *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \ \Leftrightarrow \ \langle x - z, y - z\rangle \leq 0, \ \forall y \in C$;

(ii) $z = P_C x \ \Leftrightarrow \ \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \ \forall y \in H$.

Definition 2.3. A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism); alternatively, T is firmly nonexpansive if and only if T is expressed as $T = (I + S)/2$, where S is nonexpansive on H.

Proposition 2.4 ([18]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)-(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is,*

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle$$

for any $x, y \in H$;

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex;*

(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ *for all $s, t > 0$ and $x \in H$.*

Definition 2.5. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is, $T \equiv (1 - \alpha)I + \alpha S$ where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged mappings.

Proposition 2.6 ([6]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.7 ([6, 23]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If* $T = (1-\alpha)S + \alpha V$ *for some* $\alpha \in (0,1)$ *and if* S *is firmly nonexpansive and* V *is nonexpansive, then* T *is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings* $\{T_i\}_{i=1}^N$ *is averaged, then so is the composite* $T_1 \cdots T_N$. *In particular, if* T_1 *is* α_1*-averaged and* T_2 *is* α_2*-averaged, where* $\alpha_1, \alpha_2 \in (0,1)$, *then the composite* T_1T_2 *is* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.8 ([31])**.** *Let* X *be a real inner product space. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Lemma 2.9 ([31])**.** *Let* H *be a real Hilbert space. Then the following hold:*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0,1]$ *with* $\lambda + \mu = 1$.

Let $\{T_n\}_{n=1}^\infty$ be an infinite family of nonexpansive self-mappings on C and $\{\rho_n\}_{n=1}^\infty$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a self-mapping W_n on C as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \rho_n T_n U_{n,n+1} + (1-\rho_n)I, \\ \cdots \\ U_{n,k} = \rho_k T_k U_{n,k+1} + (1-\rho_k)I, \\ \cdots \\ U_{n,2} = \rho_2 T_2 U_{n,3} + (1-\rho_2)I, \\ W_n = U_{n,1} = \rho_1 T_1 U_{n,2} + (1-\rho_1)I. \end{cases} \tag{2.2}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\rho_n, \rho_{n-1}, ..., \rho_1$.

Lemma 2.10 ([30])**.** *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of nonexpansive self-mappings on* C *such that* $\bigcap_{n=1}^\infty \mathrm{Fix}(T_n) \neq \emptyset$ *and let* $\{\rho_n\}$ *be a sequence in* $(0,\theta]$ *for some* $\theta \in (0,1)$. *Then, for every* $x \in C$ *and* $k \geq 1$ *the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists.*

Remark 2.11 ([41], Remark 3.1). It can be known from Lemma 2.10 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$, $\sup_{x\in D} \|U_{n,k}x - U_kx\| \leq \epsilon$.

Remark 2.12 ([41]). Utilizing Lemma 2.10, we define a mapping $W : C \to C$ by $Wx = \lim_{n\to\infty} W_nx = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, ...$ and $\rho_1, \rho_2, ...$ Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. If $\{x_n\}$ is a bounded sequence in C, then from Remark 2.12, one can show that $\lim_{n\to\infty} \|W_nx_n - Wx_n\| = 0$.

Lemma 2.13 ([30])**.** *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of nonexpansive self-mappings on* C *such that* $\bigcap_{n=1}^\infty \mathrm{Fix}(T_n) \neq \emptyset$, *and let* $\{\rho_n\}$ *be a sequence in* $(0,\theta]$ *for some* $\theta \in (0,1)$. *Then,* $\mathrm{Fix}(W) = \bigcap_{n=1}^\infty \mathrm{Fix}(T_n)$.

Lemma 2.14 ([24])**.** (Demiclosedness principle). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* S *be a nonexpansive self-mapping on* C. *Then* $I - S$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *weakly converging to some* $x \in C$ *and the sequence* $\{(I-S)x_n\}$ *strongly converges to some* y, *it follows that* $(I-S)x = y$. *Here* I *is the identity operator of* H.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^\lambda : C \to H$ by $T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C$, where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the condition that for all $x, y \in H$,

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2.$$

Lemma 2.15 ([36], Lemma 3.1)**.** *T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 2.16 ([34])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1 - s_n)a_n + s_n b_n + t_n, \quad \forall n \geq 1,$$

where $\{s_n\} \subset (0, 1]$ and $\{b_n\}$ are such that

(i) $\sum_{n=1}^{\infty} s_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=0}^{\infty} |s_n b_n| < \infty$;

(iii) $\sum_{n=1}^{\infty} t_n < \infty$ *where* $t_n \geq 0$, *for all* $n \geq 1$.

Then $\lim_{n\to\infty} a_n = 0$.

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply that $\langle f - g, x - y\rangle \geq 0$. A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Next we provide an example to illustrate the concept of maximal monotone mapping. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{u \in H : \langle v - p, u\rangle \geq 0, \ \forall p \in C\}$. Define

$$\widetilde{T}v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, $\widetilde{T}$ is maximal monotone (see [29]) such that $0 \in \widetilde{T}v \iff v \in \mathrm{VI}(C, A)$.

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 2.17 ([3])**.** *There holds the resolvent identity*

$$J_{R,\lambda}x = J_{R,\mu}\left(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_{R,\lambda}x\right), \quad \forall x \in H.$$

Remark 2.18. For $\lambda, \mu > 0$, there holds the following relation

$$\|J_{R,\lambda}x - J_{R,\mu}y\| \leq \|x - y\| + |\lambda - \mu|\left(\frac{1}{\lambda}\|J_{R,\lambda}x - y\| + \frac{1}{\mu}\|x - J_{R,\mu}y\|\right), \quad \forall x, y \in H. \tag{2.3}$$

In terms of Huang [25] (see also [45]), there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 2.19. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, that is,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \geq \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 2.20 ([10])**.** *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (1.5) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu).$$

Lemma 2.21 ([45])**.** *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.22 ([10])**.** *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Main Results

We now state and prove the main result of this paper. Let H be a real Hilbert space and $f : H \to \mathbf{R}$ be a function. Then the minimization problem $\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2$ is ill-posed. Xu [35] considered the following Tikhonov's regularization problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2,$$

where $\alpha > 0$ is the regularization parameter. It is clear that the gradient $\nabla f_\alpha = \nabla f + \alpha I = A^*(I - P_Q)A + \alpha I$ is $(\alpha + \|A\|^2)$-Lipschitz continuous.

We are now in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two positive integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)-(A4) and $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with restriction (B1) or (B2), where $k \in \{1, 2, ..., M\}$. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A_k : H \to H$ and $B_i : C \to H$ be μ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2, ..., M\}$, $i \in \{1, 2, ..., N\}$. Let $S : H \to H$ be a nonexpansive mapping and $V : H \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C and $\{\rho_n\}$ be a sequence in $(0, \theta]$ for some $\theta \in (0, 1)$. Let $F : H \to H$ be κ-Lipschitzian and η-strongly monotone with positive constants $\kappa, \eta > 0$ such that $0 \le \gamma < \tau$ and $0 < \mu < \frac{2\eta}{\kappa^2}$ where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \bigcap_{n=1}^\infty \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^N \mathrm{I}(B_i, R_i) \cap \Gamma \neq \emptyset$. Let $\{\lambda_n\} \subset [a, b] \subset (0, \frac{2}{\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$ with $\sum_{n=1}^\infty \alpha_n < \infty$, $\{\epsilon_n\}, \{\delta_n\}, \{\beta_n\}, \{\gamma_n\}, \{\sigma_n\} \subset (0, 1)$ with $\beta_n + \gamma_n + \sigma_n = 1$, and $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_{k,n}\} \subset [c_k, d_k] \subset (0, 2\mu_k)$ where $i \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$. For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = T^{(\Theta_M,\varphi_M)}_{r_{M,n}}(I - r_{M,n}A_M)T^{(\Theta_{M-1},\varphi_{M-1})}_{r_{M-1,n}}(I - r_{M-1,n}A_{M-1}) \cdots T^{(\Theta_1,\varphi_1)}_{r_{1,n}}(I - r_{1,n}A_1)x_n, \\ v_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n}B_{N-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ y_n = \beta_n x_n + \gamma_n P_C(I - \lambda_n \nabla f_{\alpha_n})v_n + \sigma_n W_n P_C(I - \lambda_n \nabla f_{\alpha_n})v_n, \\ x_{n+1} = \epsilon_n \gamma(\delta_n V x_n + (1 - \delta_n)Sx_n) + (I - \epsilon_n \mu F)y_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\nabla f_{\alpha_n} = \alpha_n I + \nabla f$ and W_n is the W-mapping generated by (2.2). Suppose that

(C1) $\lim_{n\to\infty} \epsilon_n = 0$, $\sum_{n=1}^\infty \epsilon_n = \infty$ *and* $\lim_{n\to\infty} \frac{1}{\epsilon_n}|1 - \frac{\delta_{n-1}}{\delta_n}| = 0$*;*

(C2) $\limsup_{n\to\infty} \frac{\delta_n}{\epsilon_n} < \infty$, $\lim_{n\to\infty} \frac{1}{\epsilon_n}|\frac{1}{\delta_n} - \frac{1}{\delta_{n-1}}| = 0$ *and* $\lim_{n\to\infty} \frac{1}{\delta_n}|1 - \frac{\epsilon_{n-1}}{\epsilon_n}| = 0$*;*

(C3) $\lim_{n\to\infty} \frac{\theta^n}{\epsilon_n\delta_n} = 0$, $\lim_{n\to\infty} \frac{|\beta_n-\beta_{n-1}|}{\epsilon_n\delta_n} = 0$ *and* $\lim_{n\to\infty} \frac{|\gamma_n-\gamma_{n-1}|}{\epsilon_n\delta_n} = 0$*;*

(C4) $\lim_{n\to\infty} \frac{|\lambda_{i,n}-\lambda_{i,n-1}|}{\epsilon_n\delta_n} = 0$ *and* $\lim_{n\to\infty} \frac{|r_{k,n}-r_{k,n-1}|}{\epsilon_n\delta_n} = 0$ *for* $i = 1, 2, ..., N$ *and* $k = 1, 2, ..., M$*;*

(C5) $\lim_{n\to\infty} \frac{|\lambda_n-\lambda_{n-1}|}{\epsilon_n\delta_n} = 0$, $\lim_{n\to\infty} \frac{|\lambda_n\alpha_n-\lambda_{n-1}\alpha_{n-1}|}{\epsilon_n\delta_n} = 0$ *and* $\lim_{n\to\infty} \frac{\alpha_n}{\delta_n} = 0$*;*

(C6) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty}(\beta_n + \sigma_n) < 1$ *and* $\liminf_{n\to\infty} \sigma_n > 0$.

Then we have:

(i) $\lim_{n\to\infty} \frac{\|x_{n+1}-x_n\|}{\delta_n} = 0$*;*

(ii) $\omega_w(x_n) \subset \Omega$*;*

(iii) $\{x_n\}$ *converges strongly to a solution* x^* *of the SFP* (1.2)*, which is a unique solution in* Ω *to the HVIP*

$$\langle(\mu F - \gamma S)x^*, p - x^*\rangle \ge 0, \quad \forall p \in \Omega.$$

Proof. First of all, observe that $\mu\eta \ge \tau \ \Leftrightarrow\ \kappa \ge \eta$, and

$$\langle(\mu F - \gamma S)x - (\mu F - \gamma S)y, x - y\rangle \ge (\mu\eta - \gamma)\|x - y\|^2, \quad \forall x, y \in H.$$

Since $0 \le \gamma < \tau$ and $\kappa \ge \eta$, we know that $\mu\eta \ge \tau > \gamma$ and hence the mapping $\mu F - \gamma S$ is $(\mu\eta - \gamma)$-strongly monotone. Moreover, it is clear that the mapping $\mu F - \gamma S$ is $(\mu\kappa + \gamma)$-Lipschitzian. Thus, there exists a unique solution x^* in Ω to the VIP

$$\langle(\mu F - \gamma S)x^*, p - x^*\rangle \ge 0, \quad \forall p \in \Omega.$$

That is, $\{x^*\} = \mathrm{VI}(\Omega, \mu F - \gamma S)$. Now, we put

$$\Delta_n^k = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})}(I - r_{k-1,n}A_{k-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)x_n$$

for all $k \in \{1, 2, ..., M\}$ and $n \ge 1$,

$$\Lambda_n^i = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I - \lambda_{i-1,n}B_{i-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)$$

for all $i \in \{1, 2, ..., N\}$, $\Delta_n^0 = I$ and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$.

In addition, in terms of condition (C6), we may assume, without loss of generality, that $\{\beta_n\} \subset [c, d] \subset (0, 1)$. Now, we show that $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged for each $\lambda \in \left(0, \frac{2}{\alpha+\|A\|^2}\right)$, where $\zeta = \left(2 + \lambda(\alpha + \|A\|^2)\right)/4 \in (0, 1)$. Indeed, since $\nabla f = A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism, it is easy to see that

$$(\alpha + \|A\|^2)\langle\nabla f_\alpha(x) - \nabla f_\alpha(y), x - y\rangle \ge \|\nabla f_\alpha(x) - \nabla f_\alpha(y)\|^2.$$

Hence, it follows that $\nabla f_\alpha = \alpha I + A^*(I - P_Q)A$ is $\frac{1}{\alpha+\|A\|^2}$-ism. Thus, by Proposition 2.6 (ii), $\lambda\nabla f_\alpha$ is $\frac{1}{\lambda(\alpha+\|A\|^2)}$-ism. From Proposition 2.6 (iii), the complement $I - \lambda\nabla f_\alpha$ is $\frac{\lambda(\alpha+\|A\|^2)}{2}$-averaged. Therefore, noting that P_C is $\frac{1}{2}$-averaged and utilizing Proposition 2.7 (iv), we obtain that for each $\lambda \in (0, \frac{2}{\alpha+\|A\|^2})$, $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged with $\zeta = \left(2 + \lambda(\alpha + \|A\|^2)\right)/4 \in (0, 1)$. This shows that $P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive. Taking into account that $\{\lambda_n\} \subset [a, b] \subset (0, \frac{2}{\|A\|^2})$ and $\alpha_n \to 0$, we get

$$\limsup_{n\to\infty} \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \le \frac{2 + b\|A\|^2}{4} < 1.$$

Without loss of generality, we may assume that $\zeta_n := \frac{2+\lambda_n(\alpha_n+\|A\|^2)}{4} < 1$ for each $n \geq 1$. So, $P_C(I-\lambda_n\nabla f_{\alpha_n})$ is nonexpansive for each $n \geq 1$. Since $\limsup_{n\to\infty} \frac{\lambda_n(\alpha_n+\|A\|^2)}{2} \leq \frac{b\|A\|^2}{2} < 1$, it is known similarly that $I-\lambda_n\nabla f_{\alpha_n}$ is nonexpansive for each $n \geq 1$. Next, we divide the rest of the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded. Indeed, take a fixed $p \in \Omega$ arbitrarily. Utilizing (2.1) and Proposition 2.4 (ii) we have

$$\|u_n - p\| \leq \|\varDelta_n^{M-1}x_n - \varDelta_n^{M-1}p\| \leq \cdots \leq \|\varDelta_n^0 x_n - \varDelta_n^0 p\| = \|x_n - p\|. \tag{3.2}$$

Utilizing (2.1) and Lemma 2.19 we have

$$\|v_n - p\| = \|\varLambda_n^N u_n - \varLambda_n^N p\| \leq \|\varLambda_n^{N-1}u_n - \varLambda_n^{N-1}p\| \leq \cdots \leq \|\varLambda_n^0 u_n - \varLambda_n^0 p\| = \|u_n - p\|. \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\|v_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

For simplicity, put $t_n = P_C(I-\lambda_n\nabla f_{\alpha_n})v_n$ for each $n \geq 0$. Note that $P_C(I-\lambda\nabla f)p = p$ for $\lambda \in (0, \frac{2}{\|A\|^2})$. Hence, from (3.4), it follows that

$$\begin{aligned} \|t_n - p\| &\leq \|P_C(I-\lambda_n\nabla f_{\alpha_n})v_n - P_C(I-\lambda_n\nabla f_{\alpha_n})p\| + \|P_C(I-\lambda_n\nabla f_{\alpha_n})p - P_C(I-\lambda_n\nabla f)p\| \\ &\leq \|v_n - p\| + \lambda_n\alpha_n\|p\| \leq \|x_n - p\| + \lambda_n\alpha_n\|p\|. \end{aligned} \tag{3.5}$$

Since $W_n p = p$ for all $n \geq 1$ and W_n is a nonexpansive mapping, we obtain from (3.1) and (3.5) that

$$\begin{aligned} \|y_n - p\| &\leq \beta_n\|x_n - p\| + \gamma_n\|t_n - p\| + \sigma_n\|W_n t_n - p\| \\ &\leq \beta_n\|x_n - p\| + (\gamma_n + \sigma_n)\left(\|x_n - p\| + \lambda_n\alpha_n\|p\|\right) \\ &\leq \|x_n - p\| + \lambda_n\alpha_n\|p\|. \end{aligned} \tag{3.6}$$

Utilizing Lemma 2.16, we deduce from (3.1), (3.6), $\{\lambda_n\} \subset [a,b] \subset (0, \frac{2}{\|A\|^2})$ and $0 \leq \gamma < \tau$ that for all $n \geq 1$

$$\begin{aligned} \|x_{n+1} - p\| &\leq \epsilon_n \|\delta_n(\gamma V x_n - \mu F p) + (1-\delta_n)(\gamma S x_n - \mu F p)\| \\ &\quad + \|(I-\epsilon_n\mu F)y_n - (I-\epsilon_n\mu F)p\| \\ &\leq \epsilon_n\left[\delta_n(\gamma\rho\|x_n - p\| + \|\gamma V p - \mu F p\| + (1-\delta_n)(\gamma\|x_n - p\| + \|\gamma S p - \mu F p\|)\right] \\ &\quad + (1-\epsilon_n\tau)\|y_n - p\| \\ &\leq \epsilon_n\gamma\|x_n - p\| + \epsilon_n \max\left\{\|\gamma V p - \mu F p\|, \|\gamma S p - \mu F p\|\right\} \\ &\quad + (1-\epsilon_n\tau)\|x_n - p\| + \alpha_n b\|p\| \\ &\leq \max\left\{\|x_n - p\|, \tfrac{\|\gamma V p - \mu F p\|}{\tau-\gamma}, \tfrac{\|\gamma S p - \mu F p\|}{\tau-\gamma}\right\} + \alpha_n b\|p\|. \end{aligned}$$

By induction, we get

$$\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau-\gamma}, \frac{\|\gamma S p - \mu F p\|}{\tau-\gamma}\right\} + \sum_{j=1}^n \alpha_j b\|p\|$$

for all $n \geq 1$. Thus, $\{x_n\}$ is bounded (due to $\sum_{n=1}^\infty \alpha_n < \infty$) and so are the sequences $\{t_n\}, \{u_n\}, \{v_n\}$, and $\{y_n\}$.

Step 2. We prove that $\lim_{n\to\infty} \frac{\|x_{n+1}-x_n\|}{\delta_n} = 0$.
Indeed, utilizing (2.1) and (2.3), we obtain that

$$\begin{aligned} \|v_{n+1} - v_n\| &= \left\|\varLambda_{n+1}^N u_{n+1} - \varLambda_n^N u_n\right\| \\ &\leq \left\|J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n+1}B_N)\varLambda_{n+1}^{N-1}u_{n+1} - J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\varLambda_{n+1}^{N-1}u_{n+1}\right\| \end{aligned}$$

$$
\begin{aligned}
&+ \left\| J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1} - J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n \right\| \\
&\le |\lambda_{N,n+1}-\lambda_{N,n}| \left(\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\| + \widetilde{M} \right) + \left\| \Lambda_{n+1}^{N-1}u_{n+1} - \Lambda_n^{N-1}u_n \right\| \\
&\le \cdots \\
&\le |\lambda_{N,n+1}-\lambda_{N,n}| \left(\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\| + \widetilde{M} \right) \\
&\quad + |\lambda_{N-1,n+1}-\lambda_{N-1,n}| \left(\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\| + \widetilde{M} \right) \\
&\quad + \cdots + |\lambda_{1,n+1}-\lambda_{1,n}| \left(\|B_1\Lambda_{n+1}^{0}u_{n+1}\| + \widetilde{M} \right) + \|\Lambda_{n+1}^{0}u_{n+1} - \Lambda_n^0 u_n\| \\
&\le \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1}-\lambda_{i,n}| + \|u_{n+1}-u_n\|,
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\sup_{n\ge1}\{ &\frac{1}{\lambda_{N,n+1}} \left\| J_{R_N,\lambda_{N,n+1}}(I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1} - (I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n \right\| \\
&+ \frac{1}{\lambda_{N,n}} \left\| (I-\lambda_{N,n}B_N)\Lambda_{n+1}^{N-1}u_{n+1} - J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n \right\| \} \le \widetilde{M}
\end{aligned}
$$

for some $\widetilde{M} > 0$ and $\sup_{n\ge0}\left\{ \sum_{i=1}^{N} \|B_i\Lambda_{n+1}^{i-1}u_{n+1}\| + \widetilde{M} \right\} \le \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$.

Utilizing Proposition 2.4 (ii), (v), we deduce that

$$
\begin{aligned}
\|u_{n+1}-u_n\| &= \|\Delta_{n+1}^M x_{n+1} - \Delta_n^M x_n\| \\
&\le \|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_{n+1}^{M-1}x_{n+1} - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_n^{M-1}x_n\| \\
&\le |r_{M,n+1}-r_{M,n}|[\|A_M\Delta_{n+1}^{M-1}x_{n+1}\| + \|\Delta_{n+1}^{M-1}x_{n+1} - \Delta_n^{M-1}x_n\| \\
&\quad + \frac{1}{r_{M,n+1}}\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} - (I-r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1}\|] \\
&\le \cdots \\
&\le |r_{M,n+1}-r_{M,n}|[\|A_M\Delta_{n+1}^{M-1}x_{n+1}\| \\
&\quad + \frac{1}{r_{M,n+1}}\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}(I-r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1} \\
&\quad - (I-r_{M,n+1}A_M)\Delta_{n+1}^{M-1}x_{n+1}\|] + \cdots + |r_{1,n+1}-r_{1,n}|[\|A_1\Delta_{n+1}^0 x_{n+1}\| \\
&\quad + \frac{1}{r_{1,n+1}}\|T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)}(I-r_{1,n+1}A_1)\Delta_{n+1}^0 x_{n+1} - (I-r_{1,n+1}A_1)\Delta_{n+1}^0 x_{n+1}\|] \\
&\quad + \|\Delta_{n+1}^0 x_{n+1} - \Delta_n^0 x_n\| \\
&\le \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1}-r_{k,n}| + \|x_{n+1}-x_n\|,
\end{aligned}
\tag{3.8}
$$

where $\widetilde{M}_1 > 0$ is a constant such that for each $n \ge 1$

$$
\sum_{k=1}^{M}[\|A_k\Delta_{n+1}^{k-1}x_{n+1}\| + \frac{1}{r_{k,n+1}}\|T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)}(I-r_{k,n+1}A_k)\Delta_{n+1}^{k-1}x_{n+1} - (I-r_{k,n+1}A_k)\Delta_{n+1}^{k-1}x_{n+1}\|] \le \widetilde{M}_1.
$$

Furthermore, we define $y_n = \beta_n x_n + (1-\beta_n)w_n$ for all $n \ge 1$. It follows that

$$
\begin{aligned}
w_{n+1} - w_n = &\frac{\gamma_{n+1}(t_{n+1}-t_n) + \sigma_{n+1}(W_{n+1}t_{n+1} - W_n t_n)}{1-\beta_{n+1}} \\
&+ (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})t_n + (\frac{\sigma_{n+1}}{1-\beta_{n+1}} - \frac{\sigma_n}{1-\beta_n})W_n t_n.
\end{aligned}
\tag{3.9}
$$

Taking into account the nonexpansivity of W_n, T_k and $U_{n,k}$, from (2.2) we get

$$\|W_{n+1}t_{n+1} - W_n t_n\| \le \|t_{n+1} - t_n\| + \prod_{j=1}^{n+1} \rho_j \|T_{n+1}t_n - t_n\|. \tag{3.10}$$

By the nonexpansivity of $P_C(I - \lambda_n \nabla f_{\alpha_n})$ we have

$$\|t_{n+1} - t_n\| \le \|v_{n+1} - v_n\| + |\lambda_{n+1}\alpha_{n+1} - \lambda_n \alpha_n| \|v_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(v_n)\|. \tag{3.11}$$

Hence it follows from (3.7)-(3.11) and $\{\rho_n\} \subset (0, \theta] \subset (0, 1)$ that

$$\begin{aligned}
\|w_{n+1} - w_n\| \le & \frac{\gamma_{n+1}\|t_{n+1} - t_n\| + \sigma_{n+1}\|W_{n+1}t_{n+1} - W_n t_n\|}{1 - \beta_{n+1}} + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| \|t_n\| \\
& + |\frac{\sigma_{n+1}}{1 - \beta_{n+1}} - \frac{\sigma_n}{1 - \beta_n}| \|W_n t_n\| \\
\le & \|t_{n+1} - t_n\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| (\|t_n\| + \|W_n t_n\|) + \prod_{j=1}^{n+1} \rho_j \|T_{n+1}t_n - t_n\| \\
\le & \|v_{n+1} - v_n\| + |\lambda_{n+1}\alpha_{n+1} - \lambda_n \alpha_n| \|v_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(v_n)\| \\
& + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| (\|t_n\| + \|W_n t_n\|) + \theta^{n+1} \|T_{n+1}t_n - t_n\| \\
\le & \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + \|x_{n+1} - x_n\| \\
& + |\lambda_{n+1}\alpha_{n+1} - \lambda_n \alpha_n| \|v_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(v_n)\| \\
& + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| (\|t_n\| + \|W_n t_n\|) + \theta^{n+1} \|T_{n+1}t_n - t_n\|.
\end{aligned} \tag{3.12}$$

Note that

$$y_{n+1} - y_n = \beta_n (x_{n+1} - x_n) + (1 - \beta_n)(w_{n+1} - w_n) + (\beta_{n+1} - \beta_n)(x_{n+1} - w_{n+1}).$$

It follows from (3.12) that

$$\begin{aligned}
\|y_{n+1} - y_n\| \le & \beta_n \|x_{n+1} - x_n\| + (1 - \beta_n)\|w_{n+1} - w_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| \\
\le & \|x_{n+1} - x_n\| + \widetilde{M}_0 \sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \widetilde{M}_1 \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| \\
& + \frac{|\gamma_{n+1} - \gamma_n|(1 - \beta_n) + \gamma_n |\beta_{n+1} - \beta_n|}{1 - \beta_{n+1}} (\|t_n\| + \|W_n t_n\|) + |\lambda_{n+1}\alpha_{n+1} - \lambda_n \alpha_n| \|v_n\| \\
& + |\lambda_{n+1} - \lambda_n| \|\nabla f(v_n)\| + \theta^{n+1} \|T_{n+1}t_n - t_n\| + |\beta_{n+1} - \beta_n| \|x_{n+1} - w_{n+1}\| \\
\le & \|x_{n+1} - x_n\| + \widetilde{M}_2 \big(\sum_{i=1}^{N} |\lambda_{i,n+1} - \lambda_{i,n}| + \sum_{k=1}^{M} |r_{k,n+1} - r_{k,n}| + |\gamma_{n+1} - \gamma_n| \\
& + |\beta_{n+1} - \beta_n| + |\lambda_{n+1}\alpha_{n+1} - \lambda_n \alpha_n| + |\lambda_{n+1} - \lambda_n| + \theta^{n+1} \big),
\end{aligned} \tag{3.13}$$

where $\sup_{n\ge1} \Big\{ \|x_{n+1} - w_{n+1}\| + \frac{\|t_n\| + \|W_n t_n\|}{1-d} + \|v_n\| + \|\nabla f(v_n)\| + \|T_{n+1}t_n - t_n\| + \widetilde{M}_0 + \widetilde{M}_1 \Big\} \le \widetilde{M}_2$ for some $\widetilde{M}_2 > 0$.

On the other hand, we define $z_n := \delta_n V x_n + (1 - \delta_n) S x_n$ for all $n \ge 1$. Then it is known that $x_{n+1} = \epsilon_n \gamma z_n + (I - \epsilon_n \mu F) y_n$ for all $n \ge 1$. Simple calculations show that

$$\begin{aligned}
z_{n+1} - z_n &= (\delta_{n+1} - \delta_n)(V x_n - S x_n) + \delta_{n+1}(V x_{n+1} - V x_n) + (1 - \delta_{n+1})(S x_{n+1} - S x_n), \\
x_{n+2} - x_{n+1} &= (\epsilon_{n+1} - \epsilon_n)(\gamma z_n - \mu F y_n) + \epsilon_{n+1}\gamma(z_{n+1} - z_n) + (I - \lambda_{n+1}\mu F) y_{n+1} - (I - \lambda_{n+1}\mu F) y_n.
\end{aligned}$$

Since V is a ρ-contraction with coefficient $\rho \in [0,1)$ and S is a nonexpansive mapping, we conclude that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le |\delta_{n+1}-\delta_n|\|Vx_n-Sx_n\|+\delta_{n+1}\|Vx_{n+1}-Vx_n\| \\ &\quad +(1-\delta_{n+1})\|Sx_{n+1}-Sx_n\| \\ &\le |\delta_{n+1}-\delta_n|\|Vx_n-Sx_n\|+\delta_{n+1}\rho\|x_{n+1}-x_n\| \\ &\quad +(1-\delta_{n+1})\|x_{n+1}-x_n\| \\ &= (1-\delta_{n+1}(1-\rho))\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Vx_n-Sx_n\|,\end{aligned}$$

which together with (3.13) and $0 \le \gamma < \tau$, implies that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le |\epsilon_{n+1}-\epsilon_n|\|\gamma z_n-\mu F y_n\|+\epsilon_{n+1}\gamma\|z_{n+1}-z_n\| \\ &\quad +\|(I-\epsilon_{n+1}\mu F)y_{n+1}-(I-\epsilon_{n+1}\mu F)y_n\| \\ &\le |\epsilon_{n+1}-\epsilon_n|\|\gamma z_n-\mu F y_n\|+\epsilon_{n+1}\gamma\big[(1-\delta_{n+1}(1-\rho))\|x_{n+1}-x_n\| \\ &\quad +|\delta_{n+1}-\delta_n|\|Vx_n-Sx_n\|\big]+(1-\epsilon_{n+1}\tau)\big[\|x_{n+1}-x_n\| \\ &\quad +\widetilde{M}_2\big(\sum_{i=1}^N|\lambda_{i,n+1}-\lambda_{i,n}|+\sum_{k=1}^M|r_{k,n+1}-r_{k,n}|+|\gamma_{n+1}-\gamma_n| \\ &\quad +|\beta_{n+1}-\beta_n|+|\lambda_{n+1}\alpha_{n+1}-\lambda_n\alpha_n|+|\lambda_{n+1}-\lambda_n|+\theta^{n+1}\big)\big] \\ &\le (1-\epsilon_{n+1}(\tau-\gamma))\|x_{n+1}-x_n\|+\widetilde{M}_3\{\sum_{i=1}^N|\lambda_{i,n+1}-\lambda_{i,n}| \\ &\quad +\sum_{k=1}^M|r_{k,n+1}-r_{k,n}|+|\epsilon_{n+1}-\epsilon_n|+|\delta_{n+1}-\delta_n| \\ &\quad +|\beta_{n+1}-\beta_n|+|\gamma_{n+1}-\gamma_n|+|\lambda_{n+1}\alpha_{n+1}-\lambda_n\alpha_n|+|\lambda_{n+1}-\lambda_n|+\theta^{n+1}\},\end{aligned}$$

where $\sup_{n\ge 0}\left\{\|\gamma z_n-\mu F y_n\|+\|Vx_n-Sx_n\|+\widetilde{M}_2\right\} \le \widetilde{M}_3$ for some $\widetilde{M}_3 > 0$. Consequently,

$$\begin{aligned}&\frac{\|x_{n+1}-x_n\|}{\delta_n} \\ &\le (1-\epsilon_n(\tau-\gamma))\frac{\|x_n-x_{n-1}\|}{\delta_n}+\widetilde{M}_3\{\sum_{i=1}^N\frac{|\lambda_{i,n}-\lambda_{i,n-1}|}{\delta_n}+\sum_{k=1}^M\frac{|r_{k,n}-r_{k,n-1}|}{\delta_n} \\ &\quad +\frac{|\epsilon_n-\epsilon_{n-1}|}{\delta_n}+\frac{|\delta_n-\delta_{n-1}|}{\delta_n}+\frac{|\beta_n-\beta_{n-1}|}{\delta_n}+\frac{|\gamma_n-\gamma_{n-1}|}{\delta_n} \\ &\quad +\frac{|\lambda_{n+1}\alpha_{n+1}-\lambda_n\alpha_n|}{\delta_n}+\frac{|\lambda_{n+1}-\lambda_n|}{\delta_n}+\frac{\theta^{n+1}}{\delta_n}\} \\ &\le (1-\epsilon_n(\tau-\gamma))\frac{\|x_n-x_{n-1}\|}{\delta_{n-1}}+\epsilon_n(\tau-\gamma)\cdot\frac{\widetilde{M}_4}{\tau-\gamma}\{\frac{1}{\epsilon_n}|\frac{1}{\delta_n}-\frac{1}{\delta_{n-1}}|+\sum_{i=1}^N\frac{|\lambda_{i,n}-\lambda_{i,n-1}|}{\epsilon_n\delta_n} \\ &\quad +\sum_{k=1}^M\frac{|r_{k,n}-r_{k,n-1}|}{\epsilon_n\delta_n}+\frac{1}{\delta_n}|1-\frac{\epsilon_{n-1}}{\epsilon_n}|+\frac{1}{\epsilon_n}|1-\frac{\delta_{n-1}}{\delta_n}|+\frac{|\beta_n-\beta_{n-1}|}{\epsilon_n\delta_n}+\frac{|\gamma_n-\gamma_{n-1}|}{\epsilon_n\delta_n} \\ &\quad +\frac{|\lambda_n\alpha_n-\lambda_{n-1}\alpha_{n-1}|}{\epsilon_n\delta_n}+\frac{|\lambda_n-\lambda_{n-1}|}{\epsilon_n\delta_n}+\frac{\theta^n}{\epsilon_n\delta_n}\},\end{aligned} \tag{3.14}$$

where $\sup_{n\ge 1}\left\{\|x_n-x_{n-1}\|+\widetilde{M}_3\right\} \le \widetilde{M}_4$ for some $\widetilde{M}_4 > 0$. From conditions (C1)-(C5) it follows that $\sum_{n=0}^\infty \epsilon_n(\tau-\gamma)=\infty$ and

$$\lim_{n\to\infty}\frac{\widetilde{M}_4}{\tau-\gamma}\{\frac{1}{\epsilon_n}|\frac{1}{\delta_n}-\frac{1}{\delta_{n-1}}|+\sum_{i=1}^N\frac{|\lambda_{i,n}-\lambda_{i,n-1}|}{\epsilon_n\delta_n}+\sum_{k=1}^M\frac{|r_{k,n}-r_{k,n-1}|}{\epsilon_n\delta_n}+\frac{1}{\delta_n}|1-\frac{\epsilon_{n-1}}{\epsilon_n}|$$

$$+\frac{1}{\epsilon_n}|1-\frac{\delta_{n-1}}{\delta_n}|+\frac{|\beta_n-\beta_{n-1}|}{\epsilon_n\delta_n}+\frac{|\gamma_n-\gamma_{n-1}|}{\epsilon_n\delta_n}+\frac{|\lambda_n\alpha_n-\lambda_{n-1}\alpha_{n-1}|}{\epsilon_n\delta_n}$$
$$+\frac{|\lambda_n-\lambda_{n-1}|}{\epsilon_n\delta_n}+\frac{\theta^n}{\epsilon_n\delta_n}\}=0.$$

Thus, utilizing Lemma 2.17, we immediately conclude that $\lim_{n\to\infty}\|x_{n+1}-x_n\|/\delta_n=0$. So, from $\delta_n\to 0$ it follows that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Step 3. We prove that $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|x_n-v_n\|=0$, $\lim_{n\to\infty}\|v_n-t_n\|=0$ and $\lim_{n\to\infty}\|t_n-Wt_n\|=0$. Indeed, by Lemmas 2.8 and 2.9 (b), from (3.1), (3.4)-(3.5) and $0\le\gamma<\tau$ one has

$$\begin{aligned}\|y_n-p\|^2&=\|\beta_n(x_n-p)+(1-\beta_n)(\frac{\gamma_nt_n+\sigma_nW_nt_n}{1-\beta_n}-p)\|^2\\&=\beta_n\|x_n-p\|^2+\gamma_n\|t_n-p\|^2+\sigma_n\|W_nt_n-p\|^2-\frac{\gamma_n\sigma_n}{1-\beta_n}\|t_n-W_nt_n\|^2\\&\quad-\frac{\beta_n}{1-\beta_n}\|y_n-x_n\|^2\\&\le\beta_n\|x_n-p\|^2+(1-\beta_n)(\|x_n-p\|+\lambda_n\alpha_n\|p\|)^2-\frac{\gamma_n\sigma_n}{1-\beta_n}\|t_n-W_nt_n\|^2\\&\quad-\frac{\beta_n}{1-\beta_n}\|y_n-x_n\|^2\\&\le(\|x_n-p\|+\lambda_n\alpha_n\|p\|)^2-\frac{\gamma_n\sigma_n}{1-\beta_n}\|t_n-W_nt_n\|^2-\frac{\beta_n}{1-\beta_n}\|y_n-x_n\|^2,\end{aligned}\tag{3.15}$$

and hence

$$\begin{aligned}\|x_{n+1}-p\|^2&=\big\|\epsilon_n[\delta_n(\gamma Vx_n-\gamma Vp)+(1-\delta_n)(\gamma Sx_n-\gamma Sp)]\\&\quad+(I-\epsilon_n\mu F)y_n-(I-\epsilon_n\mu F)p\\&\quad+\epsilon_n[\delta_n(\gamma Vp-\mu Fp)+(1-\delta_n)(\gamma Sp-\mu Fp)]\big\|^2\\&\le\big[\epsilon_n(1-\delta_n(1-\rho))\gamma\|x_n-p\|+(1-\epsilon_n\tau)\|y_n-p\|\big]^2\\&\quad+2\epsilon_n\delta_n\langle(\gamma Vp-\mu Fp),x_{n+1}-p\rangle+2\epsilon_n(1-\delta_n)\langle(\gamma Sp-\mu Fp),x_{n+1}-p\rangle\\&\le\frac{\epsilon_n\gamma^2}{\tau}\|x_n-p\|^2+(1-\epsilon_n\tau)\big[(\|x_n-p\|+\alpha_nb\|p\|)^2-\frac{\gamma_n\sigma_n}{1-\beta_n}\|t_n-W_nt_n\|^2\\&\quad-\frac{\beta_n}{1-\beta_n}\|y_n-x_n\|^2\big]\\&\quad+2\epsilon_n\delta_n\langle(\gamma Vp-\mu Fp),x_{n+1}-p\rangle+2\epsilon_n(1-\delta_n)\langle(\gamma Sp-\mu Fp),x_{n+1}-p\rangle\\&\le(1-\epsilon_n\frac{\tau^2-\gamma^2}{\tau})(\|x_n-p\|+\alpha_nb\|p\|)^2-\frac{\gamma_n\sigma_n(1-\epsilon_n\tau)}{1-\beta_n}\|t_n-W_nt_n\|^2\\&\quad-\frac{\beta_n(1-\epsilon_n\tau)}{1-\beta_n}\|y_n-x_n\|^2\\&\quad+2\epsilon_n\delta_n\langle(\gamma Vp-\mu Fp),x_{n+1}-p\rangle+2\epsilon_n(1-\delta_n)\langle(\gamma Sp-\mu Fp),x_{n+1}-p\rangle,\end{aligned}\tag{3.16}$$

which together with $\{\beta_n\}\subset[c,d]\subset(0,1)$, immediately yields

$$\begin{aligned}&\frac{\gamma_n\sigma_n(1-\epsilon_n\tau)}{1-c}\|t_n-W_nt_n\|^2+\frac{c(1-\epsilon_n\tau)}{1-c}\|y_n-x_n\|^2\\&\le(\|x_n-x_{n+1}\|+\alpha_nb\|p\|)(\|x_n-p\|+\|x_{n+1}-p\|+\alpha_nb\|p\|)\\&\quad+2\epsilon_n\delta_n\|\gamma Vp-\mu Fp\|\|x_{n+1}-p\|+2\epsilon_n\|\gamma Sp-\mu Fp\|\|x_{n+1}-p\|.\end{aligned}$$

In terms of (C6), we find that $\liminf_{n\to\infty}\gamma_n\sigma_n\ge 0$. Since $\epsilon_n\to 0$, $\delta_n\to 0$, $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and $\{x_n\}$ is bounded, we have

$$\lim_{n\to\infty}\|t_n-W_nt_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{3.17}$$

Noting that

$$\begin{aligned}y_n - x_n =& \gamma_n(t_n - x_n) + \sigma_n(W_n t_n - x_n)\\ =& (1-\beta_n)(t_n - x_n) + \sigma_n(W_n t_n - t_n),\end{aligned}$$

we conclude from (3.17) and $\{\beta_n\} \subset [c,d] \subset (0,1)$ that as $n \to \infty$,

$$(1-d)\|t_n - x_n\| \le \|(1-\beta_n)(t_n - x_n)\| \le \|y_n - x_n\| + \|W_n t_n - t_n\| \to 0. \tag{3.18}$$

Observe that

$$\|\varDelta_n^k x_n - p\|^2 \le \|x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\|A_k \varDelta_n^{k-1} x_n - A_k p\|^2, \tag{3.19}$$

$$\|\varLambda_n^i u_n - p\|^2 \le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i \varLambda_n^{i-1} u_n - B_i p\|^2 \tag{3.20}$$

for $i \in \{1,2,...,N\}$ and $k \in \{1,2,...,M\}$. Combining (3.5), (3.15), (3.19) and (3.20), we get

$$\begin{aligned}\|y_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|t_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(\|v_n - p\| + \lambda_n\alpha_n\|p\|)^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\varLambda_n^i u_n - p\|^2 + \alpha_n b\|p\|(2\|v_n - p\| + \alpha_n b\|p\|)\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\left[\|u_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\varLambda_n^{i-1}u_n - B_i p\|^2\right]\\ &\quad + \alpha_n b\|p\|(2\|v_n - p\| + \alpha_n b\|p\|)\\ &\le \|x_n - p\|^2 + (1-\beta_n)\left[r_{k,n}(r_{k,n} - 2\mu_k)\|A_k\varDelta_n^{k-1}x_n - A_k p\|^2\right.\\ &\quad \left.+ \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\|B_i\varLambda_n^{i-1}u_n - B_i p\|^2\right] + \alpha_n b\|p\|\left(2\|v_n - p\| + \alpha_n b\|p\|\right),\end{aligned} \tag{3.21}$$

which immediately leads to

$$\begin{aligned}&(1-\beta_n)\left[r_{k,n}(2\mu_k - r_{k,n})\|A_k\varDelta_n^{k-1}x_n - A_k p\|^2 + \lambda_{i,n}(2\eta_i - \lambda_{i,n})\|B_i\varLambda_n^{i-1}u_n - B_i p\|^2\right]\\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \alpha_n b\|p\|(2\|v_n - p\| + \alpha_n b\|p\|).\end{aligned}$$

Since $\{\beta_n\} \subset [c,d] \subset (0,1)$, $\{\lambda_{i,n}\} \subset [a_i,b_i] \subset (0,2\eta_i)$, $\{r_{k,n}\} \subset [c_k,d_k] \subset (0,2\mu_k)$, $i \in \{1,2,...,N\}, k \in \{1,2,...,M\}$ and $\{v_n\},\{x_n\},\{y_n\}$ are bounded sequences, from (3.17) and $\alpha_n \to 0$ we have

$$\lim_{n\to\infty}\|A_k\varDelta_n^{k-1}x_n - A_k p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_i\varLambda_n^{i-1}u_n - B_i p\| = 0 \tag{3.22}$$

for all $k \in \{1,2,...,M\}$ and $i \in \{1,2,...,N\}$.

Furthermore, by Proposition 2.4 (ii) and Lemma 2.9 (a) we have

$$\begin{aligned}\|\varDelta_n^k x_n - p\|^2 &\le \langle (I - r_{k,n}A_k)\varDelta_n^{k-1}x_n - (I - r_{k,n}A_k)p, \varDelta_n^k x_n - p\rangle\\ &\le \frac{1}{2}(\|\varDelta_n^{k-1}x_n - p\|^2 + \|\varDelta_n^k x_n - p\|^2 - \|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n - r_{k,n}(A_k\varDelta_n^{k-1}x_n - A_k p)\|^2),\end{aligned}$$

which implies that

$$\|\varDelta_n^k x_n - p\|^2 \le \|x_n - p\|^2 - \|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|^2 + 2r_{k,n}\|\varDelta_n^{k-1}x_n - \varDelta_n^k x_n\|\|A_k\varDelta_n^{k-1}x_n - A_k p\|. \tag{3.23}$$

By Lemma 2.9 (a) and Lemma 2.20, we obtain

$$\begin{aligned}\|\varLambda_n^i u_n - p\|^2 &\le \langle (I - \lambda_{i,n}B_i)\varLambda_n^{i-1}u_n - (I - \lambda_{i,n}B_i)p, \varLambda_n^i u_n - p\rangle\\ &\le \frac{1}{2}(\|x_n - p\|^2 + \|\varLambda_n^i u_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n - \lambda_{i,n}(B_i\varLambda_n^{i-1}u_n - B_i p)\|^2),\end{aligned}$$

which immediately leads to

$$\|\varLambda_n^i u_n - p\|^2 \le \|x_n - p\|^2 - \|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|^2 + 2\lambda_{i,n}\|\varLambda_n^{i-1}u_n - \varLambda_n^i u_n\|\|B_i\varLambda_n^{i-1}u_n - B_i p\|. \tag{3.24}$$

Combining (3.21) and (3.24) we conclude that

$$\|y_n-p\|^2 \le \|x_n-p\|^2-(1-\beta_n)\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|^2+2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|\|B_i\Lambda_n^{i-1}u_n-B_ip\| \\ +\alpha_nb\|p\|(2\|v_n-p\|+\alpha_nb\|p\|),$$

which yields

$$(1-\beta_n)\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|^2 \le \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)+2\lambda_{i,n}\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|\|B_i\Lambda_n^{i-1}u_n-B_ip\| \\ +\alpha_nb\|p\|(2\|v_n-p\|+\alpha_nb\|p\|).$$

Since $\{\beta_n\}\subset[c,d]\subset(0,1)$, $\{\lambda_{i,n}\}\subset[a_i,b_i]\subset(0,2\eta_i), i=1,2,...,N$, and $\{u_n\},\{x_n\}$ and $\{y_n\}$ are bounded sequences, we deduce from (3.17), (3.22) and $\alpha_n\to0$ that

$$\lim_{n\to\infty}\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\|=0, \quad \forall i\in\{1,2,...,N\}. \tag{3.25}$$

Also, combining (3.3), (3.21) and (3.23) we deduce that

$$\|y_n-p\|^2 \le \|x_n-p\|^2-(1-\beta_n)\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\|^2+2r_{k,n}\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\|\|A_k\Delta_n^{k-1}x_n-A_kp\| \\ +\alpha_nb\|p\|(2\|v_n-p\|+\alpha_nb\|p\|),$$

which yields

$$(1-\beta_n)\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\|^2 \le \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|) \\ +2r_{k,n}\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\|\|A_k\Delta_n^{k-1}x_n-A_kp\| \\ +\alpha_nb\|p\|(2\|v_n-p\|+\alpha_nb\|p\|).$$

Since $\{\beta_n\}\subset[c,d]\subset(0,1)$, $\{r_{k,n}\}\subset[c_k,d_k]\subset(0,2\mu_k)$ for $k=1,2,...,M$, and $\{v_n\},\{x_n\},\{y_n\}$ are bounded sequences, we deduce from (3.17), (3.22) and $\alpha_n\to0$ that

$$\lim_{n\to\infty}\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\|=0, \quad \forall k\in\{1,2,...,M\}. \tag{3.26}$$

Hence from (3.25) and (3.26) we obtain that as $n\to\infty$,

$$\|x_n-u_n\|\le\|\Delta_n^0x_n-\Delta_n^1x_n\|+\|\Delta_n^1x_n-\Delta_n^2x_n\|+\cdots+\|\Delta_n^{M-1}x_n-\Delta_n^Mx_n\|\to0, \tag{3.27}$$

$$\|u_n-v_n\|\le\|\Lambda_n^0u_n-\Lambda_n^1u_n\|+\|\Lambda_n^1u_n-\Lambda_n^2u_n\|+\cdots+\|\Lambda_n^{N-1}u_n-\Lambda_n^Nu_n\|\to0. \tag{3.28}$$

Thus, from (3.27) and (3.28) we obtain

$$\|x_n-v_n\|\le\|x_n-u_n\|+\|u_n-v_n\|\to0 \quad \text{as } n\to\infty, \tag{3.29}$$

which together with (3.18), attains

$$\|t_n-v_n\|\le\|t_n-x_n\|+\|x_n-v_n\|\to0 \quad \text{as } n\to\infty. \tag{3.30}$$

In addition, it is clear that $\|t_n-Wt_n\|\le\|t_n-W_nt_n\|+\|W_nt_n-Wt_n\|$. Thus, we conclude from Remark 2.12, (3.17) and the boundedness of $\{t_n\}$ that

$$\lim_{n\to\infty}\|t_n-Wt_n\|=0. \tag{3.31}$$

Step 4. We prove that $\omega_w(x_n)\subset\Omega$. Indeed, since H is reflexive and $\{x_n\}$ is bounded, there exists at least a weak convergence subsequence of $\{x_n\}$. Hence it is known that $\omega_w(x_n)\ne\emptyset$. Now, take an arbitrary $w\in\omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup w$. From (3.18), (3.25)-(3.27) and (3.29) we have that $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, $t_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$ and $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, where $m \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., M\}$. Utilizing Lemma 2.14, we deduce from $t_{n_i} \rightharpoonup w$ and (3.31) that $w \in \mathrm{Fix}(W) = \cap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ (due to Lemma 2.13). Next, we claim that $w \in \bigcap_{m=1}^{N} \mathrm{I}(B_m, R_m) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. As a matter of fact, repeating the same arguments as those of $w \in \bigcap_{m=1}^{N} \mathrm{I}(B_m, R_m) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$ in Step 4 of the proof of [15, Theorem 3.1], we obtain the desired assertion. Thus, $w \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{m=1}^{N} \mathrm{I}(B_m, R_m)$.

Furthermore, let us define

$$\widetilde{T} v = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

where $N_C v = \{u \in H : \langle v - x, u \rangle \geq 0, \ \forall x \in C\}$. Then, $\widetilde{T}$ is maximal monotone and $0 \in \widetilde{T} v$ if and only if $v \in \mathrm{VI}(C, \nabla f)$; see [29]. By standard argument we can show that $w \in \widetilde{T}^{-1} 0$ and hence, $w \in \mathrm{VI}(C, \nabla f) = \Gamma$. Consequently, $w \in \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Step 5. We prove that $x_n \to x^*$ where $\{x^*\} = \mathrm{VI}(\Omega, \gamma S - \mu F)$.

Indeed, take an arbitrary $w \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. Utilizing (3.16), we obtain that for all $p \in \Omega$

$$\begin{aligned} \|x_{n+1} - p\|^2 \leq (\|x_n - p\| + \alpha_n b \|p\|)^2 + 2\epsilon_n \delta_n \langle (\gamma V p - \mu F p), x_{n+1} - p \rangle \\ + 2\epsilon_n (1 - \delta_n) \langle (\gamma S p - \mu F p), x_{n+1} - p \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \langle (\mu F - \gamma S) p, x_n - p \rangle &\leq \langle (\mu F - \gamma S) p, x_n - x_{n+1} \rangle + \langle (\mu F - \gamma S) p, x_{n+1} - p \rangle \\ &\leq \|(\mu F - \gamma S) p\| \|x_n - x_{n+1}\| + \frac{(\|x_n - p\| + \alpha_n b \|p\|)^2 - \|x_{n+1} - p\|^2}{2\epsilon_n (1 - \delta_n)} \\ &\quad + \frac{\delta_n}{1 - \delta_n} \langle (\gamma V - \mu F) p, x_{n+1} - p \rangle. \end{aligned} \tag{3.32}$$

Since $\limsup_{n\to\infty} \frac{\delta_n}{\epsilon_n} < \infty$, $\frac{\alpha_n}{\delta_n} \to 0$ and $\frac{\|x_n - x_{n+1}\|}{\delta_n} \to 0$ (due to Step 2), from (3.32) we get $\langle (\mu F - \gamma S) p, w - p \rangle \leq \limsup_{n\to\infty} \langle (\mu F - \gamma S) p, x_n - p \rangle \leq 0$, $\forall p \in \Omega$. Since $\mu F - \gamma S$ is $(\mu\eta - \gamma)$-strongly monotone and $(\mu\kappa + \gamma)$-Lipschitz continuous, by Minty's Lemma [24] we know that $w \in \mathrm{VI}(\Omega, \mu F - \gamma S)$. Noticing $\{x^*\} = \mathrm{VI}(\Omega, \mu F - \gamma S)$, we have $w = x^*$. Thus, $\omega_w(x_n) = \{x^*\}$; that is, $x_n \rightharpoonup x^*$. Finally, we prove that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Indeed, utilizing (3.16) with $p = x^*$, we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \epsilon_n \frac{\tau^2 - \gamma^2}{\tau})(\|x_n - x^*\| + \alpha_n b \|x^*\|)^2 + 2\epsilon_n \delta_n \|(\gamma V - \mu F) x^*\| \|x_{n+1} - x^*\| \\ &\quad + 2\epsilon_n (1 - \delta_n) \langle (\gamma S - \mu F) x^*, x_{n+1} - x^* \rangle \\ &\leq (1 - \epsilon_n \frac{\tau^2 - \gamma^2}{\tau}) \|x_n - x^*\|^2 + \epsilon_n \frac{\tau^2 - \gamma^2}{\tau} \cdot \frac{2\tau}{\tau^2 - \gamma^2} [\delta_n \|(\gamma V - \mu F) x^*\| \|x_{n+1} - x^*\| \\ &\quad + (1 - \delta_n) \langle (\gamma S - \mu F) x^*), x_{n+1} - x^* \rangle] + \alpha_n b \|x^*\| (2\|x_n - x^*\| + \alpha_n b \|x^*\|). \end{aligned}$$

Note that $\sum_{n=1}^{\infty} \alpha_n b \|x^*\| (2\|x_n - x^*\| + \alpha_n b \|x^*\|) < \infty$, $\sum_{n=1}^{\infty} \epsilon_n \frac{\tau^2 - \gamma^2}{\tau} = \infty$, and

$$\lim_{n\to\infty} \frac{2\tau}{\tau^2 - \gamma^2} \big[\delta_n \|(\gamma V - \mu F) x^*\| \|x_{n+1} - x^*\| + (1 - \delta_n) \langle (\gamma S - \mu F) x^*, x_{n+1} - x^* \rangle\big] = 0.$$

So, applying Lemma 2.17 we derive $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

Remark 3.2. The scheme (3.9) in [12, Theorem 3.2] is extended to develop our scheme (3.1) for the SFP (1.2) with constraints of finite many GMEPs, finite many variational inclusions and the fixed point problem of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$. Under the lack of the assumptions similar to those in [40, Theorem 3.2], e.g., $\{x_n\}$ is bounded, $\mathrm{Fix}(T)\cap \mathrm{int}C \neq \emptyset$ and $\|x-Tx\| \geq k\mathrm{Dist}(x, \mathrm{Fix}(T)), \forall x \in C$ for some $k>0$, the sequence $\{x_n\}$ generated by (3.1) converges strongly to a point $x^* \in \Omega$, which is a unique solution of the HVIP (over the fixed point set of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$), that is, $\langle(\mu F-\gamma S)x^*, p-x^*\rangle \geq 0, \forall p \in \Omega$.

Recalling the argument process of Theorem 3.1, we can also derive the following

Theorem 3.3. *In Theorem* 3.1, *if the conditions (C1)-(C6) are replaced by the following ones (C1)-(C5):*

(C1) $\lim_{n\to\infty}\delta_n = 0$, $\lim_{n\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$;

(C2) $\sum_{n=2}^{\infty}|\epsilon_n - \epsilon_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{\epsilon_{n-1}}{\epsilon_n} = 1$;

(C3) $\sum_{n=2}^{\infty}(\sum_{i=1}^{N}|\lambda_{i,n}-\lambda_{i,n-1}| + \sum_{k=1}^{M}|r_{k,n}-r_{k,n-1}|) < \infty$ *or*

$$\lim_{n\to\infty}(\sum_{i=1}^{N}|\lambda_{i,n}-\lambda_{i,n-1}| + \sum_{k=1}^{M}|r_{k,n}-r_{k,n-1}|)/\epsilon_n = 0;$$

(C4) $\sum_{n=2}^{\infty}(|\delta_n-\delta_{n-1}| + |\beta_n-\beta_{n-1}| + |\gamma_n-\gamma_{n-1}| + |\lambda_n-\lambda_{n-1}|) < \infty$ *or*

$$\lim_{n\to\infty}(|\delta_n-\delta_{n-1}| + |\beta_n-\beta_{n-1}| + |\gamma_n-\gamma_{n-1}| + |\lambda_n-\lambda_{n-1}|)/\epsilon_n = 0;$$

(C5) $\liminf_{n\to\infty}\sigma_n > 0$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}(\beta_n+\sigma_n) < 1$.

Then $\{x_n\}$ converges strongly to a solution x^* of the SFP provided $\|x_{n+1}-x_n\| + \alpha_n = o(\epsilon_n)$, which is a unique solution in Ω to the HVIP: $\langle(\mu F-\gamma S)x^*, p-x^*\rangle \geq 0, \quad \forall p \in \Omega$.

Acknowledgment

This research was partially supported by the Program for Outstanding Academic Leaders in Shanghai City (15XD1503100), Innovation Program of Shanghai Municipal Education Commission (15ZZ068) and Ph.D. Program Foundation of Ministry of Education of China (20123127110002). This research was partially supported by a grant from NSC.

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